Validated control using intervals and flatness; The car-trailer problem

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Reachability

We have

- a mobile robot $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
- an uncertain input $\mathbf{u}(t) \in [\mathbf{u}]$
- an initial state state $\mathbf{x}(0) \in [\mathbf{x}](0)$

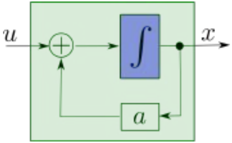
The *reachable set is*

$$\mathbb{X}(t) = \left\{ \mathbf{a} | \, \exists \mathbf{x}(0) \in [\mathbf{x}](0), \exists \mathbf{u}(\cdot) \in [\mathbf{u}], \mathbf{a} = \varphi_{t,\mathbf{u}(\cdot)}(\mathbf{x}(0)) \right\}$$

where $\varphi_{t,\mathbf{u}(\cdot)}$ is the flow.

1. Linear systems

Simple scalar linear system



$$\dot{x} = ax + u$$
$$u(t) \in [u] = [u^-, u^+]$$
$$x(0) \in [x](0)$$

This is a monotone dynamical system.

We have

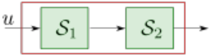
$$x(t) = e^{at} \cdot x_0 + \int_0^t e^{a\cdot(t-\tau)} u(\tau) d\tau$$

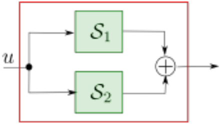
Thus

$$\begin{aligned}
\mathbb{X}(t) &= e^{at} \cdot [x_0] + \int_0^t e^{a\cdot(t-\tau)} \cdot [u] \cdot d\tau \\
&= e^{at} \cdot [x_0^-, x_0^+] + \left[\int_0^t e^{a\cdot(t-\tau)} u^- \, d\tau, \int_0^t e^{a\cdot(t-\tau)} u^+ \, d\tau\right] \\
&= e^{at} \cdot [x_0^-, x_0^+] + \frac{1}{a}(e^{a\cdot t} - 1) \cdot [u^-, u^+]
\end{aligned}$$

Denote by $\mathscr{S}_{\text{reach}}$ the class of systems for which we know how to compute an enclosure of $\mathbb{X}(t)$.

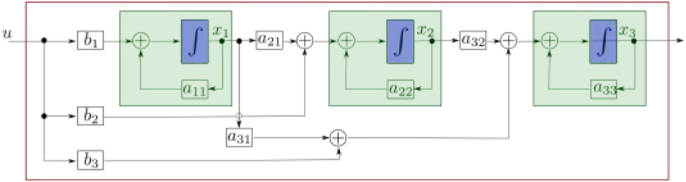
$$\begin{cases} \mathscr{S}_1 \in \mathscr{S}_{\text{reach}} \\ \mathscr{S}_2 \in \mathscr{S}_{\text{reach}} \end{cases} \Rightarrow \begin{cases} \mathscr{S}_1 \parallel \mathscr{S}_2 \in \mathscr{S}_{\text{reach}} \\ \mathscr{S}_1 \cdot \mathscr{S}_2 \in \mathscr{S}_{\text{reach}} \end{cases}$$





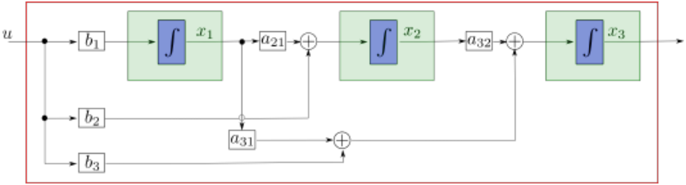
Linear triangular systems

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} u$$



Linear strictly triangular systems

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} u$$



We have

$$
\begin{aligned}
x_1(t) &= x_1(0) + b_1 \int_0^t u(\tau)d\tau \\
x_2(t) &= x_2(0) + a_{21} \int_0^t x_1(\tau)d\tau + b_2 \int_0^t u(\tau)d\tau \\
x_3(t) &= x_3(0) + a_{31} \int_0^t x_1(\tau)d\tau + a_{32} \int_0^t x_2(\tau)d\tau + b_2 \int_0^t u(\tau)d\tau
\end{aligned}
$$

$$
\begin{aligned}
x_1(t) &= x_1(0) + b_1 \int u \\
x_2(t) &= x_2(0) + a_{21} \int x_1 + b_2 \int u \\
x_3(t) &= x_3(0) + a_{31} \int x_1 + a_{32} \int x_2 + b_2 \int u
\end{aligned}
$$

$$
\begin{aligned}
x_1(t) &= x_1(0) + b_1 \int u \\
x_2(t) &= x_2(0) + a_{21} x_1(0) t + a_{21} b_1 \int^2 u + b_2 \int u \\
x_3(t) &= x_3(0) + a_{31} x_1(0) t + a_{31} b_1 \int^2 u + a_{32} x_2(0) t + a_{32} a_{21} x_1(0) t^2 \\
&\quad + a_{32} a_{21} b_1 \int^3 u + a_{32} b_2 \int u + b_2 \int u
\end{aligned}
$$

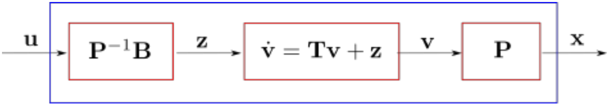
We have $(x_1, x_2, x_3) \in \mathscr{R}_0 < u >$.

Linear triangularizable systems

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

If $\mathbf{A} = \mathbf{P}^{-1}\mathbf{T}\mathbf{P}$. Set $\mathbf{v} = \mathbf{P}^{-1}\mathbf{x}$.

$$\begin{aligned} & \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \Leftrightarrow \quad & \mathbf{P}^{-1}\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{x} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u} \\ \Leftrightarrow \quad & \dot{\mathbf{v}} = \underbrace{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}}_{\mathbf{T}}\mathbf{v} + \underbrace{\mathbf{P}^{-1}\mathbf{B}\mathbf{u}}_{\mathbf{z}} \end{aligned}$$



Linear non triangularisable system

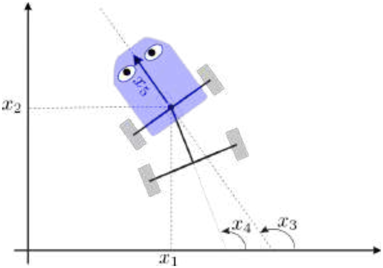
$$\begin{cases} \dot{x}_1 & = & x_2 + u \\ \dot{x}_2 & = & -x_1 \end{cases}$$

An integral formulation is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(0) + \int (u \cos t) \\ x_2(0) + \int (u \sin t) \end{pmatrix}$$

We have $(x_1, x_2) \in \mathscr{R}_0 < u >$.

For non linear systems?



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix}$$

Boatbot towing a magnetometer

We will present:

- Differential algebra: classic, suited to symbolic approaches
- Integral algebra: original ?, suited to numerical approaches

1. Differential algebra

Complex numbers

Consider the set

$$\mathbb{R} < i >= \left\{ -1, 1, 3, 3.1, \pi, \ldots, i, i^2, 1 + 2i + 5i^2, \frac{1}{1+i^5}, \ldots \right\}$$

Take the equation $i^2 + 1 = 0$. The quotient

$$\frac{\mathbb{R} < i >}{i^2 + 1} = \mathbb{C}$$

In $\mathbb{R} < i >$, i is algebraically independent.
In $\mathbb{C}$, i is algebraic (*e.g.*, $i^4 = 1$).
$\mathbb{C}/\mathbb{R}$ is a field extension.

Differential algebra

A differential ring is a ring $(\mathscr{R},+,\cdot)$ equipped with the derivative $\frac{d}{dt}$

$$\begin{array}{ll} \text{(i)} & \mathbb{R} \subset \mathscr{R} \\ \text{(ii)} & \forall a \in \mathscr{R}, \frac{d}{dt}a \in \mathscr{R} \end{array}$$

where $\mathbb{R}$ is the set of real numbers.

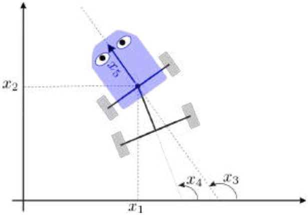
Moreover, $\frac{d}{dt}$ satisfies the classical rules. For instance

$$\begin{aligned} \frac{d}{dt}(a+b) &= \frac{d}{dt}a+\frac{d}{dt}b \\ \frac{d}{dt}(a\cdot b) &= \frac{d}{dt}a\cdot b+a\cdot\frac{d}{dt}b \end{aligned}$$

A system is finitely generated differential extension.
For instance, the system $\mathscr{S} : \dot{x} = x + u$ corresponds to the differential extension

$$\mathscr{S} : \frac{\mathbb{R} < u, x >}{\dot{x} - x - u}$$

Elimination methods are used in this context



$$\mathscr{S}: \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix}; \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The variable x_i of the system is observable if $x_i \in \mathbb{R} < y_1, y_2 >$

We have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Thus $x_1 \in \mathbb{R} < y_1, y_2 >$ and $x_2 \in \mathbb{R} < y_1, y_2 >$.

We have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \end{pmatrix}$$

Thus $x_3 = \text{atan2}(\dot{x}_2, \dot{x}_1)$ and $x_5 = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$.
Thus $x_3 \in \mathbb{R} < y_1, y_2 >$ and $x_5 \in \mathbb{R} < y_1, y_2 >$.
We can show that $x_4 \notin \mathbb{R} < y_1, y_2 >$

3. Integral algebra

An *integral ring* is a a ring $(\mathscr{R},+,\cdot)$ equipped with the integration $\int$ such that

$$\begin{array}{ll} \text{(i)} & \mathbb{R} \subset \mathscr{R} \\ \text{(ii)} & \forall a \in \mathscr{R}, \int a \in \mathscr{R} \end{array}$$

The meaning of $\int$ is the primitive which cancel for $t=0$, i.e.,

$$\int a = \int_0^t a(\tau)d\tau.$$

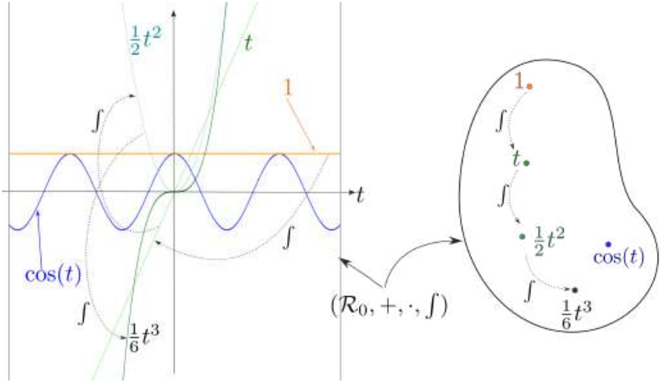
Moreover, $\int$ satisfies the classical integral rules. For instance

$$\begin{aligned}\int(a+b) &= \int a+\int b\\ \int a\cdot\int b &= \int\left(a\cdot\int b+\int a\cdot b\right)\end{aligned}$$

Consider $\mathscr{R}_0$ the smallest real integral ring.
We have

$$\begin{aligned} a = 2 \in \mathscr{R}_0 & \qquad \text{it is a constant} \\ b = 2t \in \mathscr{R}_0 & \qquad \text{since, } b = \textstyle\int a \end{aligned}$$

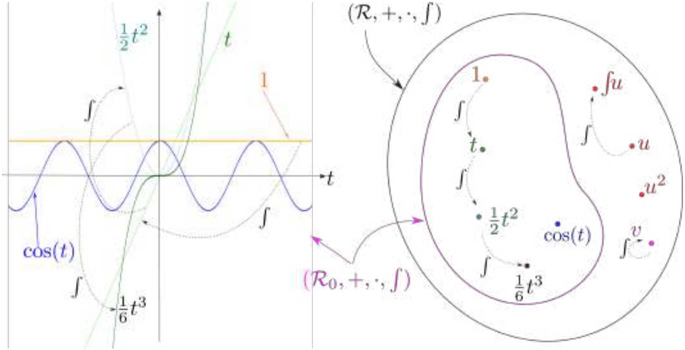
We assume that $(\mathscr{R}_0, +)$ is a topological group (*i.e.*, an infinite number of addition is allowed).


$\frac{1}{2}t^2$
t
1
$\int$
t
$\cos(t)$
$\frac{1}{6}t^3$
$(\mathcal{R}_0, +, \cdot, \int)$
1
t
$\frac{1}{2}t^2$
$\cos(t)$
$\frac{1}{6}t^3$

Consider an *integral ring extension* is $\mathscr{L}/\mathscr{R}$.
An element u of $\mathscr{L}$ is said to be integral $\mathscr{R}$-algebraic independent if

$$u, \int u, \int^2 u, \int^3 u, \ldots$$

are all independent.


$\frac{1}{2}t^2$
t
1
$\int$
t
$\int$
$\cos(t)$
$\int$
$\frac{1}{6}t^3$
$(\mathcal{R}, +, \cdot, \int)$
1
$\int$
t
$\int$
$\frac{1}{2}t^2$
$\cos(t)$
$\int$
$\frac{1}{6}t^3$
$\int u$
$\int$
u
u^2
v
$\int$
$(\mathcal{R}_0, +, \cdot, \int)$

Integral dynamical system

Given an integral ring $\mathscr{R}$.
We denote by $\mathscr{R} < u_1, u_2, \cdots >$ the integral ring generated by $\mathscr{R}$ and by a finite set $\{u_1, u_2, \dots\}$ that are integral $\mathscr{R}$-algebraic independent.

Example. Consider the integral ring $\mathscr{L} = \mathscr{R}_0 < u >$. We have

$$\begin{array}{rcl} \cos t & \in & \mathscr{L} \\ u + \int \sin u + 3 & \in & \mathscr{L} \\ u + \int \left(\sin \int^3 u \right) + 3 & \in & \mathscr{L} \end{array}$$

Definition. An *integral dynamical system* is defined as a finite subset $\{x_1,\ldots,x_n\}$ of $\mathscr{R}_0 < u_1,\ldots,u_m >$.
$x_1,x_2,\ldots$ are called the state variables
$u_1,u_2,\ldots$ are called the inputs.

Consider a system of the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

Equivalently,

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{\tau=0}^{t} \mathbf{f}(\mathbf{x}(\tau),\mathbf{u}(\tau))d\tau.$$

This system is an *integral dynamical system* if for all $i \in \{1,\ldots,n\}$, $x_i \in \mathscr{R}_0 < u_1,\ldots,u_m >$.

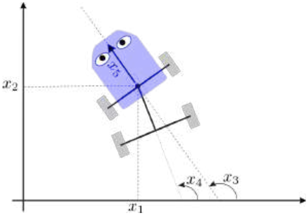
Interval extension of an integral dynamical system

Consider an integral dynamical system $\{x_1 \ldots, x_n\} \in \mathscr{R}_0 < u_1, \ldots, u_m >$.

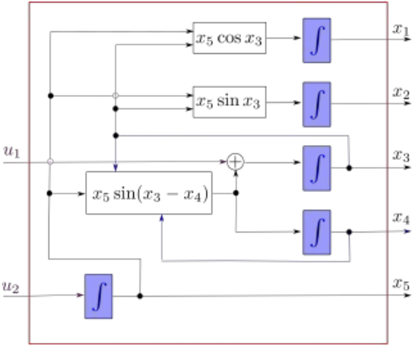
- For each x_i, we can build an expression which involves $x_1(0), \ldots, x_n(0)$, $u_1, \ldots, u_m$ as variables and $+, -, \cdot, /, \int$ as operators.
- An interval evaluation for the x_i's can be performed using the classical rules of interval arithmetic.

4. Applications

The Car-Trailer



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix}$$



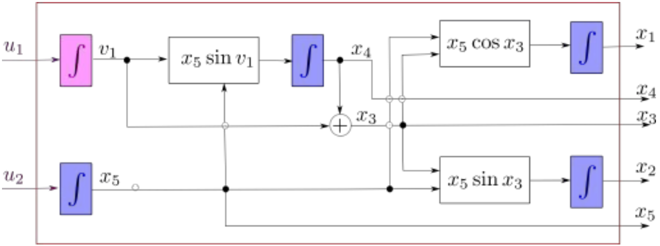
Can we conclude that x_1, x_2, x_3, x_4 belong or not to $\mathscr{R}_0 < u_1, u_2 >$?

Proposition. An integral formulation of the car-trailer is

$$\begin{cases} x_1 &= x_1(0) + \int (x_5 \cos x_3) \\ x_2 &= x_2(0) + \int (x_5 \sin x_3) \\ x_3 &= x_4 + v_1 \\ x_4 &= x_4(0) + \int (x_5 \sin v_1) \\ v_1 &= v_1(0) + \int u_1 \\ x_5 &= x_5(0) + \int u_2 \end{cases}$$

with

$$v_1(0) = x_3(0) - x_4(0).$$



Integral representation of the car-trailer

Proof. Since

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix}$$

We set $v_1 = x_3 - x_4$. We have:

$$\begin{array}{ccc} x_5 & \in & \mathscr{R}_0 < u_2 > \\ v_1 & \in & \mathscr{R}_0 < u_1 > \\ x_4 & \in & \mathscr{R}_0 < v_1, x_5 > \\ x_3 & \in & \mathscr{R}_0 < v_1, x_4 > \\ (x_1, x_2) & \in & \mathscr{R}_0 < x_3, v_5 > \end{array} \quad \Rightarrow \quad (x_1, x_2, x_3, x_4, x_5) \in \mathscr{R}_0 < u_1, u_2 >$$

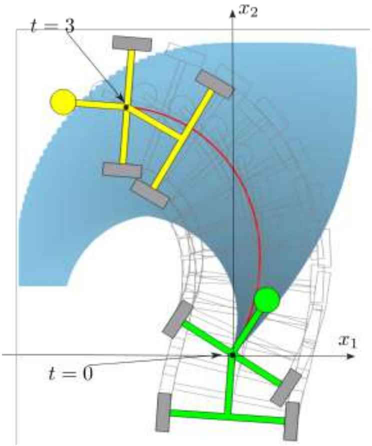
Take

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \begin{pmatrix} [u_1](t) \\ [u_2](t) \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + 10^{-2} \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}$$

$$\mathbf{x}(0) \in [\mathbf{x}](0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1.5 \\ 1 \end{pmatrix} + \begin{pmatrix} [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.2, 0.2] \\ [-0.01, 0.01] \\ [-0.001, 000.1] \end{pmatrix}.$$

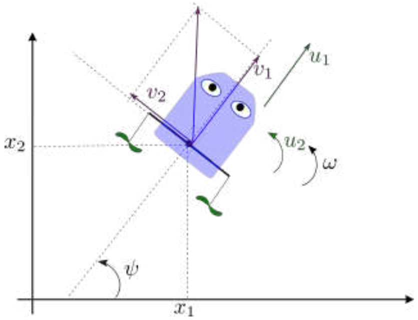
The interval trajectory is obtained by:

$$
\begin{array}{rcl}
\text{In:} & & [x_1](0),[x_2](0),[x_3](0),[x_4](0),[x_5](0),[u_1](t),[u_2](t) \\
[v_1](0) & = & [x_3](0)-[x_4](0). \\
[v_1](t) & = & [v_1](0)+\int_0^t [u_1](\tau)d\tau \\
[x_5](t) & = & [x_5](0)+\int_0^t [u_2](\tau)d\tau \\
[x_4](t) & = & [x_4](0)+\int_0^t [x_5](\tau)\cdot\cos([v_1](\tau))\cdot d\tau \\
[x_3](t) & = & [x_4](t)+[v_1](t) \\
[x_1](t) & = & [x_1](0)+\int_0^t [x_5](\tau)\cdot\cos([x_3](\tau))\cdot d\tau \\
[x_2](t) & = & [x_2](0)+\int_0^t [x_5](\tau)\cdot\sin([x_3](\tau))\cdot d\tau
\end{array}
$$



Integral simulation of the car-trailer

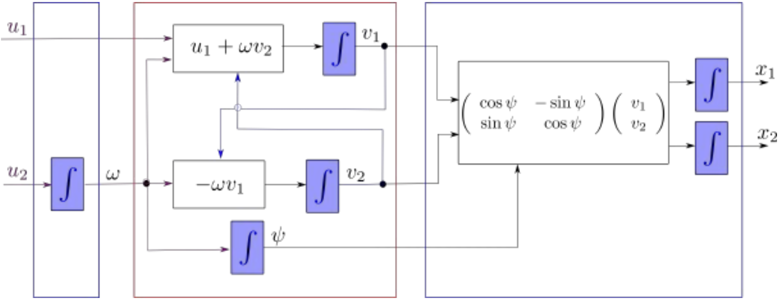
The hovercraft



The hovercraft has two propellers and can glide in all directions without any friction

The state equations are given by

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & v_1\cos\psi - v_2\sin\psi\\ \dot{x}_2 & = & v_1\sin\psi + v_2\cos\psi\\ \dot{v}_1 & = & u_1 + \omega v_2\\ \dot{v}_2 & = & -\omega v_1\\ \dot{\psi} & = & \omega\\ \dot{\omega} & = & u_2\end{array}\right.$$

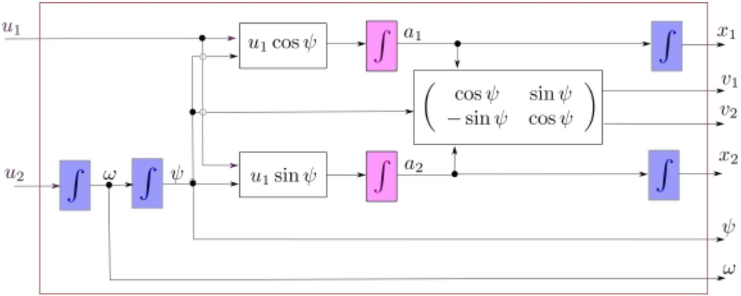

u_1
u_2
ω
$u_1 + \omega v_2$
v_1
$-\omega v_1$
v_2
ψ
$\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
x_1
x_2

Proposition. An integral formulation of the hovercraft is

$$\left\{\begin{array}{ccc}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & = & \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} \int (\cos\psi \cdot v_1 - \sin\psi \cdot v_2) \\ \int (\sin\psi \cdot v_1 + \cos\psi \cdot v_2) \end{pmatrix} \\
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} & = & \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \left(\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} + \begin{pmatrix} \int (u_1 \cos\psi) \\ \int (u_1 \sin\psi) \end{pmatrix} \right) \\
\psi & = & \psi(0) + \int \omega \\
\omega & = & \omega(0) + \int u_2
\end{array}\right.$$

where

$$\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = \begin{pmatrix} \cos\psi(0) & -\sin\psi(0) \\ \sin\psi(0) & \cos\psi(0) \end{pmatrix} \begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix}.$$


u_1
u_2
ω
ψ
$u_1 \cos\psi$
$u_1 \sin\psi$
a_1
a_2
$\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}$
x_1
v_1
v_2
x_2
ψ
ω

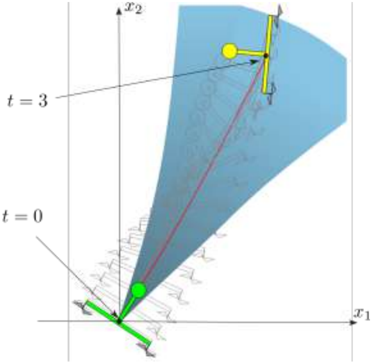
Take

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \begin{pmatrix} [u_1](t) \\ [u_2](t) \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + \begin{pmatrix} [-0.01, 0.01] \\ [-0.01, 0.01] \end{pmatrix}$$

$$\mathbf{x}(0) \in [\mathbf{x}](0) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.2, 0.2] \\ [-0.001, 0.001] \end{pmatrix}$$

The interval trajectory in the (x_1, x_2)-space is obtained by

$$
\begin{array}{rcl}
\text{In:} & & [\mathbf{x}](0), [\mathbf{v}](0), [\psi](0), [\omega](0), [\mathbf{u}](t) \\
[\mathbf{a}](0) & = & \begin{pmatrix} \cos([\psi](0)) & -\sin([\psi](0)) \\ \sin([\psi](0)) & \cos([\psi](0)) \end{pmatrix} \cdot [\mathbf{v}](0) \\
[\mathbf{a}](t) & = & [\mathbf{a}](0) + \begin{pmatrix} \int_0^t [u_1](\tau) \cdot \cos([\psi](\tau)) \cdot d\tau \\ \int_0^t [u_1](\tau) \cdot \sin([\psi](\tau)) \cdot d\tau \end{pmatrix} \\
[\omega](t) & = & [\omega](0) + \int_0^t [u_2](\tau) d\tau \\
[\psi](t) & = & [\psi](0) + \int_0^t [\omega](\tau) d\tau \\
[\mathbf{v}](t) & = & \begin{pmatrix} \cos([\psi](t)) & \sin([\psi](t)) \\ -\sin([\psi](t)) & \cos([\psi](t)) \end{pmatrix} \cdot [\mathbf{a}](t) \\
[\mathbf{x}](t) & = & [\mathbf{x}](0) + \begin{pmatrix} \cos([\psi](t)) & -\sin([\psi](t)) \\ \sin([\psi](t)) & \cos([\psi](t)) \end{pmatrix} \cdot [\mathbf{v}](t)
\end{array}
$$


x_2
$t = 3$
$t = 0$
x_1

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