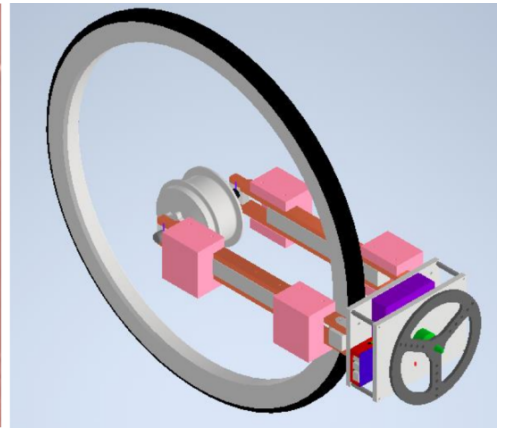


Inertial control of a flat spinning disk

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Roue autonome (Damien Esnault)

1. Flat disk

Consider a flat disk spinning in the space without any gravity.
Its inertia matrix

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

satisfies,

$$\begin{aligned} I_1 > 0, I_2 > 0, I_3 > 0 & \quad (\text{positivity}) \\ I_1 = I_2 + I_3 & \quad (\text{flatness}) \end{aligned}$$

We assume that

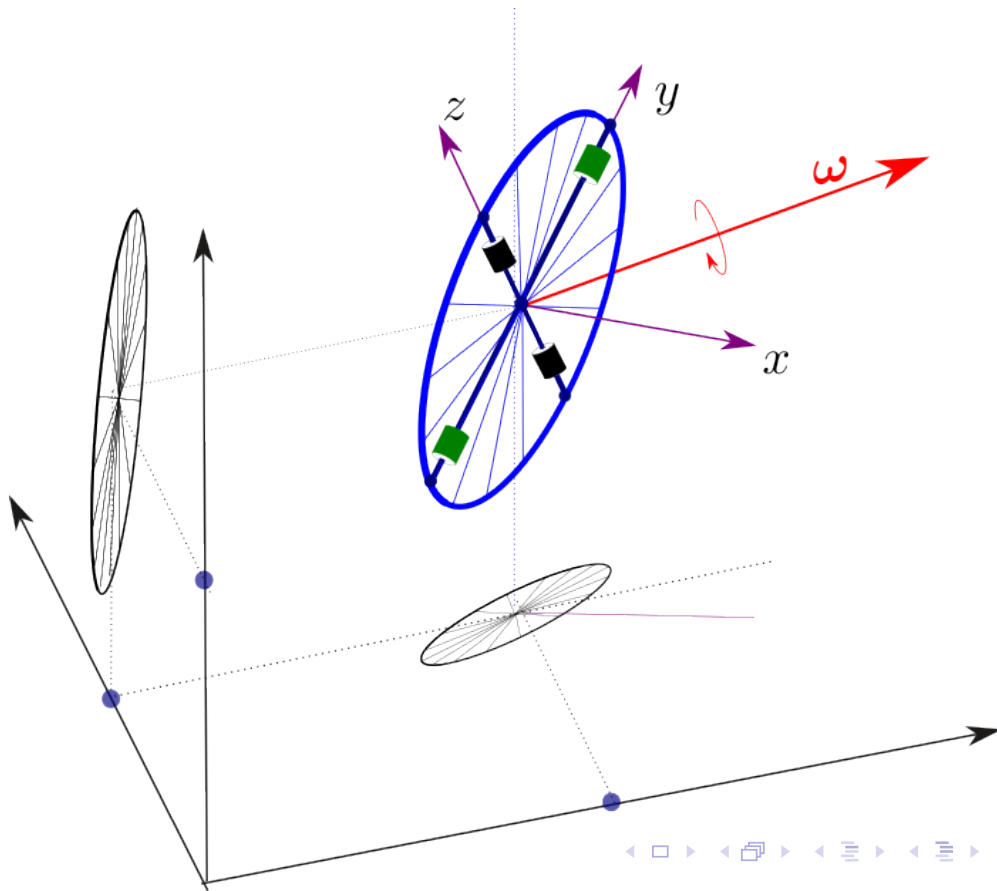
- we can control \mathbf{I} in a strap down manner,
- the center of gravity of the disk remains static.

At rest,

$$\bar{\mathbf{I}} = \begin{pmatrix} \bar{I}_1 & 0 & 0 \\ 0 & \bar{I}_2 & 0 \\ 0 & 0 & \bar{I}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}mr^2 & 0 & 0 \\ 0 & \frac{1}{4}mr^2 & 0 \\ 0 & 0 & \frac{1}{4}mr^2 \end{pmatrix}$$

where r is the radius of the disk and m is its mass.

Two symmetric pairs of masses can move along the y and z axis.



Thus

$$\mathbf{I} = \begin{pmatrix} \frac{1}{2}mr^2 + \frac{1}{2}\ell_2^2 + \frac{1}{2}\ell_3^2 & 0 & 0 \\ 0 & \frac{1}{4}mr^2 + \frac{1}{2}\ell_3^2 & 0 \\ 0 & 0 & \frac{1}{4}mr^2 + \frac{1}{2}\ell_2^2 \end{pmatrix}$$

We want to control the rotation of the disk.

State equations

The angular momentum \mathcal{L} is

$$\mathcal{L} = \mathbf{R} \cdot \mathbf{I} \cdot \omega_r$$

where ω_r is the rotation vector in the body frame, \mathbf{R} is the orientation.

Since \mathcal{L} is constant, we have

$$\begin{aligned} \dot{\mathbf{R}}\mathbf{I}\boldsymbol{\omega}_r + \mathbf{R}\cdot\dot{\mathbf{I}}\cdot\boldsymbol{\omega}_r + \mathbf{R}\cdot\mathbf{I}\cdot\dot{\boldsymbol{\omega}}_r &= \mathbf{0} \\ \Leftrightarrow \mathbf{R}^T\dot{\mathbf{R}}\mathbf{I}\boldsymbol{\omega}_r + \dot{\mathbf{I}}\cdot\boldsymbol{\omega}_r + \mathbf{I}\cdot\dot{\boldsymbol{\omega}}_r &= \mathbf{0} \end{aligned}$$

We get the Euler's rotation equation

$$\dot{\boldsymbol{\omega}}_r = \mathbf{I}^{-1} \cdot (-\dot{\mathbf{I}} \cdot \boldsymbol{\omega}_r - \boldsymbol{\omega}_r \wedge \mathbf{I} \cdot \boldsymbol{\omega}_r)$$

The state equations are thus

$$\begin{cases} \dot{\mathbf{R}} &= \mathbf{R} \cdot (\boldsymbol{\omega}_r \wedge) \\ \dot{\boldsymbol{\omega}}_r &= \mathbf{I}^{-1} \cdot (-\dot{\mathbf{I}} \cdot \boldsymbol{\omega}_r - \boldsymbol{\omega}_r \wedge \mathbf{I} \cdot \boldsymbol{\omega}_r) \end{cases}$$

Set

$$\boldsymbol{\omega}_r = (\omega_1, \omega_2, \omega_3)^\top.$$

The Euler equation $\dot{\boldsymbol{\omega}}_r = \mathbf{I}^{-1} \cdot (-\dot{\mathbf{I}} \cdot \boldsymbol{\omega}_r - \boldsymbol{\omega}_r \wedge \mathbf{I} \cdot \boldsymbol{\omega}_r)$ rewrites into

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}^{-1} \left(\begin{pmatrix} -\dot{I}_1 \omega_1 \\ -\dot{I}_2 \omega_2 \\ -\dot{I}_3 \omega_3 \end{pmatrix} - \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \wedge \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix} \right)$$

Since

$$\begin{aligned} I_1 &= \frac{1}{2}mr^2 + \frac{1}{2}l_2^2 + \frac{1}{2}l_3^2 \\ I_2 &= \frac{1}{4}mr^2 + \frac{1}{2}l_2^2 \\ I_3 &= \frac{1}{4}mr^2 + \frac{1}{2}l_3^2 \end{aligned}$$

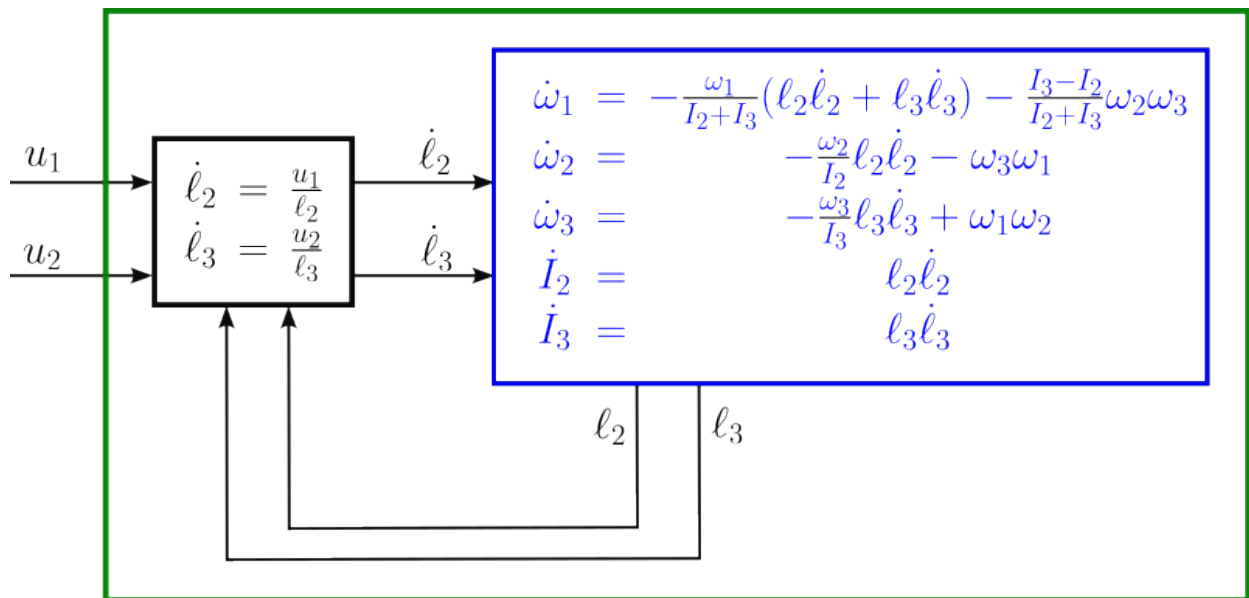
We have

$$\begin{aligned} \dot{I}_1 &= l_2\dot{l}_2 + l_3\dot{l}_3 \\ \dot{I}_2 &= l_2\dot{l}_2 \\ \dot{I}_3 &= l_3\dot{l}_3 \end{aligned}$$

The state equations of the system are thus

$$\left\{ \begin{array}{l} \dot{\omega}_1 = -\frac{\omega_1}{I_2+I_3} (l_2 \dot{l}_2 + l_3 \dot{l}_3) - \frac{I_3-I_2}{I_2+I_3} \omega_2 \omega_3 \\ \dot{\omega}_2 = -\frac{\omega_2}{I_2} l_2 \dot{l}_2 - \omega_3 \omega_1 \\ \dot{\omega}_3 = -\frac{\omega_3}{I_3} l_3 \dot{l}_3 + \omega_1 \omega_2 \\ \dot{l}_2 = l_2 \dot{l}_2 \\ \dot{l}_3 = l_3 \dot{l}_3 \end{array} \right.$$

With the inner loop



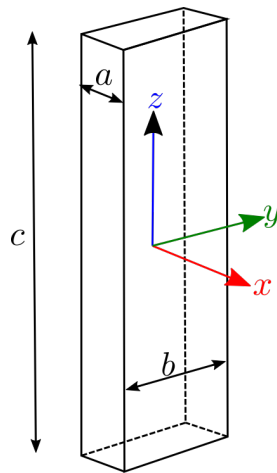
we get

$$\begin{cases} \dot{\omega}_1 &= -\frac{\omega_1}{I_2+I_3}(u_1+u_2) - \frac{I_3-I_2}{I_2+I_3}\omega_2\omega_3 \\ \dot{\omega}_2 &= -\frac{\omega_2}{I_2}u_1 - \omega_3\omega_1 \\ \dot{\omega}_3 &= -\frac{\omega_3}{I_3}u_2 + \omega_1\omega_2 \\ \dot{I}_2 &= u_1 \\ \dot{I}_3 &= u_2 \end{cases}$$

The state vector is $\mathbf{x} = (\omega_1, \omega_2, \omega_3, I_2, I_3)$ and the input vector is $\mathbf{u} = (u_1, u_2)$.

2. Dzhanibekov

The Dzhanibekov effect, discovered by Poinsot (1834):
The second principal axes is unstable



<https://youtu.be/EA1Rh5MgGKI>
(take $t=0:22$ and $t=48:19$)

The inertia matrix of the is

$$\mathbf{I} = \int_V \rho(x, y, z) \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} dx dy dz$$

i.e.,

$$\mathbf{I} = \frac{m}{12} \cdot \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}$$

The Euler equation is

$$\left\{ \begin{array}{l} \dot{\omega}_1 = \underbrace{-I_1^{-1} \cdot (I_3 - I_2)}_{\alpha_1} \omega_2 \omega_3 \\ \dot{\omega}_2 = \underbrace{-I_2^{-1} \cdot (I_1 - I_3)}_{\alpha_2} \omega_3 \omega_1 \\ \dot{\omega}_3 = \underbrace{-I_3^{-1} \cdot (I_2 - I_1)}_{\alpha_3} \omega_1 \omega_2 \end{array} \right.$$

Stability

The equilibrium points satisfy $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$. We get

$$\omega_2 \omega_3 = 0$$

$$\omega_1 \omega_3 = 0$$

$$\omega_1 \omega_2 = 0$$

i.e., among $\omega_1, \omega_2, \omega_3$ two at least are equal to zero.

Since

$$\begin{aligned}\dot{\omega}_1 &= \alpha_1 \omega_2 \omega_3 \\ \dot{\omega}_2 &= \alpha_2 \omega_1 \omega_3 \\ \dot{\omega}_3 &= \alpha_3 \omega_1 \omega_2\end{aligned}$$

at the neighborhood of $\bar{\omega}$, we have,

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} \simeq \begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \\ \bar{\omega}_3 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_1 \bar{\omega}_3 & \alpha_1 \bar{\omega}_2 \\ \alpha_2 \bar{\omega}_3 & 0 & \alpha_2 \bar{\omega}_1 \\ \alpha_3 \bar{\omega}_2 & \alpha_3 \bar{\omega}_1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 - \bar{\omega}_1 \\ \omega_2 - \bar{\omega}_2 \\ \omega_3 - \bar{\omega}_3 \end{pmatrix}.$$

Let us study the equilibrium at $\bar{\omega} = (\bar{\omega}_1, 0, 0)$:

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} \simeq \begin{pmatrix} \bar{\omega}_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{\omega}_1 \alpha_2 \\ 0 & \bar{\omega}_1 \alpha_3 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 - \bar{\omega}_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

The characteristic polynomial is

$$P(s) = s(s^2 - \bar{\omega}_1^2 \alpha_2 \alpha_3).$$

The system is unstable if $\alpha_2 \alpha_3 > 0$.

Now,

$$\begin{aligned}\alpha_2 \alpha_3 > 0 &\Leftrightarrow \frac{I_3 - I_1}{I_2} \cdot \frac{I_1 - I_2}{I_3} > 0 \\ &\Leftrightarrow (I_3 - I_1)(I_1 - I_2) > 0\end{aligned}$$

This means that I_1 is between I_2 and I_3 .

Intermediate axis theorem: the rotation of a rigid body is stable around its first and third principal axes and unstable around its second principal axis.

Simulation

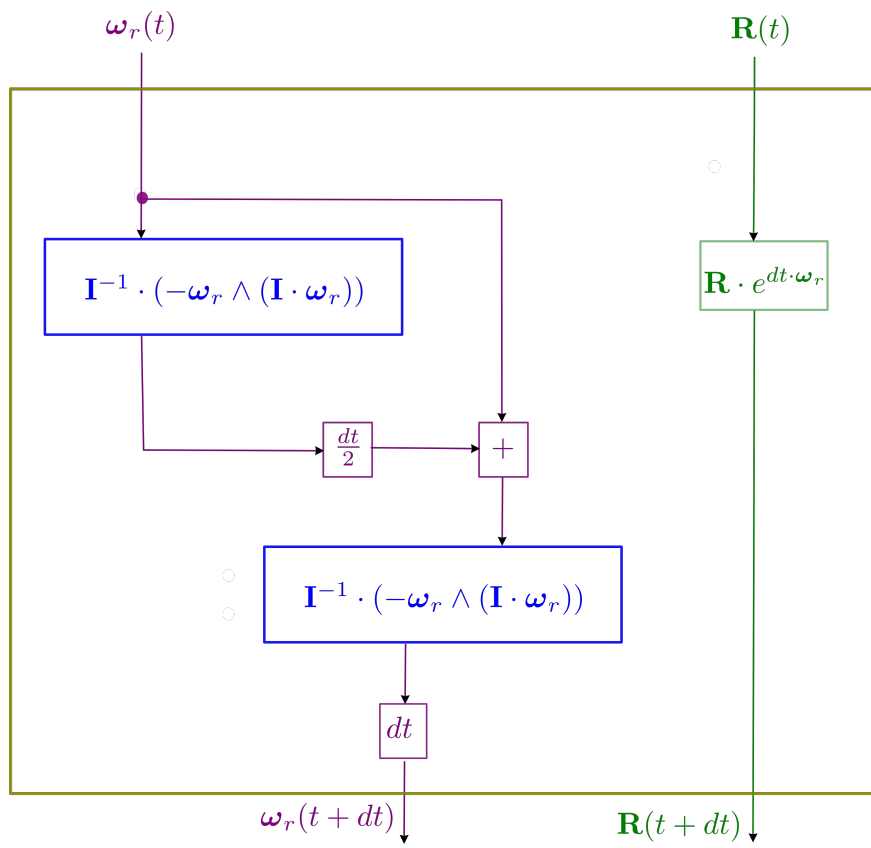
For the simulation,

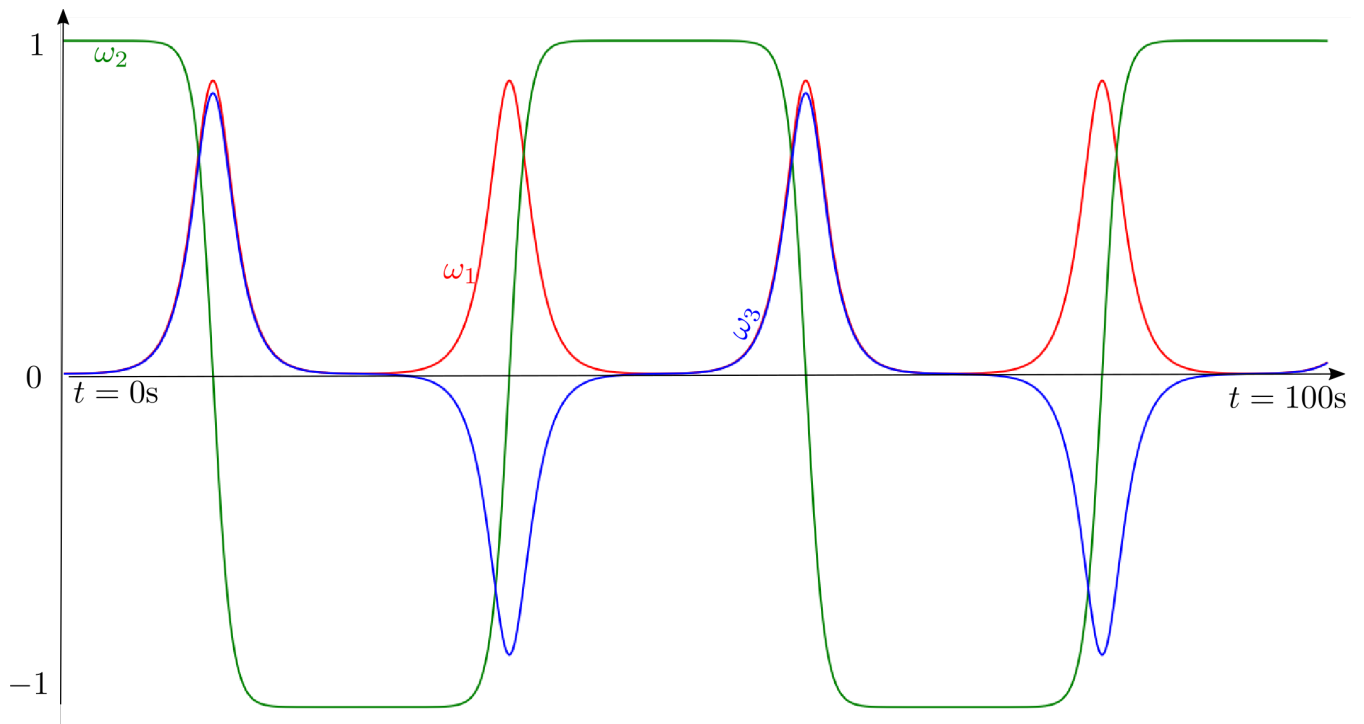
$$\begin{aligned}\dot{\boldsymbol{\omega}}_r &= \mathbf{I}^{-1} \cdot (-\boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r)) \\ \mathbf{R} &= \mathbf{R} \cdot (\boldsymbol{\omega}_r \wedge)\end{aligned}$$

we take

$$\begin{aligned}\boldsymbol{\omega}_r(t + dt) &:= \boldsymbol{\omega}_r(t) + dt \cdot \mathbf{f}(\boldsymbol{\omega}_r(t) + \frac{dt}{2} \cdot \mathbf{f}(\boldsymbol{\omega}_r(t))) \\ \mathbf{R}(t + dt) &:= \mathbf{R}(t) \cdot e^{dt \cdot \boldsymbol{\omega}_r(t)}\end{aligned}$$

where $\mathbf{f}(\boldsymbol{\omega}_r) = \mathbf{I}^{-1} \cdot (-\boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r))$.



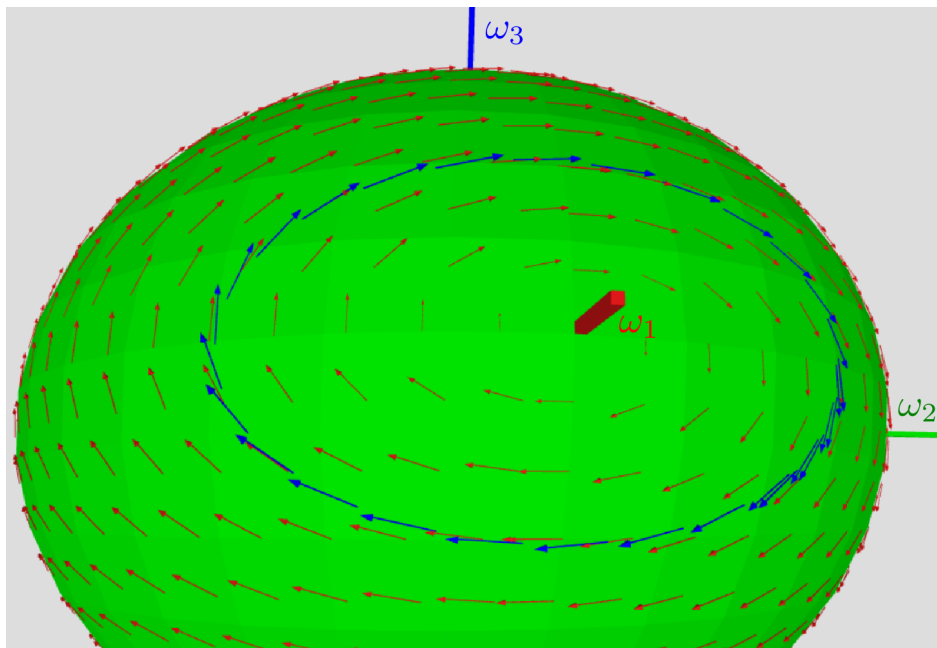


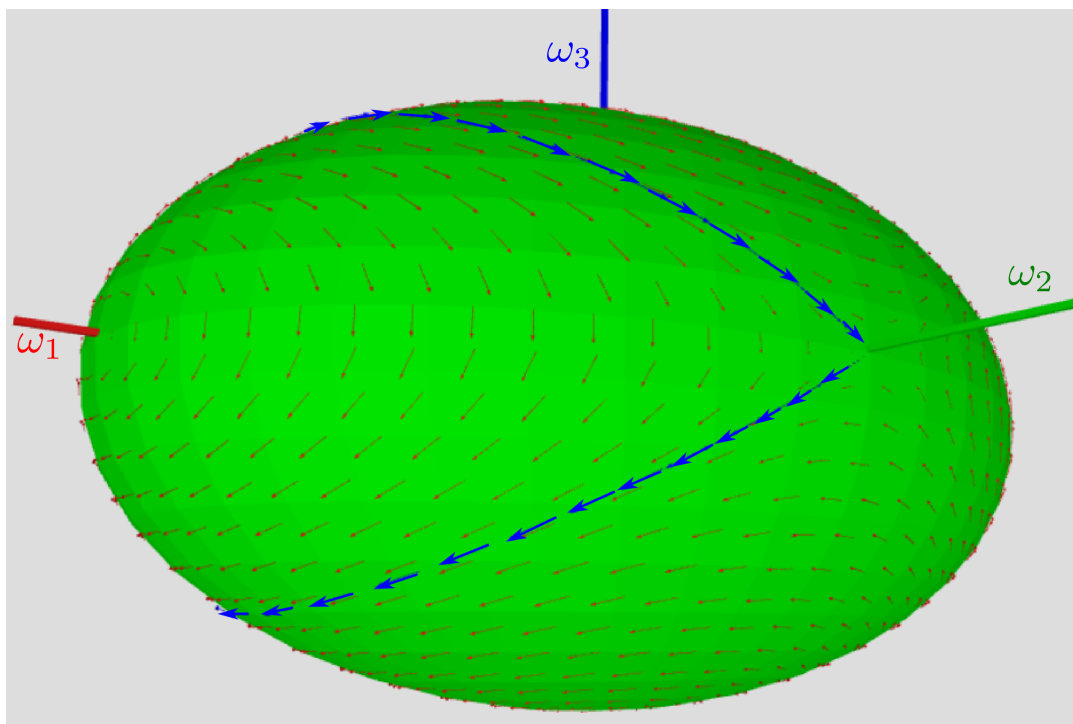
Poinsot ellipsoid

Since

$$\begin{aligned}\dot{E}_K &= \frac{d}{dt} \left(\frac{1}{2} \boldsymbol{\omega}_r^\top \mathbf{I} \boldsymbol{\omega}_r \right) \\ &= \frac{1}{2} (2 \boldsymbol{\omega}_r^\top \mathbf{I}) \dot{\boldsymbol{\omega}}_r \\ &= -\boldsymbol{\omega}_r^\top \cdot \mathbf{I} \cdot \mathbf{I}^{-1} \cdot (\boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r)) \\ &= -\boldsymbol{\omega}_r^\top (\boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r)) = 0\end{aligned}$$

we get that E_K is constant.





Dzhanibekov effect ; heteroclinic orbit

3. Controller

Recall that

$$\left\{ \begin{array}{l} \dot{\omega}_1 = -\frac{\omega_1}{I_2+I_3}(u_1+u_2) - \frac{I_3-I_2}{I_2+I_3}\omega_2\omega_3 \\ \dot{\omega}_2 = -\frac{\omega_2}{I_2}u_1 - \omega_3\omega_1 \\ \dot{\omega}_3 = -\frac{\omega_3}{I_3}u_2 + \omega_1\omega_2 \\ \dot{I}_2 = u_1 \\ \dot{I}_3 = u_2 \end{array} \right.$$

i.e.,

$$\underbrace{\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \\ \dot{I}_2 \\ \dot{I}_3 \end{pmatrix}}_{=\dot{\mathbf{x}}} = \underbrace{\begin{pmatrix} -\frac{I_3-I_2}{I_2+I_3} \omega_2 \omega_3 \\ -\omega_3 \omega_1 \\ \omega_1 \omega_2 \\ 0 \\ 0 \end{pmatrix}}_{=\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} -\frac{\omega_1}{I_2+I_3} \\ -\frac{\omega_2}{I_2} \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{=\mathbf{g}_1(\mathbf{x})} \cdot u_1 + \underbrace{\begin{pmatrix} -\frac{\omega_1}{I_2+I_3} \\ 0 \\ -\frac{\omega_3}{I_3} \\ 0 \\ 1 \end{pmatrix}}_{=\mathbf{g}_2(\mathbf{x})} \cdot u_2.$$

Choose $V(\mathbf{x}) \geq 0$ such that $V(\mathbf{x}) = 0$ when the objective is reached.

Since

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}_1(\mathbf{x}) \cdot u_1 + \mathbf{g}_2(\mathbf{x}) \cdot u_2$$

we have

$$\dot{V}(\mathbf{x}, \mathbf{u}) = \mathcal{L}_{\mathbf{f}}V(\mathbf{x}) + \mathcal{L}_{\mathbf{g}_1}V(\mathbf{x}) \cdot u_1 + \mathcal{L}_{\mathbf{g}_2}V(\mathbf{x}) \cdot u_2$$

Since the solution of

$$\begin{aligned} \min \quad & a_1 u_1 + a_2 u_2 \\ \text{s.t.} \quad & u_1^2 + u_2^2 = 1 \end{aligned}$$

where $a_1 = \mathcal{L}_{\mathbf{g}_1} V(\mathbf{x})$, $a_2 = \mathcal{L}_{\mathbf{g}_2} V(\mathbf{x})$, is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\frac{1}{\sqrt{a_1^2 + a_2^2}} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

we conclude that a Lyapunov like controller to reach our objective is

$$\mathbf{u} = -\frac{1}{\sqrt{(\mathcal{L}_{\mathbf{g}_1} V(\mathbf{x}))^2 + (\mathcal{L}_{\mathbf{g}_2} V(\mathbf{x}))^2}} \cdot \begin{pmatrix} \mathcal{L}_{\mathbf{g}_1} V(\mathbf{x}) \\ \mathcal{L}_{\mathbf{g}_2} V(\mathbf{x}) \end{pmatrix}.$$

Alignment control

We want the disk spins around one principal axis of the body.
For instance its first axis, *i.e.*, the x -axis.

We define

$$V(\mathbf{x}) = \frac{1}{2} (\omega_2^2 + \omega_3^2 + (I_2 - \bar{I}_2)^2 + (I_3 - \bar{I}_3)^2).$$

The quantity $\omega_2^2 + \omega_3^2$ corresponds to the *alignment error*.
When $V(\mathbf{x}) = 0$, we have $\omega_2 = \omega_3 = 0$ and $\mathbf{I} = \bar{\mathbf{I}}$.

We have

$$\begin{aligned}
 \dot{V}(\mathbf{x}) &= \omega_2 \dot{\omega}_2 + \omega_3 \dot{\omega}_3 + (I_2 - \bar{I}_2) \dot{I}_2 + (I_3 - \bar{I}_3) \dot{I}_3 \\
 &= -\frac{\omega_2^2 u_1}{I_2} - \omega_3 \omega_1 \omega_2 - \frac{\omega_3^2 u_2}{I_3} + \omega_1 \omega_2 \omega_3 + (I_2 - \bar{I}_2) u_1 + (I_3 - \bar{I}_3) u_2 \\
 &= \underbrace{0 \cdot \omega_1 \omega_2 \omega_3}_{\mathcal{L}_f V} + \underbrace{\left(-\frac{\omega_2^2}{I_2} + I_2 - \bar{I}_2 \right)}_{\mathcal{L}_{g_1} V} \cdot u_1 + \underbrace{\left(-\frac{\omega_3^2}{I_3} + I_3 - \bar{I}_3 \right)}_{\mathcal{L}_{g_2} V} \cdot u_2
 \end{aligned}$$

The same result is obtained using Sympy

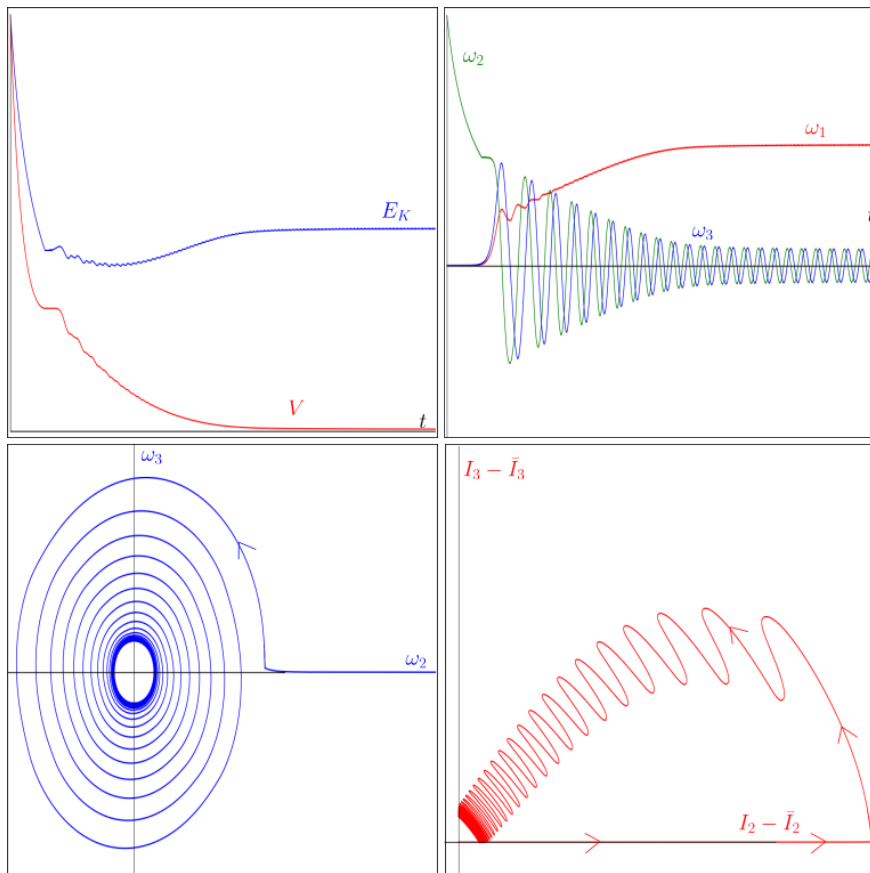
```
from sympy import *
from sympy.diffgeom import *
w1,w2,w3,I2,I3 = C.coord_functions()
E = C.base_vectors()
F=-((I3-I2)/(I2+I3))*w2*w3*E[0]-w3*w1*E[1]+w1*w2*E[2]
G1=- (w1/(I2+I3))*E[0] - (w2/I2)*E[1]+E[3]
G2=- (w1/(I2+I3))*E[0] - (w3/I3)*E[2]+E[4]
V = 1/2*(w2)**2+1/2*(w3)**2+1/2*(I2-I20)**2
    +1/2*(I3-I30)**2
LfV=LieDerivative(F,V)
Lg1V=LieDerivative(G1,V)
Lg2V=LieDerivative(G2,V)
```

Stability analysis. With our controller, we have $\dot{V}(\mathbf{x}) = 0$ if

$$\begin{cases} -\frac{\omega_2^2}{I_2} + I_2 - \bar{I}_2 = 0 \\ -\frac{\omega_3^2}{I_3} + I_3 - \bar{I}_3 = 0 \end{cases}$$

Thus

$$\dot{V}(\mathbf{x}) = 0 \Rightarrow \begin{cases} \bar{\omega}_2 = 0 \\ \bar{\omega}_3 = 0 \end{cases}$$



Passive control

If the disk loses energy it may limit the precession.

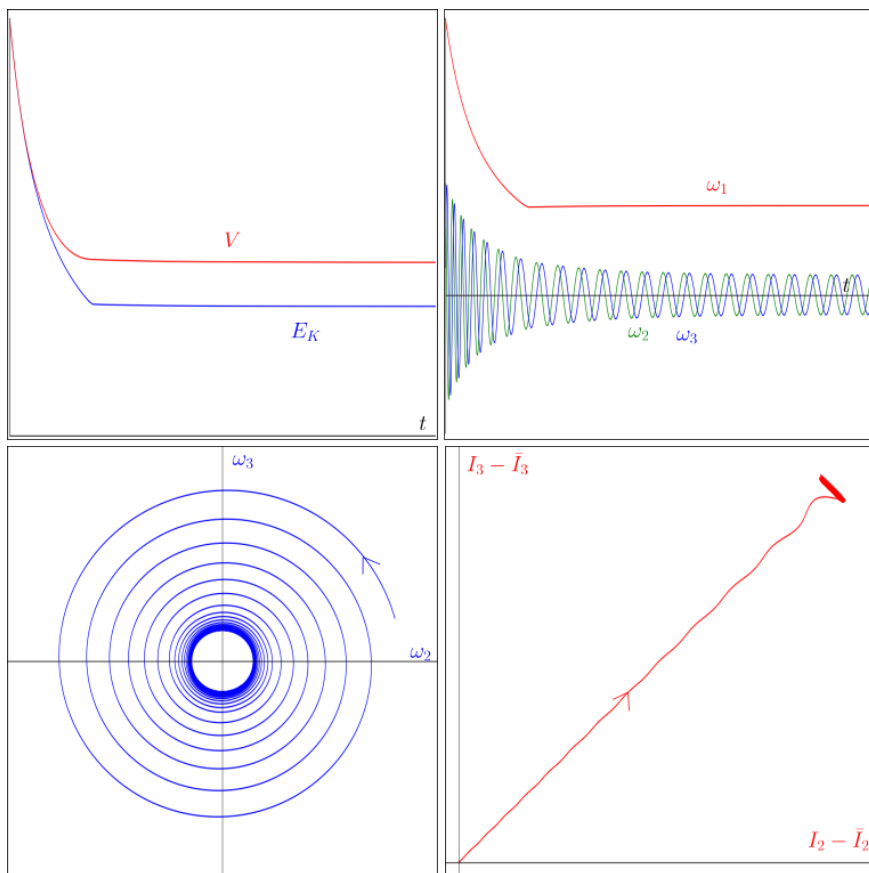
We define the *mechanical energy* as

$$V(\mathbf{x}) = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 + \frac{1}{2}(I_2 - \bar{I}_2)^2 + \frac{1}{2}(I_3 - \bar{I}_3)^2 .$$

- $\frac{1}{2}\boldsymbol{\omega}_r^\top \mathbf{I}\boldsymbol{\omega}_r$ is the kinetic energy
- $\frac{1}{2}(I_2 - \bar{I}_2)^2 + \frac{1}{2}(I_3 - \bar{I}_3)^2$ is the artificial potential energy.

We have

$$\begin{aligned}
 \dot{V}(\mathbf{x}) &= I_1 \omega_1 \dot{\omega}_1 + I_2 \omega_2 \dot{\omega}_2 + I_3 \omega_3 \dot{\omega}_3 + \frac{1}{2} \omega_1^2 \dot{I}_1 + \frac{1}{2} \omega_2^2 \dot{I}_2 + \frac{1}{2} \omega_3^2 \dot{I}_3 \\
 &\quad + (I_2 - \bar{I}_2) \dot{I}_2 + (I_3 - \bar{I}_3) \dot{I}_3 \\
 &= \underbrace{0}_{\mathcal{L}_f V} + \underbrace{\left(-\frac{1}{2} \omega_1^2 - \frac{1}{2} \omega_2^2 + I_2 - \bar{I}_2 \right)}_{\mathcal{L}_{g_1} V} \cdot u_1 \\
 &\quad + \underbrace{\left(-\frac{1}{2} \omega_1^2 - \frac{1}{2} \omega_3^2 + I_3 - \bar{I}_3 \right)}_{\mathcal{L}_{g_2} V} \cdot u_2
 \end{aligned}$$



The passivity approach does not cancel the precession

Precession control

Recall the Euler's rotation equation

$$\dot{\boldsymbol{\omega}}_r = \mathbf{I}^{-1} \cdot (-\dot{\mathbf{I}} \cdot \boldsymbol{\omega}_r - \boldsymbol{\omega}_r \wedge \mathbf{I} \cdot \boldsymbol{\omega}_r).$$

The precession energy is

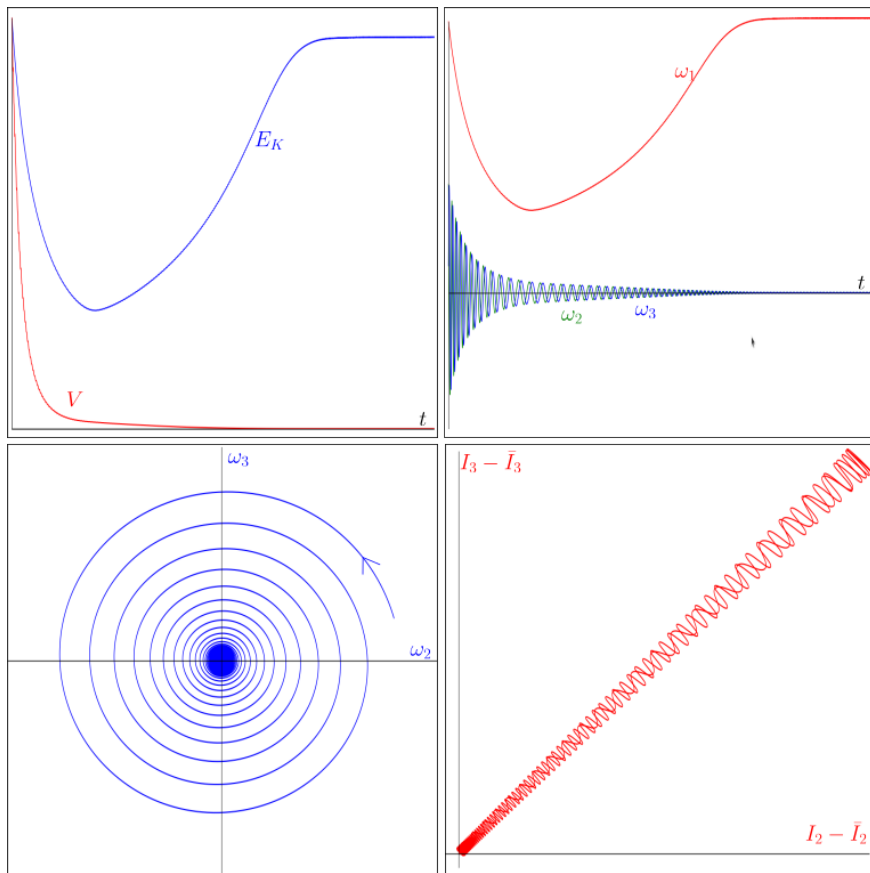
$$\begin{aligned} \|\boldsymbol{\omega}_r \wedge \mathbf{I} \cdot \boldsymbol{\omega}_r\|^2 &= \left\| \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \wedge \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix} \right\|^2 \\ &= ((I_3 - I_2)\omega_2\omega_3)^2 + ((I_1 - I_3)\omega_3\omega_1)^2 \\ &\quad + ((I_2 - I_1)\omega_1\omega_2)^2 \end{aligned}$$

Take

$$V(\mathbf{x}) = \frac{1}{2} ((I_3 - I_2) \omega_2 \omega_3)^2 + \frac{1}{2} ((I_1 - I_3) \omega_3 \omega_1)^2 + \frac{1}{2} ((I_2 - I_1) \omega_1 \omega_2)^2 + \frac{1}{2} (\bar{I}_2 - \bar{I}_2)^2 + \frac{1}{2} (\bar{I}_3 - \bar{I}_3)^2.$$

With $\omega(0) = (10, 4, 1)$, we get

Flat disk
Dzhanibekov
Controller



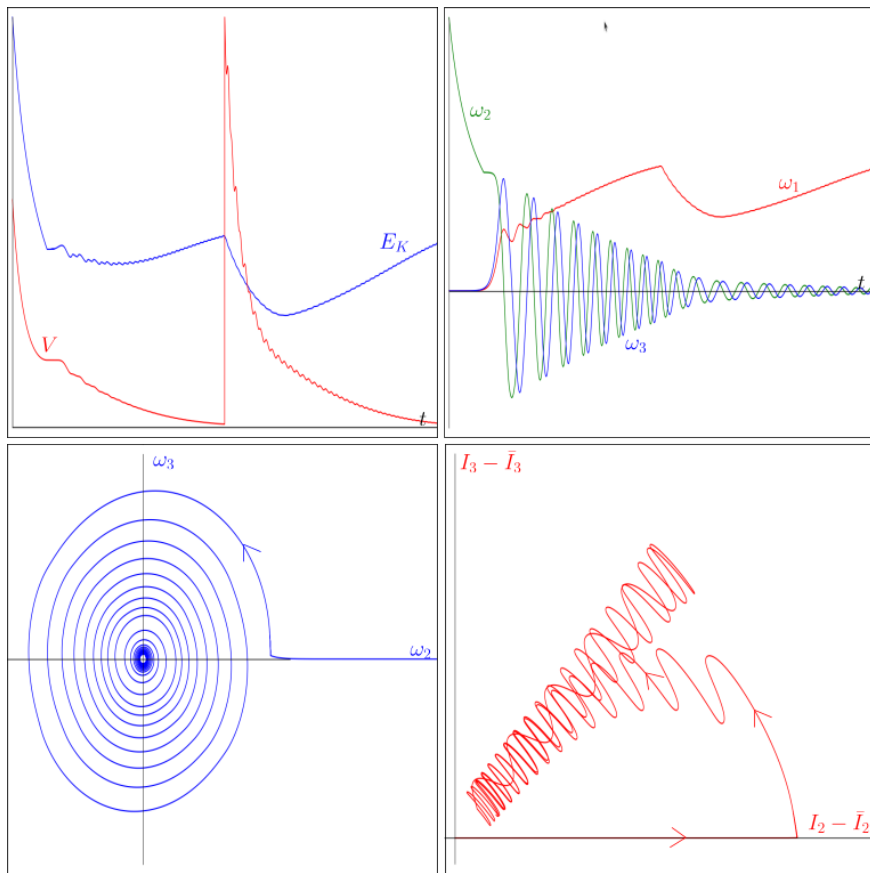
Combination

At time $t = 0$, we spin around the second axis:

$$\omega_r(0) = (10^{-5}, 10, 0)$$

and we want to spin around the second axis.

- We apply first the alignment controller
- Then we switch to the precession controller so that ω_r converges to the nearest principal axis.




An illustrative video : <https://youtu.be/gRzIDYFuMts>

Perspective


Find a unique objective function which would allow the controller to switch from one principal axes to another.


References

- 1 Flat disk problem [4, 3]
- 2 Roue autonome (Damien Esnault) [2]
- 3 Lagrangian mechanics [8]
- 4 Stability of rotating object [6]
- 5 Lyapunov control [5]
- 6 Dzhanibekov effect [7]
- 7 Passivity control [1]

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