Classifying holes of arbitrary dimensions in partially ordered cubes*

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0 Introduction

Technically, this report belongs to topology. Its motivations are, however, derived from investigations of concurrent processes. For a general discussion of concurrent processes giving rise to topological spaces, the reader is referred for instance to Pratt [7] or [8]; or to Fajstrup, Goubault, Raussen [1]; or to Goubault [6].

In the most general terms, the mathematical structures modeling cooperation between processes are endowed with a partial order and with a topology. The partial order relates events to other events they may be causes of, which captures causality and time succession. Every component of a system of processes may have its own time and these private times have only to be in tune when inter-process communication is taking place. Because of this, the order does not have to be total.

The topology in a model provides a notion of closeness between particular execution trajectories. For instance, when a process is slightly slowed down without changing the order in which it communicates with other processes (cooperates or competes for shared resources), the resulting execution is close to the original one. On the other hand, if two executions differ in the way the communication takes place, they are far from each other. In this sense, communications create "holes" in the spaces modeling concurrency. Topology makes a convenient language to discuss this.

A standard topological toolbox containing homotopies and homologies is not readily applicable for a classification of the holes, since it disregards the time- or causality-induced ordering which is paramount in computer science applications. Researchers have, therefore, attempted to adapt these concepts. Some discussion and comparison to earlier work may be found in Sec. 5.

This report is a step towards finding useful invariants of ordered topological spaces arising from concurrency considerations. The closest counterpart in algebraic topology are homotopy groups, $\pi_n(X)$. But the classical homotopy groups are based on closed trajectories, or loops, not respecting the unidirectional time flow. Without the trajectories returning to their start-points, the group operation could not be defined. For this reason, the invariants $\Omega_n X$ proposed here are not groups, they are partial orders (posets, see Sec. 1.2). They are derived from the homotopy classes of dicylinders, which are continuous and monotone mappings of n-dimensional cylinders $S^{n-1} \times I$, rather than of spheres S^n, into the space X. This reflects the fact that one of the dimensions is time, which cannot flow backwards.

The invariance of posets $\Omega_n X$ under monotone homeomorphisms is demonstrated in Sec. 4.

An important issue in setting up an invariant of this kind, is its dimension. A system of k communicating processes is modeled by a cubical complex of dimension k. A communication involving n of these processes ($2 \leq n \leq k$) is modeled as a k-dimensional rectangular hole, of which all but n coordinates are whole spaces. It is relatively well understood how to use homotopies of one-dimensional trajectories to investigate the holes for which $n = 2$, i.e., the ones corresponding to relations between pairs of processes. Relations between n-tuples of processes for $n > 2$ call for analogs of higher dimensional homotopy groups. The posets $\Omega_{n-1}X$ are useful for detecting such holes. In particular, $\Omega_2 X$ is trivial if X is a complete 3-dimensional cube; and it is the enumerable flat order (integers with a bottom) if X is a cube with a smaller subcube removed (cf. Sec. 3.3).

An important feature of the classical homotopy groups is their functoriality. The invariants proposed here are, sadly, not functorial with respect to monotone continuous maps, which seem to be the natural morphisms in the category of partially ordered topological spaces. Sec. 5.4 discusses the severity of restrictions that have to be imposed on the maps in order to win functoriality.

When a certain complex world is provided with a simpler model, that model has to be demonstrated to keep off two extremes. On one hand, it must not be so rich as too distinguish between the features which are not distinguished in the original world — this is the invariance property. On the other hand, it must not simplify out too many features which are of importance. Collapsing everything to a single point is sound but useless. Usefulness of a model can only be advocated by convincing examples. A large part of this report consists of examples showing the $\Omega_n X$ posets for various spaces X (see Sec. 2 for one-dimensional posets and Sec. 3 for two-dimensional posets).

Acknowledgements:

I am grateful to **Lisbeth Fajstrup**, **Eric Goubault** and especially to **Philippe Gaucher** for a valuable feedback on an earlier version of this report.

1 Dicylinder posets

Pospace (cf. [1]) is a topological space X with a partial order $\leq$ which is closed in the product topology of $X \times X$ (this implies that X is a Hausdorff space).

In all examples considered in this report, X is a closed subset of a unit cube I^n (for $n = 2, 3$). In most cases, it is the complement of a finite union of smaller (open) cuboids[1] whose edges are parallel to those of the large cube — such pospaces arise from investigations of finite concurrent processes (without loops). The topology on X is inherited from the cube I^n and the partial order is coordinate-wise. But technically, the definitions and results from this report apply to pospaces in their full generality.

[1] Is this a commonly used English term? *Cuboid* is explained as "rectangular parallelepiped" in the *American Heritage* dictionary.

1.1 Dicylinders and dihomotopies

By S^n (for $n \geq 0$) will be denoted the *n-dimensional sphere*, which is the quotient of the cube I^n by its boundary. In particular, S^0 is a single point (rather than two points). The Cartesian product $C^n \stackrel{\text{def}}{=} S^{n-1} \times I$ is called *n-cylinder*, for $n \geq 1$. The n-cylinder is a pospace with the usual topology and the partial order along its I-axis:

$$\langle s, t\rangle \leq \langle s', t'\rangle \stackrel{\text{def}}{\iff} s = s' \ \&\ t \leq t'$$

Every continuous and monotone mapping $f : C^n \to X$ of the unit cylinder into a pospace is called an *n-dicylinder*[2]; 1-dicylinders are referred to as *dipaths*. A dipath is, therefore, a continuous and monotone mapping $f : I \to X$ of the unit interval into a pospace.

A single dipath can be understood as an execution trajectory of a system of concurrent processes. A dicylinder of a higher dimension models many different executions of a system of processes, each corresponding to a straight-line element of the cylinder[3]. These executions are unrelated by order (no time dependency from execution to execution) but they are related by topology (some are closer to others).

A dicylinder is *trivial* if it is constant on the cylinder's elements:

$$\forall_{\langle s,t\rangle \in C^n}\ f\langle s, t\rangle = f\langle s, 0\rangle$$

Dicylinder f *precedes* dicylinder g (or: dicylinder g *extends* dicylinder f) if, for every s, $f\langle s, 1\rangle = g\langle s, 0\rangle$; i.e., if the end-sphere of f is the start-sphere for g (denoted: $f \ll g$). Whenever this is the case, the dicylinders f and g may be *concatenated* into a bigger dicylinder:

$$(f \frown g)\langle s, t\rangle \stackrel{\text{def}}{=} \begin{cases} f\langle s, 2\cdot t\rangle & \text{if } t \leq 0.5 \\ g\langle s, 2\cdot t - 1\rangle & \text{if } t \geq 0.5 \end{cases}$$

A dicylinder g is *left-maximal* if it cannot be extended to the left by concatenations, i.e., if every dicylinder f preceding g is trivial. The set of all left-maximal n-dicylinders in X will be denoted by $M^n_\ell X$.

The dicylinder posets defined in Sec. 1.2 below provide a classification of the left-maximal dicylinders. This gives an account of how different execution trajectories of a system of concurrent processes may part ways, or fork. The posets defined this way will contain information about possible futures of the system; in particular, about imminent deadlocks. It is easy to rewrite the theory presented here to a dual classification of right-maximal dicylinders. The respective posets will describe the merging of execution trajectories, possible pasts of the systems; and, in particular, the unreachable configurations. This duality corresponds conceptually to the negative and positive corner homologies studied by Gaucher in [2] and [3]. In the one-dimensional case, it has also been discussed in [1]. See also Sec. 5.1 and Sec. 5.3 below.

n-Dihomotopy in X is a continuous mapping $h : I \to (C^n \to X)$ (compact-open topology in $C^n \to X$) such that

[2] From *directed* cylinder — a reference to monotonicity.

[3] Is there a better English term? I found *element* in the *American Heritage* dictionary, accompanied by a picture.

1. for any $t \in I$, $h\,t$ is an n-dicylinder;
2. $(h\,t)\,\langle s, 0\rangle = (h\,0)\,\langle s, 0\rangle$ and $(h\,t)\,\langle s, 1\rangle = (h\,0)\,\langle s, 1\rangle$ for any $t \in I$; informally, n-dihomotopy does not move either the start-sphere or the end-sphere of the dicylinder.

Two n-dicylinders f and g are *dihomotopy equivalent* (denoted: $f\,H\,g$) if there exists an n-dihomotopy h such that $h\,0 = f$ and $h\,1 = g$.

We are mostly interested in the dihomotopy classes of left-maximal dicylinders. A necessary condition for dihomotopy equivalence is that two such dicylinders share the end-sphere. Therefore, the set of dihomotopy classes of left-maximal dicylinders in a pospace X is at least as large as the space $S^{n-1} \to Y$ of continuous mappings of the sphere S^{n-1} into the dipath-reachable part Y of X. This is too much for an useful invariant, and the identification of dicylinders will be deeper than just dihomotopic.

1.2 Collapsing dihomotopy classes of dicylinders

Consider the family of relations $R \subseteq M_\ell^n X \times M_\ell^n X$ ($M_\ell^n X$ is the set of left-maximal n-dicylinders in X) such that $f\,R\,g$ implies:

C0(R) f and g can be extended to dihomotopic dicylinders:

$$\exists_{f',g'}\ f \ll f' \ \&\ g \ll g' \ \&\ (f \frown f')\,H\,(g \frown g')$$

C1(R) for every extension g' of g there is an extension f' of f such that the concatenations are R-related:

$$\forall_{g'}\ g \ll g' \Rightarrow \exists_{f'}\ f \ll f' \ \&\ (f \frown f')\,R\,(g \frown g')$$

The union of *all* such relations in the set $M_\ell^n X$ will be denoted by $\sqsubseteq$.

Proposition 1 ($\sqsubseteq$ is well-defined)
The union of an arbitrary family of relations, satisfying the conditions C0 and C1, satisfies them too.

Therefore, $\sqsubseteq$ is the *greatest* relation satisfying the two conditions. Every two dicylinders are $\sqsubseteq$-related unless specifically ruled out either by C0($\sqsubseteq$) or by C1($\sqsubseteq$).

Proposition 2 ($\sqsubseteq$ is coarser than H)
For every dicylinders f and g, if $f\,H\,g$ then $f \sqsubseteq g$.

Proof of Proposition 2:

It is easy to see that the dihomotopy equivalence H satisfies the conditions C0(H) and C1(H). Hence, by the maximality of $\sqsubseteq$, $H \subseteq \sqsubseteq$.
□

The proof of Prop. 2 illustrates the general technique of proving properties of relation $\sqsubseteq$: define another relation R that has the required property and prove that it satisfies C0(R) and C1(R). By maximality, this will imply that R is a subset of $\sqsubseteq$.

Proposition 3 ($\sqsubseteq$ is a preorder)
Relation $\sqsubseteq$ *is a preorder on* $M_\ell^n X$ *(reflexive and transitive).*

Note that the conjunction of the conditions $\mathrm{C0}(R)$ and $\mathrm{C1}(R)$ does not guarantee the preorder property for a relation R; for instance, $\mathrm{C0}(\emptyset)$ and $\mathrm{C1}(\emptyset)$ hold, but the empty relation $\emptyset$ is not reflexive. The preorder property will follow from the maximality of $\sqsubseteq$.

Proof of Proposition 3:

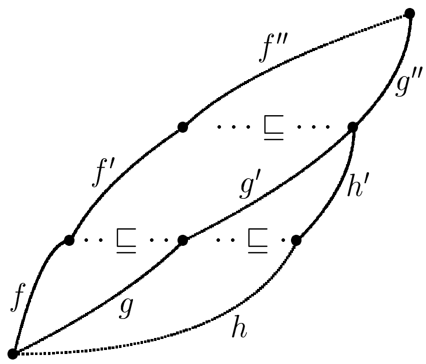
REFLEXIVITY:
Let $R \stackrel{\text{def}}{=} \{\langle f, f\rangle \mid f \in M_\ell^n X\}$. Since $\mathrm{C0}(R)$ and $\mathrm{C1}(R)$ are satisfied, $R \subseteq \sqsubseteq$, which means $\sqsubseteq$ is reflexive.

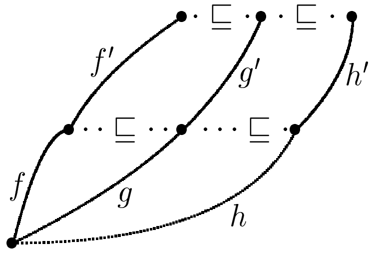
TRANSITIVITY:
Let $R \stackrel{\text{def}}{=} \sqsubseteq;\sqsubseteq \;=\; \{\langle f, h\rangle \mid \exists_g\; f \sqsubseteq g \sqsubseteq h\}$. Assume $f\,R\,h$. There exists a certain g such that $f \sqsubseteq g \sqsubseteq h$.

$\mathrm{C0}(\sqsubseteq)$ implies the existence of dicylinders g' and h' such that $(g\frown g')\,H\,(h\frown h')$. $\mathrm{C1}(\sqsubseteq)$ implies the existence of a dicylinder f' such that $f\frown f' \sqsubseteq g\frown g'$. Again by $\mathrm{C0}(\sqsubseteq)$, there exist f'' and g'' such that

$$(f\frown f'\frown f'')\,H\,(g\frown g'\frown g'')\,H\,(h\frown h'\frown g'')$$

This results in $\mathrm{C0}(R)$, the required dicylinders being $f'\frown f''$ and $h'\frown g''$.





Assume now that $h \ll h'$ for a certain dicylinder h'. By virtue of $\mathrm{C1}(\sqsubseteq)$, there exists a g' extending g such that $g\frown g' \sqsubseteq h\frown h'$ and there exists an f' extending f such that $f\frown f' \sqsubseteq g\frown g'$, resulting in $(f\frown f')\,R\,(h\frown h')$. This demonstrates $\mathrm{C1}(R)$.

By the maximality of $\sqsubseteq$, $R \;=\; \sqsubseteq;\sqsubseteq \;\subseteq\; \sqsubseteq$ which is the required transitivity of $\sqsubseteq$.
$\square$

Relation $\sqsubseteq$ will be called *collapsing preorder*. Relation

$$f \mathrel{\sqsubseteq\!\!\!\sqsupseteq} f' \stackrel{\text{def}}{\iff} f \sqsubseteq f' \;\&\; f' \sqsubseteq f$$

is, obviously, an equivalence, called *collapsing equivalence* or *collapse*. The quotient

$$\Omega_n X \stackrel{\text{def}}{=} M_\ell^n X \big/ \mathrel{\sqsubseteq\!\!\!\sqsupseteq}$$

will be called the *n-dimensional dicylinder poset* of pospace X.

Of course, $\Omega_n X$ is partially ordered (hence: poset) by the quotient of $\sqsubseteq$; this ordering relation on $\Omega_n X$ will also be denoted by $\sqsubseteq$.

The one-dimensional poset $\Omega_1 X$ will be referred to as the *fundamental poset* of X.

1.3 Intuition behind collapsing

Collapsing in general partial orders has been described in [10]. Here it is recalled in the setting adapted to the problem at hand, to make this report self-contained.

Informally, we may view dicylinders as partial realizations of maximal (not further extensible) dicylinders. While time flows, dicylinders may grow and this growth is irreversible. They may grow in different ways and end up as different maximal dicylinders. At a given moment, a dicylinder has already "made" some irreversible decisions where to be heading, while some other decisions may still be open.

The collapsing preorder $\sqsubseteq$ compares the numbers of decisions already taken. Informally, $f \sqsubseteq g$ means that f can still make all the decisions that g can make and, possibly, some more. The simplest case is when there exists an extension f' of f such that

$$(f \frown f')\, H\, g \tag{1}$$

Then f may follow the g's way by growing along f', or it may grow in a different direction. But (1) is not a necessary condition for $f \sqsubseteq g$. The end-spheres of f and g may be incomparable, or even the end-sphere of f may be above this of g, and f may still $\sqsubseteq$-precede g, when g has no possibility to reach its maximal grown-up stage inaccessible to f.

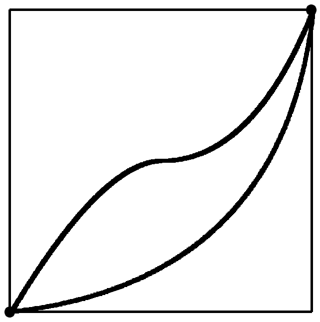
In analysing examples, I will often refer to this informal decision-making view of the $\sqsubseteq$-relation[4].

2 Examples of fundamental posets

The fundamental poset of X, $\Omega_1 X$, is derived from 1-dicylinders, or dipaths. A dipath is a continuous and monotone mapping $f : I \to X$ of the unit interval into a pospace. The dipaths that go into $\Omega_1 X$ are left-maximal, i.e. they begin in a local minimum of X. They are identified by the $\underline{\sqsubseteq}$ relation which holds of two dipaths if they both can be extended to maximal dipaths in the same (up to dihomotopy) ways. A dihomotopy in question must keep the ends of the dipaths fixed.

2.1 Full I^2

If there are no "holes", i.e., if pospace X is the complete square I^2 ordered in the picture from left to right and from bottom upwards, then all left-maximal dipaths begin in the left-bottom corner of the square and all are $\sqsubseteq$-smaller to each other. Indeed, there are no "decisions" to make. All dipaths have to grow towards the upper right corner of I^2 and once they meet there, they are dihomotopic. Therefore, the fundamental poset of I^2 consists of exactly one point.

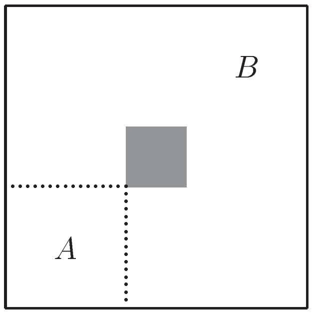


[4]Similar intuitions, but without an explicit ordering $\sqsubseteq$, are discussed in [1].

2.2 Single square hole

With one square hole, this is no longer the case. Let *Sqhole* denote the pospace:

$$Sqhole \stackrel{\text{def}}{=} I^2 \smallsetminus (0.4\,..\,0.6) \times (0.4\,..\,0.6)$$



i.e., the white part of the square depicted opposite. The left-maximal dipaths ending in zone A of the square can still grow either to the right, or upwards. The left-maximal dipaths ending in zone B have already made the decision whether to grow to the right or upwards and cannot take it back. Hence the ones ending in A are strictly $\sqsubseteq$-smaller than the ones ending in zone B. Moreover, the left-maximal dipaths entering zone B under the hole are not $\sqsubseteq$-comparable to the ones entering it to the left of the hole. Therefore, $\Omega_1 Sqhole$ is

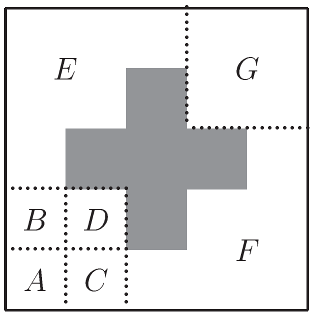
$$\begin{tikzcd} B & & B \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

In the above poset, and also in other posets depicted elsewhere in this report, the letters A, B and others have no technical significance. What counts is just the poset's shape and the letters are an informal explanation where it comes from.

2.3 Swiss flags

The shape of the hole *does* matter. Consider the "Swiss flag" space defined as follows:

$$Swiss \stackrel{\text{def}}{=} I^2 \smallsetminus (0.4\,..\,0.6) \times (0.2\,..\,0.8) \smallsetminus (0.2\,..\,0.8) \times (0.4\,..\,0.6)$$

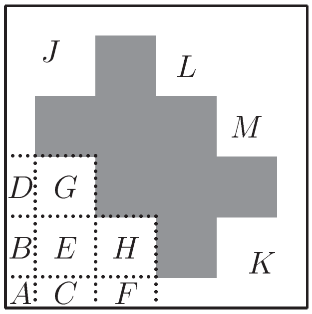


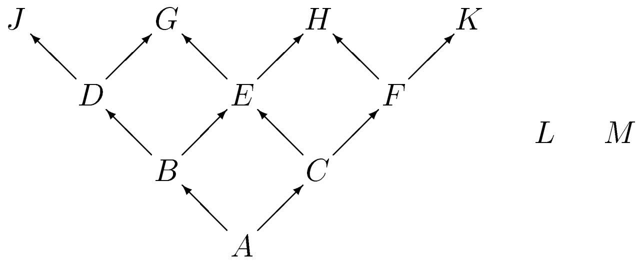
(depicted opposite). The zones in the left-bottom corner represent limitations on possible extensions of the respective dipaths — A: all extensions still possible; B: can either continue upwards or deadlock; C: can either continue to the right or deadlock; D: must deadlock. There exist left-maximal dipaths in zone G which do not begin in $\langle 0, 0\rangle$; these dipaths are not $\sqsubseteq$-comparable to any dipaths beginning in $\langle 0, 0\rangle$. Thus, $\Omega_1 Swiss$ is

$$\begin{tikzcd} E & & D & & F & \\ & B \arrow[ul] \arrow[ur] & & C \arrow[ul] \arrow[ur] & & G \\ & & A \arrow[ul] \arrow[ur] & & & \end{tikzcd}$$

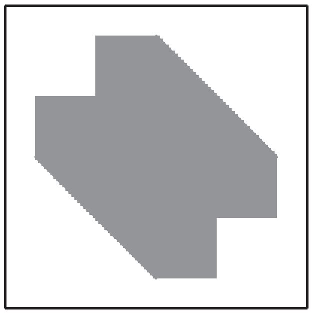
Now consider the square with two superimposed Swiss flag crosses removed, $Swiss_2$. The obvious and clumsy definition is not given, I trust the picture is enough.

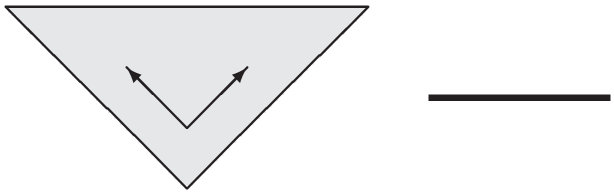
The two deadlock points give rise to a more complicated mesh of zones. Also, there are two rather than one additional start-point for left-maximal dipaths (zones L and M). Thus, $\Omega_1 Swiss_2$ is





When the number of crosses superimposed on each other grows, the resulting fundamental poset grows too. This picture may be considered a “limit” of such pospaces, $Swiss_\infty$. Its fundamental poset is a triangle with the ordering as indicated by the arrows, and a continuum of incomparable points. The top two corners of this triangle correspond to the paths that go around the hole and arrive safely at the top-right corner of the square. The remaining points of the triangle's top side correspond to the various deadlock points on the bottom-left side of the hole. The incomparable points correspond to the paths which start at the top-right side of the hole.

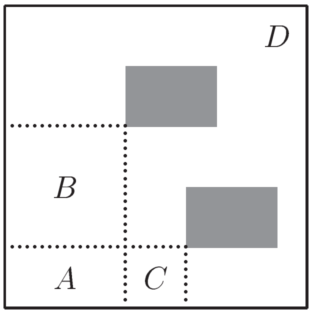




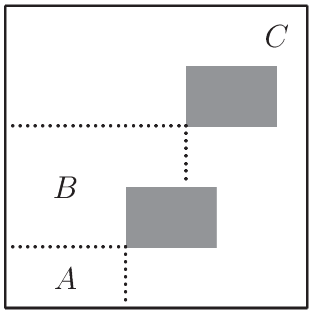
This is only to show that the fundamental poset of a relatively simple pospace may fail to be finite. In the remainder of this report I will however stick to the examples of pospaces with simpler fundamental posets.

2.4 Multiple holes

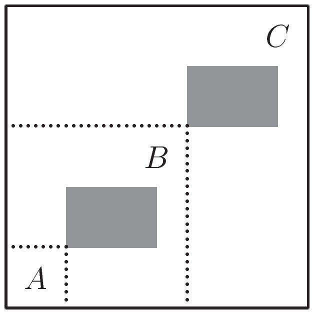
Where there is more than one hole, their mutual positioning may also affect the fundamental poset. Consider the following three examples:



$Double_1$

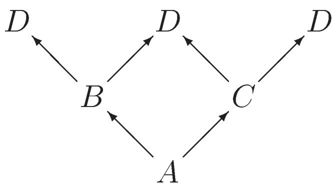


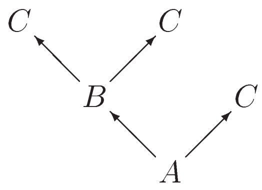
$Double_2$

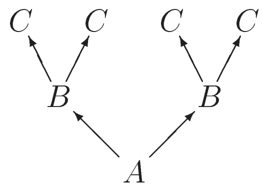


$Double_3$

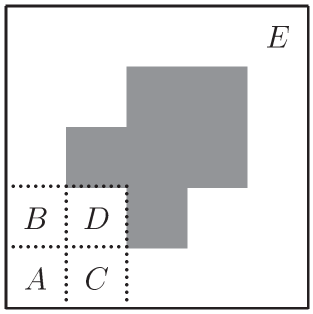
The respective posets are:







The first of these posets is the same as the fundamental poset of the Swiss-flag-derived space $Swiss_D$ depicted opposite. Admittedly, this is a weakness of Ω_1. Pospaces $Double_1$ and $Swiss_D$ are very different and it is a bad news that their fundamental posets are the same.

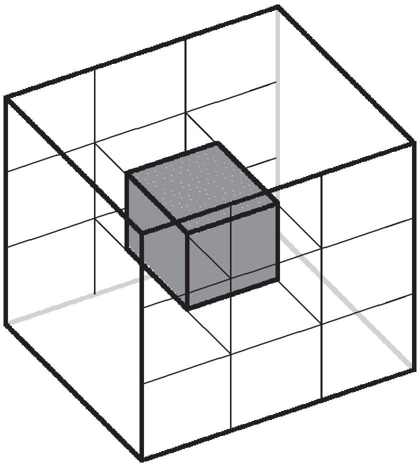


2.5 Higher dimensional pospaces

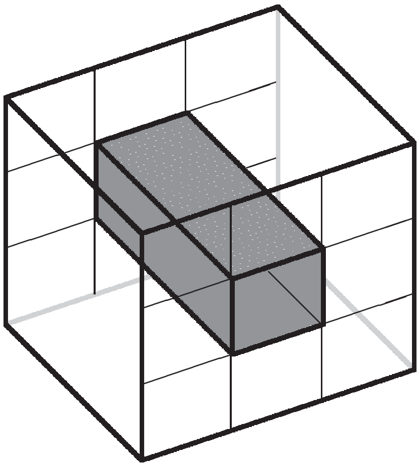
Consider two cases of I^3 with a cuboid removed:

$$Cbhole_1 \stackrel{\text{def}}{=} I^3 \setminus (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \times (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \times (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \qquad \text{and}$$

$$Cbhole_2 \stackrel{\text{def}}{=} I^3 \setminus (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \times (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \times I$$



Cbhole$_1$

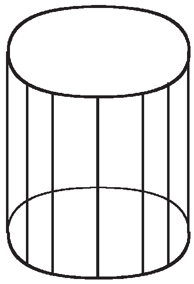


Cbhole$_2$

It is easy to see that $\Omega_1 \mathit{Cbhole}_1$ is one-point (all dipaths are collapse equivalent) while $\Omega_1 \mathit{Cbhole}_2 = \Omega_1 \mathit{Sqhole}$ (cf. Sec. 2.2). In fact, $\mathit{Cbhole}_2 = \mathit{Sqhole} \times I$, while Cbhole_1 contains a "true" three-dimensional hole.

3 Examples of two-dimensional dicylinder posets

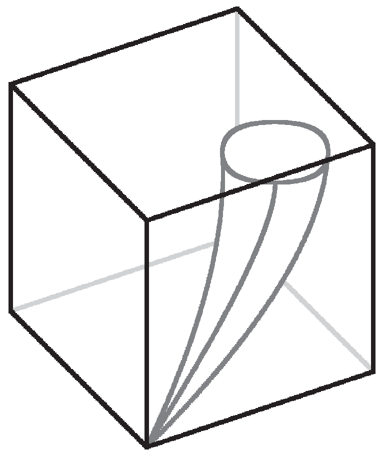
The 2-dimensional counterpart of dipath is dicylinder of dimension 2, i.e. a continuous and monotone mapping $S^1 \times I \to X$. Such dicylinders cannot be collapsed to a dipath if they go around a 3-dimensional hole. They can easily be concatenated, so as to keep the notion of extension meaningful. The concatenation corresponds to the succession in time. A dicylinder may be thought of as a family of separate execution trajectories arranged in a circle.



This section contains a number of examples of $\Omega_2 X$ for some pospaces $X \subseteq I^2$ and $X \subseteq I^3$. It is difficult to draw pictures in 3 dimensions. The reader will forgive if the ones below are not very convincing.

3.1 Full I^3

If there are no "holes", i.e., if pospace X is the complete cube I^3, all left-maximal dicylinders are dicones, which means their start-circles are glued to the least point in the cube: $\langle 0,0,0\rangle$. And they are $\sqsubseteq$-smaller to each other. The second dicylinder poset of I^3 consists of exactly one point.

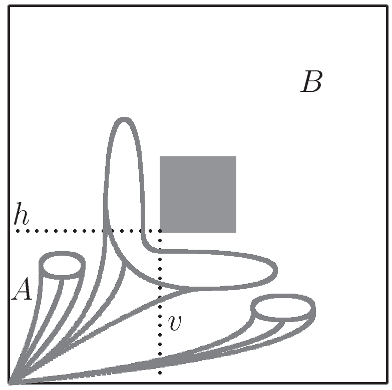


3.2 Subsets of I^2

Contrary to homotopy groups, the second dicylinder poset of a planar pospace (i.e., of a subset of the two-dimensional plane) is not trivial and may be significantly larger than its fundamental poset.

Let us try to classify the left-maximal 2-dicylinders in pospace *Sqhole* from Sec. 2.2.

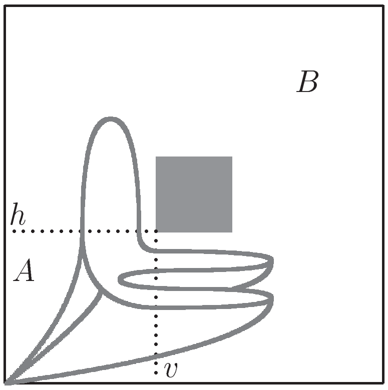
The dicylinders totally contained in area A (the left one in the picture below), are mutually equivalent. I will denote their class by 0. It makes the least element of the poset.



The ones, that intersect the vertical dotted line and whose end-circles are totally contained in area B (the right-hand one in the picture), are mutually equivalent and bigger than 0. All their dipaths have already decided to go to the maximal point of I^2 below the hole. I will denote their class by *BELOW*. *BELOW* is a maximal point in the poset.

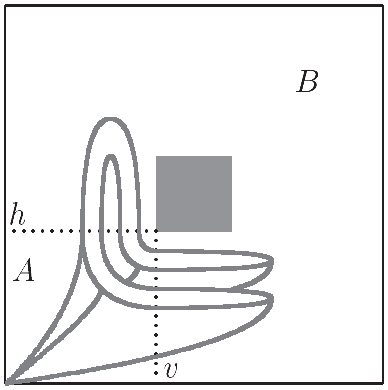
Of course, there exists another maximal point (call it *ABOVE*) consisting of the dicylinders that intersect the horizontal dotted line and whose end-circles are totally contained in area B (not depicted).

But there are many more elements in $\Omega_2\,Sqhole$: the dicylinders, whose end-circles intersect one or both dotted lines. For instance, in the one depicted above, some dipaths have decided to go above the hole and some others below the hole. I will call such dicylinders *DC* for "dog's catch", because they remind me the jaws of a dog catching a horizontally held stick. By continuity, the lack of consent in making the decision forces some dipaths to remain in the "not yet decided" area A. They can still choose either option, so the *DC*-dicylinder depicted above is not maximal.



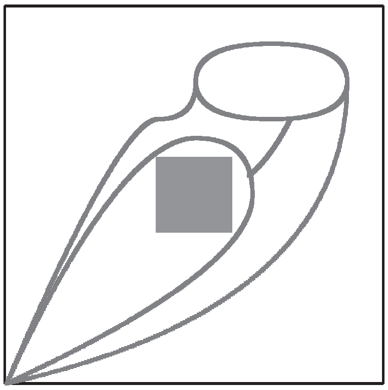
I now have to classify the equivalence classes of the dog's catch dicylinders. Consider a DC-dicylinder $f : S^1 \times I \to Sqhole$ and the trajectory of point $f\langle e^{i\alpha}, 1\rangle$ in the end-circle of f when the angle α grows from 0 to 2π. Each time this point intersects the horizontal dotted line leaving (or entering) the area A and then intersects it again returning to (or leaving) A, record this fact by letter h; record an analogous pair of intersections of the vertical dotted line by letter v. This gives an *intersection sequence* of letters h and v, which characterizes DC-dicylinders. For instance, the intersection sequence of the dicylinder depicted above is hv; in other words, the dicylinder belongs to $DC(hv)$. And the dicylinder opposite belongs to $DC(hvv)$.

If the sequence is empty, $DC(\varepsilon)$, we are back to either 0, or *BELOW*, or *ABOVE*. The simplest non-empty sequence is one-letter.



All dicylinders in $DC(v)$ are strictly $\sqsubseteq$-bigger than the ones in 0 and strictly $\sqsubseteq$-smaller than *BELOW*. Likewise, if $f \in DC(h)$ then $0 \sqsubseteq f \sqsubseteq ABOVE$ and the inequalities are strict. Dicylinders in both classes $DC(h)$ and $DC(v)$ are strictly smaller than the ones in $DC(hv)$ too.

Class $DC(hvv)$ is the same as $DC(hv)$: both dicylinders can be extended to each other. But, the dicylinders from $DC(hvv)$ can also be extended to $DC(hv\boldsymbol{hv})$ (see opposite) which is not $\sqsubseteq$-smaller than $DC(hv)$. This gives rise to an infinite increasing sequence of DC-cylinder classes in the poset.



Doubling a letter in the intersection sequence (such as from hv to hvv) leaves a dicylinder in the same class, whereas inserting a h between two v's, or inserting a v between two h's (such as from hvv to $hvhv$), makes the dicylinder strictly $\sqsubseteq$-larger. The equivalence classes of DC-dicylinders are therefore characterized by the number of letter alternations in their intersection sequences.

This exhausts all left-maximal dicones, i.e., the dicylinders with the start-circles glued to a point. But pospace *Sqhole* may contain dicylinders which are not dicones, reaching to the lower left corner of I^2 both left and under the removed square. Such dicylinders are, however, not left-maximal, so we do not take them into account. Indeed, for every such dicylinder, an appropriate change in the parameterization in its start-circle will produce another dicylinder that extends further to the left.

All in all, $\Omega_2\, \mathit{Sqhole}$ is the following partial order:

$$\begin{tikzcd}
 & & \mathit{ABOVE} & & & \\
 & v \arrow[ur] \arrow[dr] & & & & \\
0 \arrow[ur] \arrow[dr] & & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \\
 & h \arrow[ur] \arrow[dr] & & & & \\
 & & \mathit{BELOW} & & &
\end{tikzcd}$$

where h abbreviates $DC(h)$, v abbreviates $DC(v)$ and each number $n \geq 1$ abbreviates $DC((hv)^n)$.

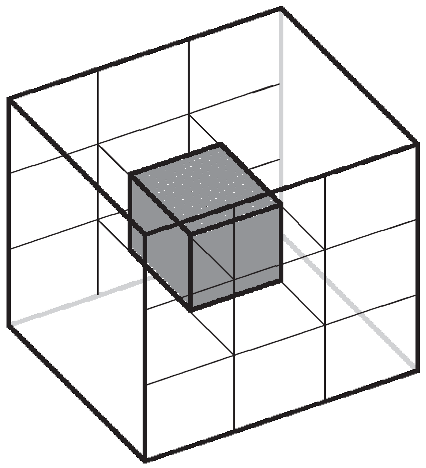
This result may seem unnatural and counterintuitive to a topologist used to the idea that two-dimensional paths in a two-dimensional plane must all be trivial. But the latter is true only because of very special properties of two-dimensional spheres used as paths for the construction of the group $\pi_2(X)$.

3.3 Single cuboidal hole

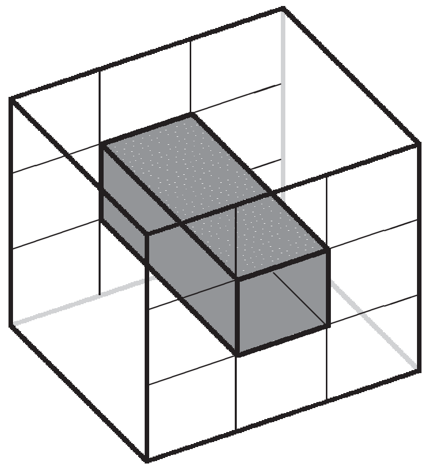
Now consider the two pospaces from Sec. 2.5:

$$\begin{aligned}
\mathit{Cbhole}_1 &\stackrel{\text{def}}{=} I^3 \setminus (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \times (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \times (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \qquad \text{and} \\
\mathit{Cbhole}_2 &\stackrel{\text{def}}{=} I^3 \setminus (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \times (\tfrac{1}{3} \,..\, \tfrac{2}{3}) \times I
\end{aligned}$$

Cbhole_1 has a “true” three-dimensional hole, while the hole in Cbhole_2 is the product of a two-dimensional hole by the entire I:



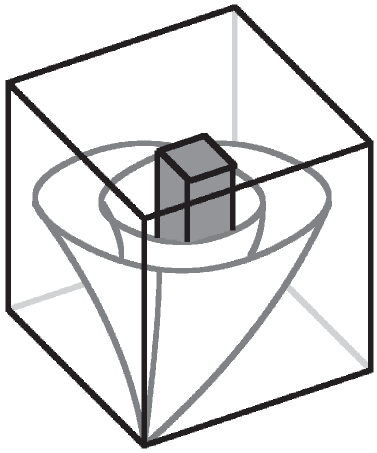
Cbhole_1

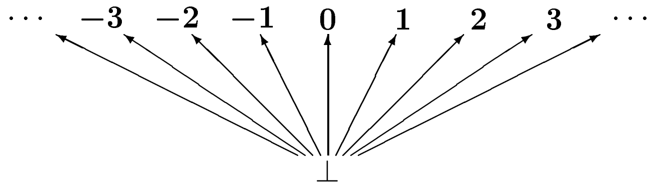


Cbhole_2

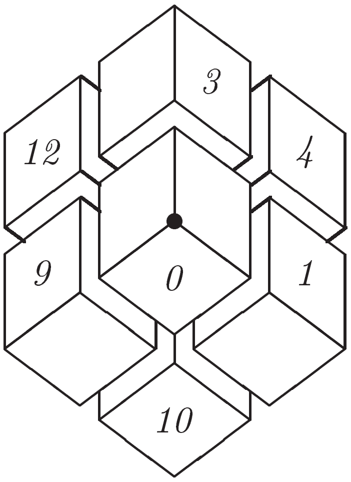
In the second case, we have $\Omega_2 Cbhole_2 = \Omega_2 Sqhole$ — one- or two-dimensional dihomotopies do not distinguish between $Cbhole_2$ and $Sqhole$.

A more interesting case is $\Omega_2 Cbhole_1$. A dicylinder may wrap around the missing cube an arbitrary finite number of times — the picture opposite presents a dicylinder that wraps two times. The second dicylinder poset is therefore the set of integers lifted by an additional bottom element $\perp$, the partial order commonly denoted as $\mathbb{Z}_\perp$. Element $\perp$ represents the class of small dicylinders, that can still be extended to either wrap around the removed cube or avoid it altogether.





To see that this is indeed the entire poset $\Omega_2 Cbhole_1$, consider the following view onto the cube's minimal point (fat dot in the figure below).



The removal of the central small cube causes $Cbhole_1$ to fall apart into 26 small cubes numbered by triples $\langle i, j, k\rangle$ with i, j, k ranging through 0, 1 and 2 and satisfying $\langle i, j, k\rangle \neq \langle 1, 1, 1\rangle$. In the picture, these triples are treated as ternary numbers; e.g.,

$$\langle 1, 1, 0\rangle \;=\; 1 \cdot 3^2 + 1 \cdot 3^1 + 0 \cdot 3^0 \;=\; \mathit{12}$$

Consider a left-maximal dicylinder f and the sequence of cubes intersected by the point $f\,\langle e^{i\alpha}, 1\rangle$ while α ranges from 0 to 2π. If this sequence does not contain *0* then the dicylinder f belongs to a maximal point of $\Omega_2 Cbhole_1$. For instance, sequence *3 12 9 10 1 4 3* represents the dicylinder from class **1** and sequence *3 4 1 10 9 12 3 4 1 10 9 12 3* represents **−2**. If the sequence is the one-element *0* (the end-circle of f totally contained in the initial small cube) then f belongs to $\perp$. We just have to make sure that the mixed sequences, the ones that contain *0* but also some non-zero cubes, represent $\perp$ too.

To see this, note the following simple "laws":

a. $w\;\mathit{0}\;v \;\sqsubseteq\; w\;\mathit{0}\;\mathit{3}^p\;\mathit{0}\;v$ for $p = 0, 1, 2$

b. $w\ \mathit{3}^p\ \mathit{0}\ \mathit{3}^p\ v \ \sqsubseteq\ w\ \mathit{3}^p\ v$ for $p = 0, 1, 2$

c. $w\ \mathit{3}^p\ \mathit{0}\ \mathit{3}^r\ v \ \sqsubseteq\ w\ \mathit{3}^p\ (\mathit{3}^p{+}\mathit{3}^r)\ \mathit{3}^r\ v$ for $p, r = 0, 1, 2$, and $p \neq r$

Law (a) should be read as follows: any dicylinder represented by a sequence $w\ \mathit{0}\ v$ can be $\sqsubseteq$-increased by replacing its subsequence $\mathit{0}$ with a longer sequence $\mathit{0}\ \mathit{3}^p\ \mathit{0}\ v$. In that latter sequence, the dicylinder's end-circle enters one of the adjacent cubes ($\mathit{3}^0$, $\mathit{3}^1$ or $\mathit{3}^2$) and immediately returns to cube $\mathit{0}$. Laws (b) and (c) should be read in the similar fashion.

Consider a sequence $\mathit{3\ 0\ 9\ 10\ 1\ 4\ 3}$. On one hand,

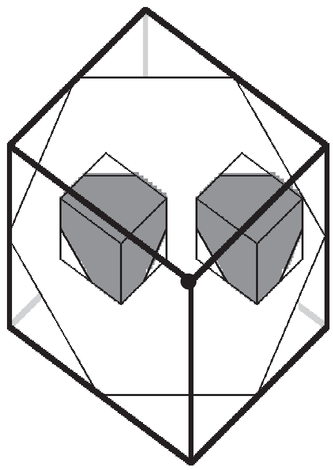
$$\mathit{3\ 0\ 9\ 10\ 1\ 4\ 3} \ \overset{\text{(c)}}{\sqsubseteq}\ \mathit{3\ 12\ 9\ 10\ 1\ 4\ 3}$$

Therefore, it is $\sqsubseteq$-smaller than $\mathbf{1}$. On the other hand,

$$\begin{aligned}\mathit{3\ 0\ 9\ 10\ 1\ 4\ 3} &\overset{\text{(a)}}{\sqsubseteq}\ \mathit{3\ 0\ 1\ 0\ 9\ 10\ 1\ 4\ 3} \ \overset{\text{(c)}}{\sqsubseteq}\ \mathit{3\ 4\ 1\ 0\ 9\ 10\ 1\ 4\ 3} \\ &\overset{\text{(c)}}{\sqsubseteq}\ \mathit{3\ 4\ 1\ 10\ 9\ 10\ 1\ 4\ 3} \ =\ (\mathit{3\ 4\ 1\ 10\ 9})\,(\mathit{3\ 4\ 1\ 10\ 9})^{-1}\end{aligned}$$

which belongs to class $\mathbf{0}$ in $\Omega_2\mathit{Cbhole}_1$. In a similar way, it may be shown that *every* sequence containing both $\mathit{0}$ and non-zero cubes represents a dicylinder which can be $\sqsubseteq$-increased to *any* maximal point in the second homotopy poset. All these dicylinders are therefore glued to $\perp$.

3.4 Multiple cuboidal holes

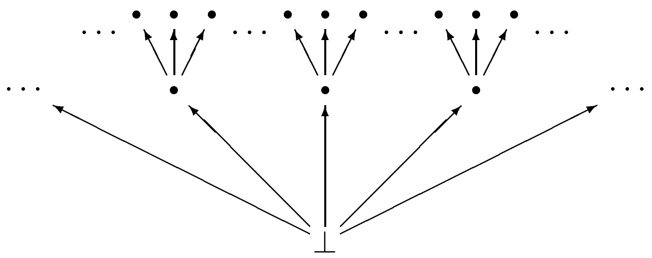


The last example from Sec. 3.3 seems to suggests a similarity between the second dicylinder poset and the first homotopy group of appropriate "cuts" of the pospace. By that similarity, the dicylinder poset would be the fundamental group of the cut lifted by a bottom element.

This similarity also holds when two disjoint appropriately located subcubes are removed — the picture opposite shows the view towards the maximal vertex of I^3 (fat dot). The fundamental poset is the lifted free group with two generators.

In general, however, I cannot state any reasonable connection between the dicylinder posets and the fundamental groups. If the two holes are located differently, the fundamental poset is different.

For instance, if they are both small, one is very close to the minimal vertex $\langle 0, 0, 0\rangle$ and the other is very close to the maximal vertex $\langle 1, 1, 1\rangle$, then the poset is



which is the partial order $\mathbb{Z}_\perp$ ($\Omega_2 Cbhole_1$ — cf. the end of Sec. 3.3) with the numbers replaced by new instances of $\mathbb{Z}_\perp$. As pointed out by Eric Goubault, this partial order is the "half-smash" product of $\mathbb{Z}_\perp$ by itself. This means the Cartesian product factored by the equalities $\langle\perp, n\rangle = \langle\perp,\perp\rangle$ for $n \in \mathbb{Z}$; but leaving $\langle n,\perp\rangle \neq \langle\perp,\perp\rangle$. Goubault suggests that this kind of directed product may prove useful for combining dicylinder posets.

See [1] for a discussion of cuts in a pospace.

4 Invariance of dicylinder posets

Since the dicylinder posets depend on the shape and location of the holes, they are not homeomorphic invariants. This does not preclude their being invariants of monotone homeomorphisms. I will show this is indeed the case. For the theorems that follow, I will need some auxiliary terminology.

A function $\varphi : X \to Y$ between two pospaces is called an *n-dicylinder surjection* if every n-dicylinder in Y, which extends the image via φ of a certain n-dicylinder f in X, is itself the image via φ of a certain n-dicylinder in X which extends f:

$$\begin{array}{l} \forall_{f:C^n\to X} \quad \forall_{g:C^n\to Y} \\ \quad f \text{ is a dicylinder in } X \ \ \& \ \ g \text{ is a dicylinder in } Y \ \ \& \ \ \varphi\circ f \ll g \implies \\ \quad \exists_{f':C^n\to X} \quad f' \text{ is a dicylinder in } X \ \ \& \ \ f \ll f' \ \ \& \ \ \varphi\circ f' = g \end{array}$$

Theorem 1 (preservation of dicylinder preorder)
Assume $\varphi : X \to Y$ is a function between two pospaces satisfying the following conditions:

- *φ is continuous and monotone;*
- *φ is an n-dicylinder surjection.*

Then for any n-dicylinders f_1 and f_2 in X such that $f_1 \sqsubseteq f_2$, $\varphi\circ f_1 \sqsubseteq \varphi\circ f_2$.

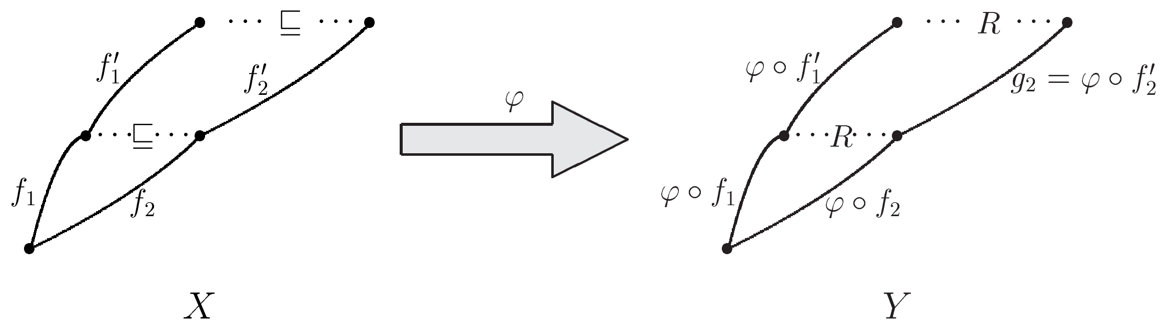
Proof of Theorem 1:

Let the relation $R \subseteq M^n_\ell Y \times M^n_\ell Y$ be defined as follows:

$$R \stackrel{\text{def}}{=} \left\{\langle\varphi\circ f_1\,,\,\varphi\circ f_2\rangle \,\middle|\, f_1 \sqsubseteq f_2 \text{ in } X\right\}$$

The property C0(R) (cf. Sec. 1.2) is obvious. To prove C1(R), assume $(\varphi\circ f_1)\,R\,(\varphi\circ f_2)$, i.e. $f_1 \sqsubseteq f_2$, for certain n-dicylinders f_1 and f_2 in X. Let g_2 extend $\varphi\circ f_2$. Since φ is a dicylinder surjection, there exists a corresponding dicylinder f_2' in X which extends f_2 and such that $\varphi \circ f_2' = g$. The property C1($\sqsubseteq$) in X implies the existence of a dicylinder f_1' in X which extends f_1 and such that $(f_1 \frown f_1') \sqsubseteq (f_2 \frown f_2')$. By the definition of relation R, this implies

$$\begin{aligned}(\varphi \circ f_1) \frown (\varphi \circ f_1') = \; & \varphi \circ (f_1 \frown f_1') \;\; R \;\; \varphi \circ (f_2 \frown f_2') \\ = \; & (\varphi \circ f_2) \frown (\varphi \circ f_2') \; = \; (\varphi \circ f_2) \frown g\end{aligned}$$



The dicylinder, the existence of which is required by C1(R), is therefore $\varphi \circ f_1'$.

Since C0(R) and C1(R) are satisfied, $R \subseteq \sqsubseteq$.

□

A *local minimum* in a pospace X is any point $x \in X$ which has an open neighborhood $U \subseteq X$ such that $\forall_{y \in U}\; x \leq y$. If X and Y are pospaces, then a function $\varphi : X \to Y$ is said to *preserve local minima* if φx is a local minimum in Y for every local minimum $x \in X$.

Assume that a certain $\varphi : X \to Y$ satisfies the premises of Thm. 1 and, additionally, φ preserves local minima. Define a corresponding mapping on the fundamental posets:

$$\begin{aligned}&\Omega_n \varphi : \Omega_n X \to \Omega_n Y \\ &(\Omega_n \varphi)[f] \;\overset{\text{def}}{=}\; [\varphi \circ f]\end{aligned} \tag{2}$$

(the equivalence classes are wrt relation $\underset{\sqsupseteq}{\sqsubseteq}$). Since φ preserves local minima, the composition $\varphi \circ f \in M_\ell^n\, X$. By Thm. 1, the definition is correct.

Theorem 2 (monotonicity of $\Omega_n \varphi$)
Assume $\varphi : X \to Y$ is a function between two pospaces satisfying the following conditions:

- φ *is continuous and monotone;*
- φ *preserves local minima;*
- φ *is an n-dicylinder surjection.*

Then the function $\Omega_n\varphi : \Omega_n X \to \Omega_n Y$ *given by* (2) *above is well-defined and monotone wrt* $\sqsubseteq$.

Theorem 3 (functoriality of Ω_n)
Ω_n *is a functor from the category of pospaces with the morphisms satisfying the assumptions of Thm. 2 to the category of partially ordered sets with monotone mappings.*

A function $\varphi : X \to Y$ is a *monotone homeomorphism* if it is a homeomorphism and it is monotone both ways (both φ and φ^{-1} are monotone). A function $\psi : A \to B$ between two partially ordered sets is an *order-isomorphism* if it is one-to-one, monotone and its reverse function $\psi^{-1} : B \to A$ is monotone.

Theorem 4 (invariance of Ω)
If $\varphi : X \to Y$ *is a monotone homeomorphism of pospaces then* $\Omega_n\varphi : \Omega_n X \to \Omega_n Y$ *is an order-isomorphism, for* $n = 1, 2, \ldots$.

Proof of Theorem 4:
This follows from Thm. 3 and from the fact that every monotone homeomorphism satisfies the assumptions of Thm. 2.
□

5 Discussion and comparison to other research

5.1 Dihomotopy sets

Fajstrup, Goubault and Raussen [1][5] define the *dihomotopy set* $\vec{\pi}(X; A, B)$ of dipaths from A to B, where A and B are subsets of pospace X. Although related, their notion and mine differ a lot. The dipaths, which go into $\vec{\pi}(X; A, B)$, are identified by the dihomotopies that are not required to keep the start- and the end-point of the dipaths fixed — they can move within the sets A, resp. B. In general, this will glue together more than the dihomotopies introduced here. On the other hand, there is some flexibility in selection of the sets A and B that delimit the dipaths.

As an example, consider the dihomotopy sets $\vec{\pi}(X; \{\mathbf{0}\}, B)$ for all pospaces X described in Sec. 2 with the subsets B ranging over X, $\{\mathbf{1}\}$ and M, where $\mathbf{0}$ and $\mathbf{1}$ are, resp., the least and the greatest points in the pospace X; and M is the set of local maxima in X— it contains the final point and all deadlocks. The table opposite gives the cardinalities of the respective sets $\vec{\pi}(X; \{\mathbf{0}\}, B)$ for particular sets B named in its upper row.

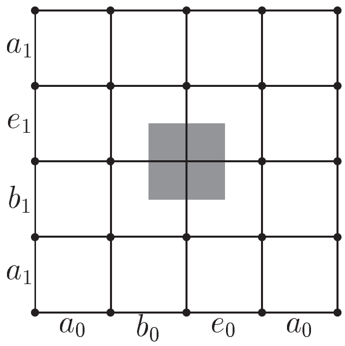
[5] Actually, this is where I am borrowing some ideas and much of my terminology. Thank you.

It is apparent from the table, that the dihomotopy sets are not a very fine tool for classifying the holes in pospaces. For instance, $\vec{\pi}(X; \{\mathbf{0}\}, X)$ is the same for all the example pospaces; $\vec{\pi}(X; \{\mathbf{0}\}, \{\mathbf{1}\})$ is the same for 6 out of the 11 example pospaces; and $\vec{\pi}(X; \{\mathbf{0}\}, M)$ is the same for 5 out of the 11 example pospaces.

	X	$\{\mathbf{1}\}$	M
I^2	1	1	1
Sqhole	1	2	2
Swiss	1	2	3
$Swiss_2$	1	2	4
$Swiss_\infty$	1	2	3
$Double_1$	1	3	3
$Double_2$	1	3	3
$Double_3$	1	4	4
$Swiss_D$	1	2	3
$Cbhole_1$	1	1	1
$Cbhole_2$	1	2	2

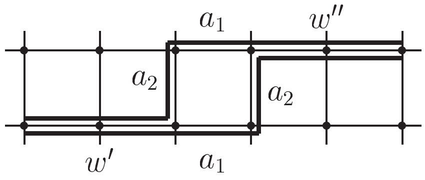
A different choice of subsets A and B might improve this, but it is difficult to make such a choice without a deep knowledge of the pospace X to begin with. Apparently, more structure in the dihomotopy sets is needed. In $\Omega_1 X$, an extra structure is provided by the partial order derived from dipath concatenation.

5.2 A discrete alternative



The idea to keep the ends of the dipaths fixed but to identify all the dipaths having dihomotopic extensions comes from my earlier [9]. That paper was written in a different, discrete style. It is customary in computer science to view everything as discrete. To comply with this fashion, I have come up with discrete analogs of the concepts derived from algebraic topology, such as homotopy.

In [9], discrete partially ordered grid of isolated points played the rôle of pospace. Actions performed by the processes were instant jumps from a grid point to its right neighbor or to its upper neighbor. Dipath (in [9], called *path*) was a finite or countably infinite sequence of such jumps. Dihomotopy (called *path homotopy*) was generated by transpositions of neighboring actions: $(w' \frown a_1 a_2 \frown w'')\; H\; (w' \frown a_2 a_1 \frown w'')$.Note that such transpositions could never be performed over a missing point in the grid, so the missing points were taking place of the holes.

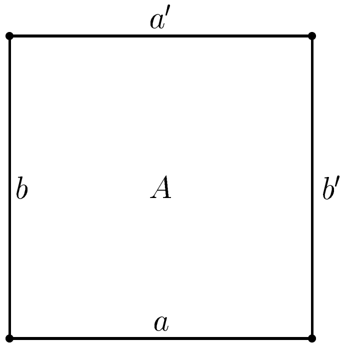


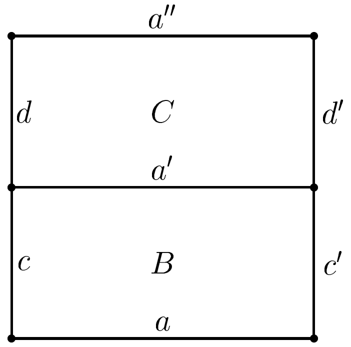
Similarly to the case described in this report, the set of path homotopy classes was at least as large as the reachable part of the whole (discrete) space and it needed a factorization. How to perform such a factorization (called *collapsing*), was shown in [10]. Informally, the (homotopy classes of) paths were identified whenever they could be extended in the same way — this is similar to the $\subsetneq$ equivalence used in this report. The resulting partial orders were exactly the ones presented in this report. A corollary is that similar approaches yield the same end-results regardless of whether the initial spaces are discrete or continuous. The choice of the one or the other is a matter of taste.

After reading [1] and after discussions with the authors, my taste changed and I now prefer to translate discrete processes to continuous pospaces. In a sense, the one-dimensional case of this report is [9] in a continuous disguise.

A discrete approach similar in spirit to that of [9] appears in Goubault [6] (Sec. 7.4). It is based on higher dimensional automata — cf. van Glabbeek [4], (or a short version [5]). The dihomotopies (called simply *homotopies*) between dipaths (called *paths*) are defined by (an algebraic counterpart of) transpositions of neighboring actions. This implies the fixed ends of the dihomotopies. But the left-maximality of dipaths is not required. Goubault [6] stops short of introducing a partial order on the obtained set of dihomotopy classes; instead, he blows it up to an Abelian group $\Pi_1(X)$ by an algebraic manipulation.

Since there is no counterpart of equivalence $\overset{\sqsubset}{\sqsupset}$, the fundamental group thus obtained is very large. More disturbingly, it depends on a particular "cubicalation"[6] of the pospace. For instance, if I^2 is decomposed into one square, four sides and four corners, as in the left-hand side picture below, then $\Pi_1(X)$ is freely generated by 3 dipaths: a, b and a' (ab' is equivalent to ba' and to $b + a'$, while b' is equivalent to $b + a' - a$). If, however, it is cubicalated into two rectangles, seven sides and six corners, as in the right-hand side picture below, then $\Pi_1(X)$ is freely generated by 5 dipaths: a, c, d, a' and a''.





It may be argued that, when investigating higher dimensional automata rather than pospaces, a different cubicalation is a different automaton, is a different behavior. After all, [6] does not claim to deal with pospaces. I am, however, attracted to a more topological point of view, which may be expressed in the terminology of concurrent processes as follows. In the left-hand side picture above, b is an indivisible action independent of a — this is marked by the existence of the two-dimensional square A. In view of this independence, the behavior of the system does not change dramatically when b is replaced by two lower-level actions c and d that, collectively, do the same as b. So an invariant should remain the same.

[6] studies also particular subgroups of $\Pi_1(X)$; notably, the one corresponding to the paths between the initial and the final points: $\Pi_1^{\mathbf{0},\mathbf{1}}(X)$. This subgroup does not depend on cubicalation. But it suffers from the same drawback as the dihomotopy set $\vec{\pi}(X; \{\mathbf{0}\}, \{\mathbf{1}\})$ (cf. Sec. 5.1): it is too indiscriminating. It cannot distinguish, for instance, between

[6] This is how decomposition into cubes is called in [6]; probably by analogy to triangulation.

different shapes of a single hole (examples *Sqhole*, *Swiss*, $Swiss_2$, $Swiss_\infty$ and $Swiss_D$ in Sec. 2).

5.3 Higher dimensions

I do not know about any mature proposal of a counterpart for pospaces of the classical homotopy groups in dimensions higher than 1. Goubault [6] puts forward two different definitions but he admits that they are not well understood yet. The first definition, by higher dimensional dipaths, depends on cubicalation and leads to enormous groups even for simple pospaces with minimal decompositions. The second definition, by suspensions, is conjectured to yield isomorphic groups; if this is indeed the case then its shortcomings are isomorphic too. Besides, both proposals have a strong flavor of homologies rather than homotopies.

On the other hand, higher dimensional directed homologies have been studied by Goubault [6] and by Gaucher [2] and [3]. The latter author defines three homology theories: globular H_*^{gl}, negative corner H_*^- and positive corner H_*^+. Negative corner homology is based on similar intuitions as the dicylinder classes presented here. Very informally, the cycles correspond to dicones (dicylinders with their start-spheres contracted to a point); while the boundaries are those dicones which can be extended to continuous and monotone maps of full cones (with the interior). Such extensions can only be done when no holes get in the way.

The relation between the negative corner homology and the dicylinder posets is unclear as yet. The conceptual starting point is similar indeed, but the factorizations performed (Gaucher: quotient of the group of cycles by the subgroup of boundaries; me: quotient of the poset by its maximal collapse) yield very different results.

My personal feeling is that the approach presented here is a close counterpart for the directed case of classical homotopy groups, even though the structures obtained are not groups. Similarly to the standard homotopy groups, the elements that make the $\Omega_n X$ posets are the mappings of simple geometric objects identified by an equivalence which has a simple geometric meaning. On the other hand, negative corner homologies H_*^- are in the spirit of the classical homologies: they are made of linear combinations of mappings identified by an algebraically defined equivalence.

The proposition, that relation between the negative corner homologies and the dicylinder posets is similar to the relation between the classical homology and the classical homotopy groups, requires more study. But there is one *caveat*: the negative corner homologies of concrete pospaces seem to be more difficult to calculate than the dicylinder posets.

5.4 The issue of functoriality

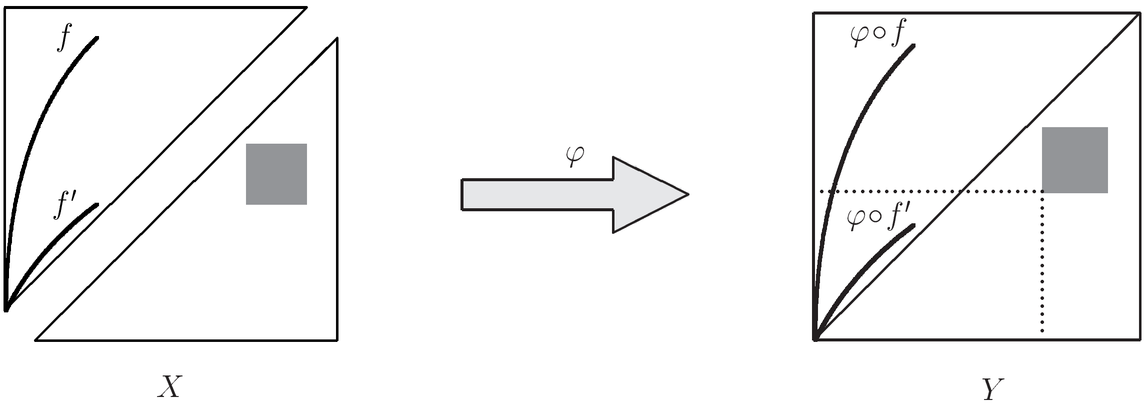
The reader may wonder why I cared to lead her/him through a pretty standard proof of invariance (Sec. 4). The reason is my dream about functoriality of defined constructions with respect to a reasonable class of pospace morphisms. Invariance is just a side product.

A goal on its own is finding a largest possible class of morphisms for which Ω_n are still functors. Thm. 3 in Sec. 4 points to the functions $\varphi : X \to Y$ which satisfy the following conditions:

- φ is continuous and monotone;
- φ preserves local minima;
- φ is an n-dicylinder surjection for every $n = 1, 2, \ldots$.

Luckily, monotone homeomorphisms belong to this class. Sadly, the class is very restrictive and far from natural. The problem is the last condition.

Dicylinder surjection may be just an ill-conceived notion. But more importantly, there do exist many seemingly harmless mappings of pospaces, which do not preserve the equivalence $\mathrel{\overset{\sqsubset}{\sqsupset}}$ on the set of left-maximal dicylinders. For such mappings, the definition (2) in Sec. 4 of $\Omega_n\varphi$ is incorrect. As an example, consider the following mapping $\varphi : X \to Y$:



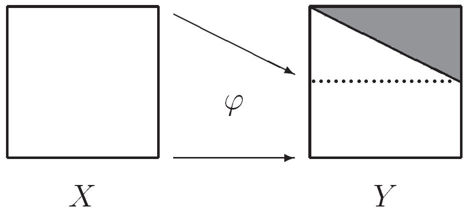
(two triangular connected components are glued together along the diagonal). This mapping is monotone, continuous, minima preserving and surjective, but not 1-dicylinder surjective. Relation $f \mathrel{\overset{\sqsubset}{\sqsupset}} f'$ holds in X, but relation $\varphi \circ f \sqsubseteq \varphi \circ f'$ does not hold in Y.

The following example shows that the dicylinder posets are not invariant under dihomotopy equivalences. Two pospaces X and Y are *dihomotopy equivalent* provided there exist monotone and continuous mappings $X \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Y$ such that $\psi \circ \varphi \simeq Id_X$ and $\varphi \circ \psi \simeq Id_Y$. The relation $\simeq$ of dihomotopy equivalence between two mappings $\alpha, \beta : A \to B$ is defined as follows: $\alpha \simeq \beta$ if there exists a continuous $h : I \to (A \to B)$ s.t.

- $h\,0 = \alpha$ and $h\,1 = \beta$,
- $h\,t$ is continuous and monotone for every $t \in I$.

Let $X \overset{\text{def}}{=} I^2$ and $Y \overset{\text{def}}{=} \{\langle x, y\rangle \in I^2 \mid y \leq 1 - \frac{x}{2}\}$. Thus Y is the unit square with the upper right corner clipped off. Define $\varphi : X \to Y$ by

$$\varphi \langle x, y\rangle \overset{\text{def}}{=} \left\langle x\,,\,\frac{y}{2}\right\rangle$$



This mapping is continuous and monotone; besides, it is a monotone homeomorphic injection but not a surjection. Define $\psi : Y \to X$ by

$$\psi \langle x, y\rangle \overset{\text{def}}{=} \left\langle x\,,\,\frac{2}{2-x}\cdot y\right\rangle$$

This mapping is continuous and monotone too; besides, it is surjective and homeomorphic, but it is not a monotone homeomorphism, since ψ^{-1} is not monotone. The composition is:

$$(\psi \circ \varphi)\ \langle x, y\rangle = \left\langle x\,,\,\frac{y}{2-x}\right\rangle \tag{3}$$

The dihomotopy equivalence between $\psi \circ \varphi$ and the identity on X is given by

$$h\,t\ \langle x, y\rangle \overset{\text{def}}{=} \left\langle x\,,\,\left(1 + t\cdot\frac{x-1}{2-x}\right)\cdot y\right\rangle$$

which is continuous and monotone, and reduces to Id_X and to $\psi \circ \varphi$ for $t = 0$ and $t = 1$, respectively. Since the other composition, $\varphi \circ \psi$, is expressed by the same formula (3), a similar dihomotopy will do.

But $\Omega_1 X$ is one-point, while $\Omega_1 Y$ has a continuum of incomparable points corresponding to the maximal (deadlock) points on its slanted upper edge.

The above example is another evidence that the morphisms need to have more properties than just continuity and monotonicity.

I believe the issue of functoriality will have to be given more thought in future.

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