Exam CTRLVERIFMPRI 2026

3rd March 2026

1 Backwards reachability

In the course, we saw how to pave inner and outer-approximations of the backwards reachable set of a neural network. Here we want to find a more direct method using polyhedra instead of boxes.

We recall the definition of a convex polyhedron in $\mathbb{R}^n$:

Definition 1. *A convex polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ is defined* externally *by a set of ($m \geq 0$) affine inequalities:*

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

where A is a $n \times m$-matrix and b is a vector of dimension m.

We recall that a convex polyhedron, when bounded, can also be represented *internally* as the convex combination of k points $p_1, \ldots, p_k$, i.e can be described by convex combinations:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{k} \lambda_i p_i,\ \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

We consider a ReLU feedforward neural network NN given by weights W^j and biaises b^j, $j = 1, \ldots, l$, for each of the l layers it is composed of. We suppose it has the same activation function σ in all layers, so that given inputs $y_1, \ldots, y_n$, the values of neurons at layer $k = 0, \ldots, l$, layer 0 corresponding to the input neurons, is given inductively by:

$$\begin{array}{rcl} y^0 & = & (y_1, \ldots, y_n) \\ y^{k+1} & = & \sigma(W^k y^k + b^k) \end{array}$$

During the course, we have seen how to do a "forward analysis" of such neural networks, using polyhedra, in particular. This means that, given some polyhedric representation of the values of the input neurons $\mathcal{P}^0 = (A^0, c^0)$, we infer a polyhedric outer approximation of the set of values that neurons at each layer k can take, given by a polyhedron $\mathcal{P}^k = (A^k, c^k)$.

In this exercise, we are concerned with a so-called "backward" analysis. Given $\mathcal{P}^{l+1} = (A^{l+1}, c^{l+1})$ a polyhedron giving the values of the neurons of the last layer, we want to infer inductively, from $k = l$ down to 0, a set of values that neurons at layer k should take so that neurons at layer l have the prescribed values. Formally, we want to compute the polyhedra $\mathcal{P}^k = (A^k, c^k)$ from $k = l$ to 0 such that:

$$\left\{\sigma(W^k y^k + b^k) \ \mid\ y^k \in \mathcal{P}^k\right\} \subseteq \mathcal{P}^{k+1} \qquad (1)$$

In this exercise, the only dependency is question 5 (depending on 4).

Question 1.
Consider $\mathcal{P}^0$ as defined inductively in Equation (1). Is it an inner or outer approximation of the set of initial inputs to the neural network NN that are (exactly) mapped to $\mathcal{P}^{l+1}$?

Solution 1.
Inner-approximation, by induction on k and Equation (1).

Question 2.
Consider any layer of a neural network, given by weights and biaises (W, b). Suppose we are given an hypercube $\mathcal{Q}$ and want to find an hypercube $\mathcal{P}$ such that:

$$\{Wy + b \mid y \in \mathcal{P}\} \subseteq \mathcal{Q} \tag{2}$$

Formulate the solution $\mathcal{P}$ of Equation (2) as a quantified first-order logical formula. Can you use direcly the algorithm of course 4 (quantifier elimination) to compute $\mathcal{P}$?

Solution 2.
Such sets $\mathcal{P}$ are defined as:

$$\mathcal{P} = \{y \in \mathbb{R}^n \mid \exists z \in \mathcal{Q},\ z = Wy + b\}$$

We cannot use directly the algorithm we have seen in the quantifier elimination course (part 4), since the matrix involved is not of the form $y = f(x)$ but rather $x = f(y)$.

Question 3. 1 P.
We consider the simple ReLU neural network NN with a single layer, with two input neurons $(x_1, x_2) \in [-1, 1]^2$ and two output neurons (y_1, y_2), defined as follows:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} ReLU(2(x_1 - x_2)) \\ ReLU(2(x_1 + x_2) + 1) \end{pmatrix}$$

we write $y(x) = (y_1(x), y_2(x))$ for $y = (y_1, y_2)$ computed from $x = (x_1, x_2)$ as above.

Suppose $\mathcal{Q}$ is the hyperrectangle $[0, 1] \times [0, 1]$. Observing that the linear layer is invertible, compute the corresponding hyperrectangle $\mathcal{P}$ defined by Equation (2) for the linear layer (not considering the ReLU computation).

Solution 3.
We write the quantified formulas describing $\mathcal{P}$ as:

$$\mathcal{P} = \{(x_1, x_2) \mid \exists y_1 \in [0, 1],\ \exists y_2 \in [0, 1],\ y_1 = 2(x_1 - x_2) \wedge y_2 = 2(x_1 + x_2) + 1\}$$

We note that the linear layer is invertible, and we have:

$$x_1 = 1/4(y_1 + y_2) - 1/4 \tag{3}$$
$$x_2 = 1/4(y_2 - y_1) - 1/4 \tag{4}$$

Now we are using plain interval arithmetic and find $x_1 \in [-1/4, 1/4]$ and $x_2 \in [-1/2, 0]$.

Question 4.
Consider any layer of a neural network, given by weights and biaises (W, b). Determine a polyhedron $\mathcal{P} = (A, c)$ such that, for any polyhedron $\mathcal{Q} = (C, d)$:

$$\{Wy + b \mid y \text{ s.t. } Ay + c \leq 0\} \subseteq (C, d)$$

Solution 4.
Take $A = CW$ and $c = Cb + d$: let y be such that $Ay + c = CWy + Cb + d \leq 0$. Then, $C(Wy + b) + d \leq 0$ and $Wy + b$ is in the polyhedron defined by (C, d).

Question 5.
We are in the same situation as in Question 4, but we suppose here that W is invertible, with inverse W^{-1}. Show that the polyhedron $\mathcal{P} = (A, c)$ determined in Question 4 is such that, for any polyhedron $\mathcal{Q} = (C, d)$:

$$\{Wy + b \mid y \text{ s.t. } Ay + c \leq 0\} = (C, d)$$

Solution 5.
Let x such that $Cx + d \leq 0$ and consider $y = W^{-1}(x - b)$, $A = CW$, $c = Cb + d$. We have:

$$\begin{aligned} Ay + c &= CWW^{-1}(x-b) + Cb + d & (5) \\ &= C(x-b) + Cb + d & (6) \\ &= Cx + d \leq 0 & (7) \end{aligned}$$

Therefore, $(C, d) \subseteq \{Wy + b \mid y \text{ s.t. } Ay + c \leq 0\}$. Combined with Question 4, we get the equality.

Question 6.
In the situation of Question 5, when W is invertible, do you think we can we find polyhedron $\mathcal{P}$ with an internal representation when $\mathcal{Q}$ is given also with an internal representation? (Give a short explanation, no need to prove anything there)

Solution 6.
If $\mathcal{Q}$ is the convex enveloppe of points $q_1, \ldots, q_m$, consider points $p_j = W^{-1}(q_j - b)$, $j = 1, \ldots, m$. They are such that $Wp_j + b = q_j$. Polyhedron $\mathcal{P}$ is necessarily the convex hull of point $p_1, \ldots, p_m$.

A priori, without resorting to computing the external representation of $\mathcal{Q}$ from its internal representation, it is not feasible to get directly the internal representation of $\mathcal{P}$ from the one of $\mathcal{Q}$, hence the natural use of the external representation in Question 3.

Question 7.
Consider the particular neural network defined in Question 3, and the polyhedron $\mathcal{P}^1$ defined by $y_1 \geq y_2$. Determine a polyhedron $\mathcal{P}^0$ such that Equation (1) is satistied, for inputs x_1, x_2 between -1 and 1. Is that the best one possible? What is missing is of course the backwards interpretation of the ReLU function, that we do not ask to treat in full generality.

Solution 7.
By case analysis on the ReLU function, the condition $y_1(x) \geq y_2(x)$ is satisfied, for $x_1, x_2 \in [-1, 1]$ iff:

$$\begin{cases} x_2 \leq -1/4 \\ x_2 \leq x_1 \end{cases}$$

These inequalities, together with $x_1 \in [-1, 1]$ and $x_2 \geq -1$ define the best polyhedron possible.

2 Conley index

Consider the following dynamical system in $\mathbb{R}^2$, given in polar coordinates:

$$\dot{r} = r(1 - r), \qquad \dot{\theta} = 1. \tag{8}$$

where r is the radius from point $(0, 0)$ and θ is the angle with respect to the abscissae. We call φ the flow of this ODE: $\varphi(x, t)$ is the value at time t of the solution of Equation (8) starting at $t = 0$ at x.

Consider the grid defined in polar coordinates, made up of five boxes in the coordinates (r, θ): $A_0 = [0, 1/2] \times \mathbb{R}$, $B = [1/2, 3/4] \times \mathbb{R}$, $C = [3/4, 1] \times \mathbb{R}$, $D = [1, 5/4] \times \mathbb{R}$ and $E = [5/4, 3/2] \times \mathbb{R}$ and pictured in Figure 3 on the plane.

We consider the discretized dynamical system $\varphi_\tau(x, k) = \varphi(x, k\tau)$, $k \in \mathbb{Z}$, with $\tau = 1/8$. The objective of this exercise is to study the invariant sets of the dynamical system given by Equation (8), by finding index pairs and computing Conley indexes of the discretized system, further discretized in space on the grid (A, B, C, D, E).

The second component θ of Equation (8) has a trivial behavior, we concentrate on the first component r of the ODE and recall that a guaranteed outer-approximation of the first component of the image of $R_0 \times \Theta_0$ by $\varphi_\tau(., 1)$ is R such that:

$$R_0 + \tau R(1 - R) \subseteq R \tag{9}$$

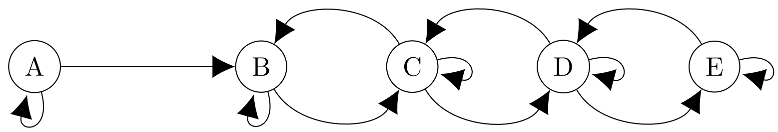

Figure 1: The graph of the multivalued map corresponding to φ_τ.

(everything being computed using interval arithmetics).

Question 1.

Briefly justify Equation (9).

Hint: you can rewrite Equation (8) in integral form: $\varphi(x,t) = \varphi(x,0) + \int_0^t \frac{d\varphi}{dt}(x,s)ds$ for all $t \in [0,\tau]$.

Solution 1.

Concentrating on the first component of the ODE, we write, for all $x \in R$, and for the solution r of Equation (8):

$$\begin{aligned} r(t) &= r(0) + \int_0^t \dot{r}(s)ds \\ &= r(0) + \int_0^t r(s)(1-r(s))ds \end{aligned}$$

Thus r is the least fixed point (by uniqueness of the solution of Equation (8)) of the functional $r \rightarrow r(0) + \int_0^t r(s)(1-r(s))ds$ in the space of continuously differentiable functions from $[0,1]$ to $\mathbb{R}$. Using the Galois connection that abstracts the set of subsets of such functions to intervals of $\mathbb{R} \cup \{-\infty, \infty\}$ (by taking the union of all ranges in that subset), we get that the abstraction of the least fixed point is included in any postfixed point R of functional $R \rightarrow r(0) + tR(1-R)$. This is true for all $r(0) \in R_0$ hence $R_0 + \tau R(1-R) \subseteq R$ means we get an outer-approximation of the range of solutions to Equation (8) starting at any point in R_0.

Question 2.

Compute the images of each element of the grid, in terms of the elements of the grid itself.

Indications: This means you will need to find union of grid elements among A, B, C, D, and E, A (resp. B, C, D and E) is mapped to. You can either do the first steps of a Kleene iteration on the interval map $F_{R_0}: R \rightarrow R_0 + \tau R(1-R)$ to infer a postfixed point and check that some unions of elements of the grid are postfixed-point of F_{R_0}, or check directly for a postfixed point if you have a good intuition of what it should be.

Solution 2.

We check:

- $\varphi_\tau(A) \subseteq A \cup B$ since $F_A(A \cup B) = [0,1/2] + 1/8[0,3/4][1/4,1] \subseteq A \cup B$,
- $\varphi_\tau(B) \subseteq B \cup C$ since $F_B(B \cup C) = [1/2,3/4] + 1/8[1/2,1][0,1/2] \subseteq B \cup C$,
- $\varphi_\tau(C) \subseteq B \cup C \cup D$ since $F_C(B \cup C \cup D) = [3/4,1] + 1/8[1/2,5/4][-1/4,1/2] \subseteq B \cup C \cup D$,
- Symmetrically, $\varphi_\tau(D) \subseteq C \cup D \cup E$, $\varphi_\tau(E) \subseteq D \cup E$.

From Question 2, we get the graph pictured in Figure 1 of the multivalued map which is the representation of the discretization in space, on the grid, of the discrete dynamical system φ_τ.

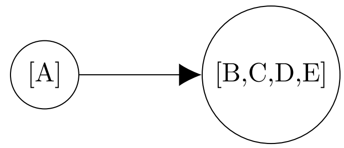

Figure 2:

Question 3.
What is the condensation of the graph (the graph of strongly connected components) pictured in Figure 1?

Solution 3.
The condensation of the graph is pictured in Figure 2.

Question 4.
Deduce the index pairs for the continuous flow of Equation (8).

Solution 4.
There are two strongly connected components, one is A and gives the index pair $(N_1, L_1) = (|A| \cup |B| \cup |C| \cup |D| \cup |E|, |B| \cup |C| \cup |D| \cup |E|)$. The other is $(N_2, L_2) = (|B| \cup |C| \cup |D| \cup |E|, \emptyset$.

Question 5.
What is the homological Conley index of the index pairs you found in last question? These will be computed for the geometric realization of these index pairs in the plane deduced from the respective polar coordinates.

Solution 5.
For the first index pair (N_1, L_1), the geometric realization in the plane of $|A| \cup |B| \cup |C| \cup |D| \cup |E|/|B| \cup |C| \cup |D| \cup |E|$ is homeomorphic to S^2, hence

$$CH_i(N_1, L_1) = \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{array} \right.$$

For the second index pair (N_2, L_2), the geometric realization in the plane of $|B| \cup |C| \cup |D| \cup |E|$ is homotopy equivalent to S^1, hence

$$CH_i(N_2, L_2) = \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } i = 0, 1 \\ 0 & \text{otherwise} \end{array} \right.$$

Question 6.
What can you deduce about the existence of invariant sets within these index pairs, and their shape?

Solution 6.
In both cases, by Wazewki's property, we deduce that there is a non empty invariant within, respectively, $|A|$, and $|B| \cup |C| \cup |D| \cup |E|$.

For (N_1, L_1) we recognize the typical Conley index for a fixpoint with an unstable manifold of dimension 2. It could be something else, but we easily check that this is $(0, 0)$, which is repulsive.

For (N_2, L_2), the Conley index is that of a circle, and we can prove the invariant is a attractive limit cycle by exhibiting a Poincaré section. the abscissae axis is obviously one such, and we can check easily on the differential equation that the unit circle is an attracting periodic orbit, as shown in Figure 3.

3 Euler-Poincaré characteristic and sensor networks

In the course we showed that reachable sets could be characterized geometrically, e.g. using the Euler-Poincaré characteristic, for some applications. Here we are going to show that this characteristic gives

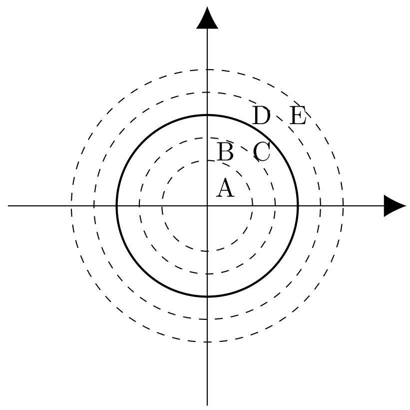

Figure 3: The grid on the phase space of the dynamical system of Equation 8, in the plane (after transformation from polar coordinates).

enough geometric information for being use for localizing and counting a number of targets that a network of sensors with limited visibility can spot.

For this, we start by showing that the Euler-Poincaré characteristic is some kind of measure, and are going to study an integration operator against it.

Question 1.
Prove $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ for finite simplicial complexes A and B.

Solution 1.
Suppose $A \cap B$ (respectively A, B) has $N^i_{A\cap B}$ simplexes of dimension i (respectively N^i_A, N^i_B). Then for all i, $N^i_{A\cup B} = N^i_A + N^i_B - N^i_{A\cap B}$ since i-simplexes that are in $A \cap B$ are counted twice in A and in B. Therefore, using the alternating sum definition of the Euler-Poincaré characteristic, we get the equality that we need.

We admit now that this is true also for any topological space, geometric realization of finite simplicial complexes and recall that the Euler-Poincaré characteristic of a topological space X can be defined as

$$\chi(X) = \sum_{i\in\mathbb{N}} (-1)^i rk(H_i(X)) \tag{10}$$

where $rk(M)$ is the rank of the $\mathbb{Z}$-module M, which is $m - r$ in the canonical decomposition of M as $\mathbb{Z}^{m-r} \oplus \mathbb{Z}/d_1 \oplus \ldots \mathbb{Z}/d_r$ (with $d_j > 0$ and $d_j | d_{j+1}$).

Question 1 implies that χ is finitely additive, making it similar to a measure, at least in a finite context, and we wish to define an integral with respect to the *geometric measure* χ of a class of function $h : X \to \mathbb{Z}$ as:

$$\int_X h d\chi = \sum_{s=-\infty}^{\infty} s\chi(h^{-1}(s))$$

when this is well-defined (in particular all sums considered here and in the sequel are with finitely non-zero or non-empty summands). We will restrict to the class of χ-integrable functions that are *simple functions*, written in *canonical form* as follows:

$$h = \sum_{s=-\infty}^{\infty} s\mathbb{1}_{X_s}$$

where $\mathbb{1}_{X_s}$ denotes the characteristic map of X_s and where $X_s = h^{-1}(s)$ form a finite collection of disjoint (non-empty) sets with $X = \underset{s\in\mathbb{Z}}{\cup} X_s$. In that case, the integral of h becomes trivially:

$$\int_X h d\chi = \sum_{s=-\infty}^{\infty} s\chi(X_s) \tag{11}$$

Question 2.
What is $\int_X \mathbb{1}_U$ where U is any closed subset of X? Briefly justify Equation (11).

Solution 2.
$\int_X \mathbb{1}_U = \chi(U)$. As all X_s are disjoint, using the additivity seen in Question 1, we get, as $h^{-1}(s) = \mathbb{1}_{X_s}$, $\int_X h d\chi = \sum\limits_{s=-\infty}^{\infty} s \int_X \mathbb{1}_{X_s} d\chi = \sum\limits_{s=-\infty}^{\infty} s\chi(X_s)$.

Question 3.
Using additivity, proven in Question 1, show that, for a simple function not written in canonical form:

$$h = \sum_{s,t=-\infty}^{\infty} s\mathbb{1}_{X_{s,t}}$$

we still have a similar formula:

$$\int_X h d\chi = \sum_{s,t=-\infty}^{\infty} s\chi(X_{s,t})$$

with finitely many non-empty disjoint sets $X_{s,t}$ with $\bigcup\limits_{s,t=-\infty}^{\infty} X_{s,t} = X$.

Solution 3.
We write $X_s = \bigcup\limits_{t=-\infty}^{\infty} X_{s,t}$ which forms a disjoint union and we see that:

$$h = \sum_{s=-\infty}^{\infty} s\mathbb{1}_{X_s}$$

so:

$$\int_X h d\chi = \sum_{s=-\infty}^{\infty} s\chi(X_s)$$

and by additivity:

$$\chi(X_s) = \sum_{t=-\infty}^{\infty} \chi(X_{s,t})$$

so:

$$\begin{aligned} \int_X h d\chi &= \sum_{s=-\infty}^{\infty} s\chi(X_s) \\ &= \sum_{s,t=-\infty}^{\infty} s\chi(X_{s,t}) \end{aligned}$$

Question 4.
Prove that $\int_X$ is a linear form on the space of simple functions, i.e. that $\int_X (\lambda h) d\chi = \lambda \int_X h d\chi$ for all $\lambda \in \mathbb{Z}$, and $\int_X (h+g) d\chi = \int_X h d\chi + \int_X g d\chi$.

(hint: note that writing $X_s = h^{-1}(s)$ and $Y_t = g^{-1}(t)$ for $s \in \mathbb{Z}$ and $t \in \mathbb{Z}$, $h + g = \sum\limits_{s=-\infty}^{\infty} \sum\limits_{t=-\infty}^{\infty} (s + t)\mathbb{1}_{X_s \cap Y_t}$)

Solution 4.
First, for any $\lambda \in \mathbb{Z}$:

$$\begin{aligned} \int_X (\lambda h) d\chi &= \sum_{s=-\infty}^{\infty} s\chi((\lambda h)^{-1}(s)) \\ &= \sum_{s'=\lambda s, s=-\infty}^{\infty} s'\chi(h^{-1}(s'/\lambda)) \\ &= \sum_{s=-\infty}^{\infty} \lambda s\chi(h^{-1}(s)) \\ &= \lambda \int_X h d\chi \end{aligned}$$

Now, for h and g two χ-integrable functions, let us write $X_s = h^{-1}(s)$ and $Y_t = g^{-1}(t)$ for $s \in \mathbb{Z}$ and $t \in \mathbb{Z}$. Then $X = \underset{s\in\mathbb{Z}}{\cup} X_s$, $X = \underset{t\in\mathbb{Z}}{\cup} Y_t$, $h = \sum_{s=-\infty}^{\infty} s\mathbb{1}_{X_s}$ and $g = \sum_{t=-\infty}^{\infty} t\mathbb{1}_{Y_t}$. Therefore:

$$\begin{aligned}
h+g &= \sum_{s=-\infty}^{\infty} s\mathbb{1}_{X_s} + g = \sum_{t=-\infty}^{\infty} t\mathbb{1}_{Y_t} \\
&= \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} s\mathbb{1}_{X_s\cap Y_t} + \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} t\mathbb{1}_{X_s\cap Y_t} \\
&= \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} (s+t)\mathbb{1}_{X_s\cap Y_t}
\end{aligned}$$

since $X = \underset{s\in\mathbb{Z}}{\cup} X_s$ as well as $X = \underset{t\in\mathbb{Z}}{\cup} Y_t$. We can apply the formula for the integral of Question 2:

$$\begin{aligned}
\int_X (h+g)d\chi &= \sum_{s,t=-\infty}^{\infty} (s+t)\chi(X_s\cap Y_t) \\
&= \sum_{s,t=-\infty}^{\infty} s\chi(X_s\cap Y_t) + \sum_{s,t=-\infty}^{\infty} t\chi(X_s\cap Y_t) \\
&= \sum_{s=-\infty}^{\infty} s\chi\left(\bigcup_{t=-\infty}^{\infty} X_s\cap Y_t\right) + \sum_{t=-\infty}^{\infty} t\chi\left(\bigcup_{s=-\infty}^{\infty} X_s\cap Y_t\right) \\
&= \sum_{s=-\infty}^{\infty} s\chi\left(X_s\right) + \sum_{t=-\infty}^{\infty} t\chi\left(Y_t\right) \\
&= \int_X hd\chi + \int_X gd\chi
\end{aligned}$$

by additivity of χ (Question 1) and the fact that $X = \underset{t\in\mathbb{Z}}{\cup} Y_t$ and $X = \underset{s\in\mathbb{Z}}{\cup} X_s$.

Because of linearity proven in last question and Question 1, the integral of functions of the form:

$$h = \sum_{\alpha\in A} c_\alpha \mathbb{1}_{U_\alpha}$$

where the sets U_α may not be disjoint, contrarily to the case of simple functions presented in a canonical manner, can still be written as:

$$\int_X hd\chi = \sum_{\alpha\in A} c_\alpha \chi(U_\alpha)$$

Suppose there is a certain (finite) number of *targets* in $\mathbb{R}^n$, that we want to count using *sensors*. To keep things simple, we suppose that we have as many sensors as we may need, and for this exam, this means we consider all points $x \in \mathbb{R}^n$ to be a sensor. Each target $\alpha \in A$ is going to be seen by sensor $x \in U_\alpha$ which is a closed set of $\mathbb{R}^n$.

In this situation, we consider now the counting function

$$h = \sum_{\alpha\in A} \mathbb{1}_{U_\alpha} \tag{12}$$

i.e. is such that $h(x)$ is the number of targets belonging to A that sensor x sees. An example is given with four targets at Figure 4.

We suppose that all U_α have the *same shape*, i.e. in what is of concerned to us here, that there exists $N \in \mathbb{Z}$ such that for all $\alpha \in A$, $\chi(U_\alpha) = N$.

Question 5.

Suppose $N \neq 0$. Prove that the total number of targets is:

$$\frac{1}{N}\int_X hd\chi$$

where h is the counting function of Equation (12).

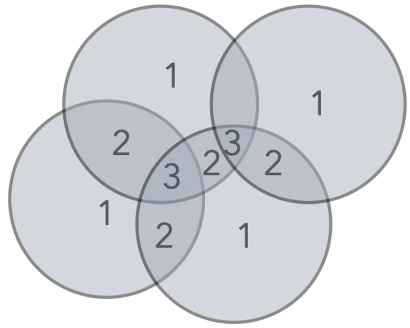

Figure 4: Regions U_α for each of 4 targets, and the global counting function.

Solution 5.
We have

$$\begin{aligned} \int_X h d\chi &= \int_X \sum_{\alpha \in A} \mathbb{1}_{U_\alpha} \\ &= \sum_{\alpha \in A} \chi(U_\alpha) \\ &= \alpha N \end{aligned}$$

by linearity (Question 4).

This provides us with a simple algorithm for a sensor network to determine the total number of targets: each sensor broadcasts its position and its number of targets seen, the rest is a simple algebraic calculation (generalizing the inclusion-exclusion principle).

Still, the level sets $h^{-1}(s)$, $s \in \mathbb{Z}$, are not closed nor open sets in general, making it a bit tricky to compute their Euler-Poincaré characteristic. Instead, we can prove, but do not ask you to prove, that we can compute the Euler integral of h as:

$$\int_X h d\chi = \sum_{s=0}^{\infty} \left(\chi(h^{-1}((s,\infty)) - \chi(h^{-1}((-\infty,-s))\right)$$

where (a, b) denote the open interval, not including a nor b.

Question 6.
Consider the 4 targets of Figure 4 together with their respective U_α, which are all closed discs in $\mathbb{R}^2$ as pictured. Compute the Euler integral $\int_X h d\chi$ for h the corresponding counting function $h = \sum_{\alpha \in A} \mathbb{1}_{U_\alpha}$. Is this consistent with the result of Question 5?

Solution 6.
We look at the only non-empty sublevel sets of function h:

- the set $h^{-1}((0,\infty))$ is contractible hence has as Euler characteristic 1,
- the set $h^{-1}((1,\infty))$ is contractible hence has as Euler characteristic 1,
- the set $h^{-1}((2,\infty))$ has two contractible components, hence has as Euler characteristic 2.

Moreover, $\chi(U_\alpha) = 1$ as they are closed discs. Therefore we do have $1/1 \int_X h d\chi = 4$ as expected.

Question 7.
Compute the Euler-Poincaré characteristic of the n-sphere $S^n \subseteq \mathbb{R}^{n+1}$.

Solution 7.
We have $H_i(S^n) = \mathbb{Z}$ iff $i = 0$ or $i = n$, otherwise is 0. Hence $\chi(S^n) = 1 + (-1)^n$.

Question 8.
Suppose each target α can be seen by sensors in U_α, homotopy equivalent to $S^n \subseteq \mathbb{R}^{n+1}$. Is this *target counting* algorithm, consequence of the formula obtained at Question 5, applicable in all $\mathbb{R}^{n+1}$, whatever $n \in \mathbb{N}$ is?

Solution 8.
The Euler characteristic is non zero only when n is even. So typically, this is applicable in $\mathbb{R}^3$ but not in $\mathbb{R}^2$.