Exercise sheet - courses 5-8 CTRLVERIF

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1 Euler characteristic and homology

During the course we have seen the Euler-Poincaré characteristic for surfaces. This can be defined more generally directly on finite simplicial sets. Given a simplicial set X of finite dimension n, we set:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i card(X_i)$$

where *card* denotes the cardinal of a set.

Question 1.
In which sense this definition generalizes the definition you saw in the course?

Solution 1.
We saw that, for surfaces presented by polyhedric cells, $\chi(X) = V - E + F$, with V the number of vertices, E the number of edges and F the number of faces. When the polyhedric cells are actually simplicial, making the surface into a simplicial set, we note that $V = card(X_0)$, $E = card(X_1)$, $F = card(X_2)$ and $card(X_i) = 0$ for all $i > 2$.

We recall that the homology groups of X are

$$H_k(X) = Ker\ \partial_{k-1}/Im\ \partial_k$$

where $\delta_k :\ C(X)_{k+1} \to C(X)_k$ is the boundary morphism from the free $\mathbb{Z}$-module generated by the $(k+1)$-simplexes X_{k+1} of X to the free $\mathbb{Z}$-module generated by the k-simplexes X_k of X.

As $\mathbb{Z}$ is a principal ideal domain, submodules (such as $Ker\ \partial$ and $Im\ \partial$) of free $\mathbb{Z}$-modules are free: we note that $C(X)_k = Ker\ \partial_{n-1} \oplus Im\ \partial_n$, where $\oplus$ denotes the direct sum of $\mathbb{Z}$-modules.

We note also $rk(M)$ ("rank of M") of a $\mathbb{Z}$-module M to be $m - r$, if $M = \mathbb{Z}^{m-r} \oplus \mathbb{Z}/d_1 \oplus \ldots \mathbb{Z}/d_u$ is the canonical form of M, for some $d_1, \ldots, d_u > 1$, i.e. the dimension of its free sub-$\mathbb{Z}$-module. As for vector spaces and the notion of dimension, $rk(M \oplus N) = rk(M) + rk(N)$ and $rk(M/N) = rk(M) - rk(N)$.

We are going to prove that χ is a homotopy invariant.

Question 2.
Prove that

$$rk(C(X)_i) = rk(Ker\ \partial_{i-1}) + rk(Im\ \partial_{i-1})$$

(hint: consider the Smith-Normal Form of the matrix representing the linear map ∂_{i-1})

Solution 2.
$C(X)_i$ and $C(X)_{i-1}$ are free $\mathbb{Z}$-modules generated by $x_1, \ldots, x_n \in X_i$ and $y_1, \ldots, y_m \in X_{i-1}$, respectively.

Now, because the SNF exists for linear maps between $\mathbb{Z}$-modules, there exist invertible integer matrices

$P \in GL_m(\mathbb{Z})$, $Q \in GL_n(\mathbb{Z})$ such that

$$P\partial Q = \begin{pmatrix} d_1 & 0 & \cdots & 0 & & 0 \\ 0 & d_2 & \cdots & 0 & & 0 \\ \vdots & & \ddots & \vdots & & \\ 0 & 0 & \cdots & d_r & & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & & \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

with $d_i > 0$ and $d_i \mid d_{i+1}$. This means that the image of δ can be written on a base with r elements, and that the kernel can be written on a base of $m - r$ elements, hence the result.

Another way of proving this is to use the first isomorphism theorem of abelian groups (and $\mathbb{Z}$-modules are exactly the same as abelian groups): $Im\ \partial_{i-1} \simeq C_{i-1}/Ker\ \partial_{i-1}$, and applying the identity on ranks that we gave.

Question 3.
Prove that

$$card(X_i) = rk(H_i(X)) + rk(Im\ \partial_i) + rk(Im\ \partial_{i-1})$$

Solution 3.
We note that:

$$\begin{aligned} card(X_i) &= rk(C(X)_i) \\ &= rk(Ker\ \partial_{i-1}) + rk(Im\ \partial_{i-1}) \end{aligned}$$

by Question (2).

Also, we know that $rk(H_i(X)) = rk(Ker\ \partial_{i-1}) - rk(Im\ \partial_i)$, because $H_i(X) = Ker\ \partial_{i-1}/Im\ \partial_i$ hence:

$$card(X_i) = rk(H_i(X)) + rk(Im\ \partial_i) + rk(Im\ \partial_{i-1})$$

Question 4.
Prove that:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i rk(H_i(X)) \tag{1}$$

Solution 4.
We use Question (2) and write:

$$\begin{aligned} \chi(X) &= \sum_{i=0}^{n} (-1)^i card(X_i) \\ &= \sum_{i=0}^{n} (-1)^i \left(rk(H_i(X)) + rk(Im\ \partial_i) + rk(Im\ \partial_{i-1}) \right) \\ &= \sum_{i=0}^{n} (-1)^i rk(H_i(X)) + \sum_{i=0}^{n} (-1)^i \left(rk(Im\ \partial_i) + rk(Im\ \partial_{i-1}) \right) \\ &= \sum_{i=0}^{n} (-1)^i rk(H_i(X)) + \sum_{i=0}^{n} (-1)^i rk(Im\ \partial_i) - \sum_{i=0}^{n} (-1)^i rk(Im\ \partial_i) \\ &= \sum_{i=0}^{n} (-1)^i rk(H_i(X)) \end{aligned}$$

Question 5.
Deduce from last question that the Euler-Poincaré characteristic of a finite simplicial set is a homotopy invariant.

Solution 5.
For finite simplicial sets X, as, from last question, the Euler-Poincaré characteristic depends only on the homology groups of X which are homotopy invariants. Hence χ is an homotopy invariant.

Note: finiteness/compactness is required. Think of $(0,1)$ and $[0,1]$: $\chi((0,1)) = -1$ and $\chi([0,1]) = 1$ whereas $(0,1)$ and $[0,1]$ are homotopy equivalent.

Equation (1) allows for defining the Euler-Poincaré characteristic for all topological spaces X with finite homology.

We recall that a convex subset of $\mathbb{R}^n$ is a set $X \subseteq \mathbb{R}^n$ such that for all x, y in X, the line $\lambda \in [0,1] \to \lambda x + (1-\lambda)y$ is entirely included in X.

Question 6.
Prove that a convex set if contractible, i.e. that it is homotopy equivalent to a point.

Solution 6.
First, every pair of points is obviously connected by an arc (a line in fact). So X is connected.

Then, take any $x_0 \in X$ and define $H : X \times [0,1] \to X$ by $H(x,t) = tx + (1-t)x_0$. H is a homotopy between the constant map with value x_0 and the identity on X, and makes x_0 into a strong deformation retract of X.

Question 7.
Deduce that the Euler-Poincaré characteristic of any compact convex set is 1.

Solution 7.
By the previous question, all homotopy groups are 0 except for $H_0 = \mathbb{Z}$ (it is connected). By the formula giving the Euler-Poincaré characteristic in terms of homology, we get $\chi = 1$.

Question 8.
What is the Euler-Poincaré characteristic of a disc minus k open discs?

Solution 8.
Digging k discs in a disc makes π_1 into the free group generated by k generators, or, homologically, make $H_1 = \mathbb{Z}^k$. We have $H_0 = \mathbb{Z}$ and all higher groups being zero, so, using the formula that gives the Euler-Poincaré characteristic in terms of homology, we get $\chi = 1 - k$.

2 Euler-Poincaré characteristics of graphs

We consider here graphs as one-dimensional simplicial sets. Therefore $G = (V, E, d_0, d_1)$ is a graph when V is a set of vertices, E is a set of edges E, and d_0 and d_1 are boundary maps.

A bouquet of circles is a graph with one vertex.

Question 1.
What is the set of connected components $\pi_0(G)$ when G is a bouquet of circles? What is the fundamental group of G when G is a bouquet of circles?

Solution 1.
G being a bouquet of circles means that all edges define one circle, all linked by the only vertex.

$\pi_0(G)$ is the singleton set (one connected component).

$\pi_1(G)$ is the free group generated by $|E|$ generators.

Question 2.
What about the homology groups in any dimension, of a bouquet of circles?

Solution 2.
By Hurewitz theorem, and the previous question, $H_0(G) = \mathbb{Z}$ and $H_1(G)$ is the abelianization of the free group on $|E|$ generators, i.e. is $\mathbb{Z}^{|E|}$. Higher homology groups are trivially 0 (there is no cell of dimension higher or equal than 2).

Question 3.
Using the result of Exercise 1, question 4 (the Euler-Poincaré characteristic in terms of homology groups) and what you found in last question, what is the Euler-Poincaré characteristic of a bouquet of circles?

Solution 3.
We have, for a bouquet of circles:

$$\begin{array}{rcl} \chi(G) & = & \sum_{i=0}^{n}(-1)^i rk(H_i(X)) \\ & = & 1-|E| \end{array}$$

Question 4.
Prove that two connected finite graphs G and H have isomorphic fundamental groups if and only if $\chi(G) = \chi(H)$.

You can take for granted the following property: a connected graph is homotopy equivalent to a bouquet of circles. Remember that, from exercise 1, χ is a homotopy invariant (for finite simplicial sets).

Solution 4.
Let G' and H' be the two bouquet of circles, homotopy equivalent to, respectively, G and H. We know that $\chi(G') = \chi(G)$ and $\chi(H') = \chi(H)$. Now, from previous questions, we know that $\chi(G') = \chi(G) = 1 - |E_{G'}|$, $\chi(H') = \chi(H) = 1 - |E_{H'}|$ and $\pi_1(G')$ (respectively $\pi_1(H')$) is the free group on $|G'|$ (respectively on $|H'|$) generators. Free groups are isomorphic iff they have the same number of generators, QED.

In fact we could prove that G and H are homotopy equivalent if and only if their Euler-Poincaré characteristics are equal.

3 Homological calculations and Smith normal forms

3.1 Reminder: SNF algorithm

We recall below the Smith normal form algorithm:

Let $A \in M_{m\times n}(\mathbb{Z})$ be an integer matrix. The *Smith normal form* (SNF) of A is a diagonal matrix

$$D = \operatorname{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$$

such that

$$UAV = D$$

for some unimodular matrices

$$U \in GL_m(\mathbb{Z}), \qquad V \in GL_n(\mathbb{Z}),$$

with

$$d_i > 0, \qquad d_i \mid d_{i+1}.$$

The SNF is unique and determines the structure of finitely generated abelian groups arising as kernels and cokernels of A.

The algorithm uses only operations corresponding to multiplication by unimodular matrices:

- Row operations

1. Swap two rows
2. Multiply a row by -1
3. Replace a row by itself plus an integer multiple of another row

- Column operations
  1. Swap two columns
  2. Multiply a column by -1
  3. Replace a column by itself plus an integer multiple of another column

Note that division is allowed only when exact, we use everywhere the Euclidean division to determine the factors that we have to use to eliminate entries with row and column operations.

We now describe the algorithm inductively.

- Step 0: Initialization: Set $k = 1$. Restrict attention to the submatrix consisting of rows $k, \ldots, m$ and columns $k, \ldots, n$.
- Step 1: Choose a pivot If the submatrix is zero, stop.

  Otherwise, choose a nonzero entry of *minimal absolute value* and move it to position (k, k) using row and column swaps.
- Step 2: Clear the pivot column: For each row $i > k$, use the Euclidean algorithm:
$$R_i \leftarrow R_i - qR_k$$
  to reduce the entry a_{ik}. Repeat until all entries below the pivot are divisible by a_{kk}, then eliminate them.
- Step 3: Clear the pivot row: Apply the same procedure to columns $j > k$, using column operations.
- Step 4: Enforce divisibility: If any remaining entry is not divisible by a_{kk}, use the Euclidean algorithm to replace the pivot by a smaller one and return to Step 2.
- Step 5: Lock the pivot: Once all other entries in row k and column k are zero and divisibility holds, increment $k \leftarrow k + 1$ and repeat.
- Step 6: Normalize: Ensure all diagonal entries are positive.

3.2 Example

Consider the matrix
$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 6 & 8 \end{pmatrix}.$$

- Step 1: Choose pivot: The smallest nonzero entry is 2, already in position $(1, 1)$.
- Step 2: Clear below the pivot:
$$R_2 \leftarrow R_2 - 2R_1$$
$$\begin{pmatrix} 2 & 4 & 6 \\ 4 & 6 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 4 & 6 \\ 0 & -2 & -4 \end{pmatrix}$$
- Step 3: Clear to the right of the pivot:
$$C_2 \leftarrow C_2 - 2C_1, \qquad C_3 \leftarrow C_3 - 3C_1$$
$$\begin{pmatrix} 2 & 4 & 6 \\ 0 & -2 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & -4 \end{pmatrix}$$

- Step 4: Reduce submatrix: Multiply the second row by -1:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \end{pmatrix}$$

- Step 5: Clear right of second pivot:

$$C_3 \leftarrow C_3 - 2C_2$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

- Step 6: Verify divisibility:

$$2 \mid 2 \quad \text{(satisfied)}$$

- Final Smith Normal Form:

$$\mathrm{SNF}(A) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

In this course, we use SNF on linear maps which are boundary operators, so that we can compute the corresponding homology groups. If A represents a boundary map

$$\partial : \mathbb{Z}^3 \to \mathbb{Z}^2,$$

then

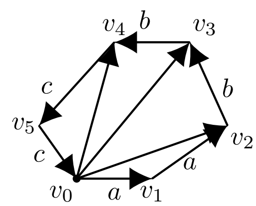
$$\ker \partial \cong \mathbb{Z}, \qquad \operatorname{coker} \partial \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

3.3 A simple polygon model

We consider now the following polygon model:

$$a\,a\,b\,b\,c\,c$$

triangulated from a single vertex, and its triangularization $X = (X_0, X_1, X_2)$:



The triangles are:

$$T_1 = [v_0, v_1, v_2], \quad T_2 = [v_0, v_2, v_3], \quad T_3 = [v_0, v_3, v_4], \quad T_4 = [v_0, v_4, v_5]$$

and the edges are:

$$a, b, c, d_1, d_2, d_3$$

We remind you that in this depiction, all edges with the same label are identified, and all vertices are identified together, hence the chain groups are given by

$$C_2 = \mathbb{Z}^4, \quad C_1 = \mathbb{Z}^6, \quad C_0 = \mathbb{Z}$$

and the boundaries are given by, using

$$\partial[v_i, v_j, v_k] = (v_j v_k) - (v_i v_k) + (v_i v_j) :$$

with $(v_0 v_1) = a$, $(v_1 v_2) = a$, $(v_2, v_3) = b$ etc and $(v_i v_j) = -(v_j v_i)$.

Question 1.
Compute the boundary matrix of $\partial_2 : \ C_2 \to C_1$.

Solution 1.
We compute:

$$\begin{aligned}
\partial T_1 &= a - d_1 + a = 2a - d_1 \\
\partial T_2 &= b - d_2 + d_1 \\
\partial T_3 &= b - d_3 + d_2 \\
\partial T_4 &= c + c + d_3 = 2c + d_3
\end{aligned}$$

So with basis (a, b, c, d_1, d_2, d_3), we have:

$$\partial_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Question 2.
Compute the Smith Normal Form of the matrix of Question 6.

Solution 2.
We obtain successively:

By swapping columns and rows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Now we do: $R_4 \leftarrow R_4 - R_1$, $R_5 \leftarrow R5 + R_1$:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

The next pivot comes from row 4, that we multiply by -1 and swap with R_2:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Now we do $R_4 \leftarrow R_4 - 2R_2$:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

We swap minus row 6 with row 3:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

We do $R_4 \leftarrow R_4 + 2R_3$ and $R_5 \leftarrow R_5 - 2R_3$:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

We do $R_5 \leftarrow R_5 + R_4$, $R_6 \leftarrow R_6 + R_4$ and then multiply R_4 by -1:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We finish by the obvious column operations to get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus

$$\Rightarrow \text{SNF}(\partial_2) = \text{diag}(1, 1, 1, 2, 0, 0)$$

Question 3.
Deduce from last question the first homology of the polygon model X.

Solution 3.

$$H_1 = \mathbb{Z}^{6-4} \oplus \mathbb{Z}/2 = \mathbb{Z}^2 \oplus \mathbb{Z}/2$$

Question 4.
What are $H_0(X)$ and $H_2(X)$? Give just a brief explanation (no need for lengthy calculations)

Solution 4.

$$H_2(X) = 0, \quad H_0(X) = \mathbb{Z}$$

Question 5.
Find the fundamental group of the polygon model X from the boundary of the given triangularization. Is that compatible also with the homological calculation you made?

Solution 5.
Look at the boundary word, going counter-clockwise, we get that $\pi_1(X)$ is generated by 3 generators a, b and c together with the relation $a^2b^2c^2 = 1$. Abelianizing this we get $\mathbb{Z}^3$ modulo the relation $2a+2b+2c = 0$, hence this is $\mathbb{Z}^2 \oplus \mathbb{Z}/2$, which is compatible with what we computed.

4 Conley Index and Equilibria

Consider the system

$$\dot{x} = 1 - x^2, \qquad \dot{y} = -y \tag{2}$$

on $R = [0, 2] \times [-2, 2]$.

Equilibrium states are the ones which are fixed points of the flow of an ODE.

Question 1.

Given an ODE $\dot{x} = f(x)$ in $\mathbb{R}^n$ $(n \geq 1)$, what should be the value of the vector field f at the equilibrium states of this ODE?

Solution 1.

We should have $f(x) = 0$ on these points.

An equilibrium point x of an ODE $\dot{x} = f(x)$ is:

- a saddle point if the Jacobian $J(x) = \left(\frac{\partial f_{x_i}}{\partial x_j}\right)_{i,j}$ of the vector field f is indefinite, i.e. it has both negative and positive eigenvalues,
- a sink if the Jacobian has only negative real parts (one of which at least having a strictly negative real part).

Question 2.

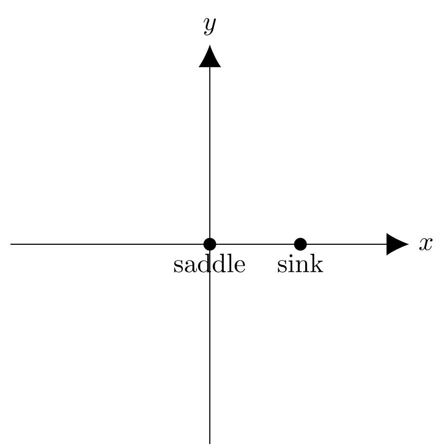
Find and classify equilibria for the system defined by Equation 2.

Solution 2.

Equilibria satisfy $y = 0$ and $1 - x^2 = 0$, giving $x = \pm 1$. The Jacobian is

$$J(x, y) = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}.$$

At $(-1, 0)$ eigenvalues have opposite signs 2 and -1, hence is a saddle. At $(1, 0)$ eigenvalues have negative real part: -2 and -1 hence is a sink.



We partition R into 16 unit squares $A = [0,1] \times [-2,-1]$, $B = [1,2] \times [-2,-1]$, $C = [0,1] \times [-1,0]$, $D = [1,2] \times [-1,0]$, $E = [0,1] \times [0,1]$, $F = [1,2] \times [0,1]$, $G = [0,1] \times [1,2]$, $H = [1,2] \times [1,2]$, $A' = [2,1] \times [-2,-1]$, $B' = [3,2] \times [-2,-1]$, $C' = [2,1] \times [-1,0]$, $D' = [3,2] \times [-1,0]$, $E' = [2,1] \times [0,1]$, $F' = [3,2] \times [0,1]$, $G' = [2,1] \times [1,2]$, $H' = [3,2] \times [1,2]$ as shown in Figure 1, and consider the discretized flow φ_τ of the Equation (2) at times $k\tau$ with $\tau = 1/8$.

The objective of this exercise is to study the invariant sets of the dynamical system given by Equation (2), by finding index pairs and computing Conley indexes of the discretized system, further discretized in space on the grid (A, B, C, D, E, F, G, H).

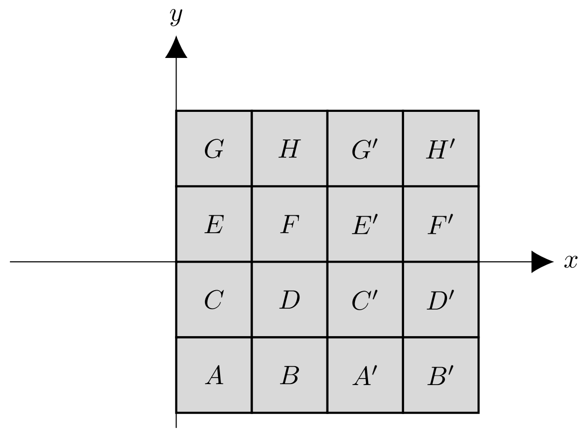


Figure 1: The grid on R.

We recall that, using the mean-value theorem, a guaranteed outer-approximation of the image of S_0 of φ_τ is $S = S_x \times S_y$ such that:

$$S_0 + \tau(1 - S_x^2, -S_y) \subseteq S \tag{3}$$

(everything being computed using interval arithmetics).

Question 3.

Compute the images of elements A and D of the grid, in terms of the elements of the grid itself.

Indications: This means you will need to find union of grid elements among A, B, C, D, E, F, G and H, A (resp. D) is mapped to. You can either do the first steps of a Kleene iteration on the interval map F_{S_0} : $R \to S_0 + \tau(1 - S_x^2, -S_y)$ to infer a postfixed point and check that some unions of elements of the grid are postfixed-point of F_{S_0}, or check directly for a postfixed point if you have a good intuition of what it should be.

Solution 3.

We easily get:

- $\varphi_\tau(A) \subseteq A \cup B \cup C \cup D$ since $F_A(A \cup B \cup C \cup D) = ([0, 1/2] + 1/8(1 - [0, 1]^2), [-2, -1] - 1/8[-2, 0]) \subseteq A \cup B \cup C \cup D$,
- $\varphi_\tau(D) \subseteq D \cup C' \cup F \cup E'$ since $F_D(D \cup C' \cup F \cup E') = ([1/2, 1] + 1/8(1 - [1/2, 3/2]^2, [-1, 0] - 1/8[-1, 1]) \subseteq D \cup C' \cup F \cup E'$.

Overall, the graph of the corresponding multivalued map is indicated in Figure 2.

Question 4.

What is condensation of the graph of Figure 2?

Solution 4.

It is given at Figure 3.

Question 5.

From last question, can you give all index pairs?

Solution 5.

Index pairs are $([G] \cup [H, G'] \cup [D, C', F, E'] \cup [C, E], [H, G'] \cup [D, C', F, E'] \cup [C, E])$ and three others (symmetric in x and y).

There are also $([H, G'] \cup [D, C', F, E'], [D, C', F, E'])$ and three other symmetric ones.

And there is $([D, C', F, E'], \emptyset)$.

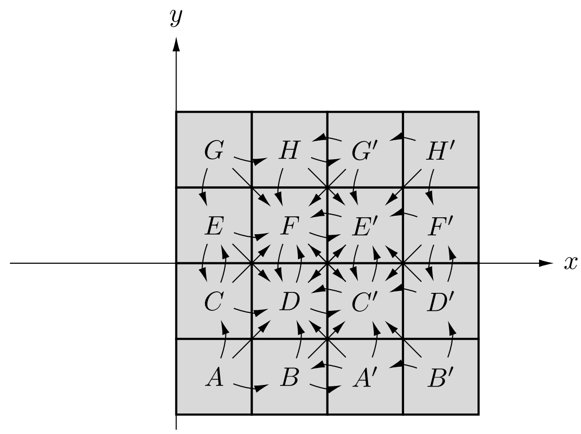


Figure 2: Graph of the multivalued map representing system given by Equation 2.

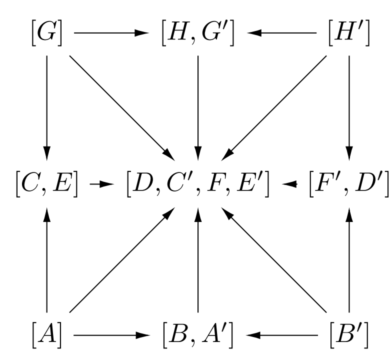


Figure 3: Graph of the multivalued map representing system given by Equation 2.

Question 6.

Determine the homogical Conley index of the index pairs you have determined and the existence of invariants. Did we spot the saddle point? The sink?

Solution 6.

All index pairs but one, have proven non-empty invariant: $([D, C', F, E'], \emptyset)$ (by Wazewski property). The last one gives a contractible quotient hence it may contain a stable fixpoint, which is indeed the case here. We managed to find the sink, but not the saddle point, which is not in the interior of any index pair we found.