

# Exam CTRLVERIFMPRI 2026

3rd March 2026

## 1 Backwards reachability

In the course, we saw how to pave inner and outer-approximations of the backwards reachable set of a neural network. Here we want to find a more direct method using polyhedra instead of boxes.

We recall the definition of a convex polyhedron in  $\mathbb{R}^n$ :

**Definition 1.** A convex polyhedron  $\mathcal{P}$  in  $\mathbb{R}^n$  is defined externally by a set of ( $m \geq 0$ ) affine inequalities:

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

where  $A$  is a  $n \times m$ -matrix and  $b$  is a vector of dimension  $m$ .

We recall that a convex polyhedron, when bounded, can also be represented *internally* as the convex combination of  $k$  points  $p_1, \dots, p_k$ , i.e can be described by convex combinations:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^k \lambda_i p_i, \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1 \right\}$$

We consider a ReLU feedforward neural network NN given by weights  $W^j$  and biases  $b^j$ ,  $j = 1, \dots, l$ , for each of the  $l$  layers it is composed of. We suppose it has the same activation function  $\sigma$  in all layers, so that given inputs  $y_1, \dots, y_n$ , the values of neurons at layer  $k = 0, \dots, l$ , layer 0 corresponding to the input neurons, is given inductively by:

$$\begin{aligned} y^0 &= (y_1, \dots, y_n) \\ y^{k+1} &= \sigma(W^k y^k + b^k) \end{aligned}$$

During the course, we have seen how to do a “forward analysis” of such neural networks, using polyhedra, in particular. This means that, given some polyhedral representation of the values of the input neurons  $\mathcal{P}^0 = (A^0, c^0)$ , we infer a polyhedral outer approximation of the set of values that neurons at each layer  $k$  can take, given by a polyhedron  $\mathcal{P}^k = (A^k, c^k)$ .

In this exercise, we are concerned with a so-called “backward” analysis. Given  $\mathcal{P}^{l+1} = (A^{l+1}, c^{l+1})$  a polyhedron giving the values of the neurons of the last layer, we want to infer inductively, from  $k = l$  down to 0, a set of values that neurons at layer  $k$  should take so that neurons at layer  $l$  have the prescribed values. Formally, we want to compute the polyhedra  $\mathcal{P}^k = (A^k, c^k)$  from  $k = l$  to 0 such that:

$$\{\sigma(W^k y^k + b^k) \mid y^k \in \mathcal{P}^k\} \subseteq \mathcal{P}^{k+1} \tag{1}$$

In this exercise, the only dependency is question 5 (depending on 4).

### Question 1.

Consider  $\mathcal{P}^0$  as defined inductively in Equation (1). Is it an inner or outer approximation of the set of initial inputs to the neural network NN that are (exactly) mapped to  $\mathcal{P}^{l+1}$ ?

**Question 2.**

Consider any layer of a neural network, given by weights and biases  $(W, b)$ . Suppose we are given an hypercube  $\mathcal{Q}$  and want to find an hypercube  $\mathcal{P}$  such that:

$$\{Wy + b \mid y \in \mathcal{P}\} \subseteq \mathcal{Q} \quad (2)$$

Formulate the solution  $\mathcal{P}$  of Equation (2) as a quantified first-order logical formula. Can you use directly the algorithm of course 4 (quantifier elimination) to compute  $\mathcal{P}$ ?

**Question 3.**

1 P.

We consider the simple ReLU neural network  $NN$  with a single layer, with two input neurons  $(x_1, x_2) \in [-1, 1]^2$  and two output neurons  $(y_1, y_2)$ , defined as follows:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} ReLU(2(x_1 - x_2)) \\ ReLU(2(x_1 + x_2) + 1) \end{pmatrix}$$

we write  $y(x) = (y_1(x), y_2(x))$  for  $y = (y_1, y_2)$  computed from  $x = (x_1, x_2)$  as above.

Suppose  $\mathcal{Q}$  is the hyperrectangle  $[0, 1] \times [0, 1]$ . Observing that the linear layer is invertible, compute the corresponding hyperrectangle  $\mathcal{P}$  defined by Equation (2) for the linear layer (not considering the ReLU computation).

**Question 4.**

Consider any layer of a neural network, given by weights and biases  $(W, b)$ . Determine a polyhedron  $\mathcal{P} = (A, c)$  such that, for any polyhedron  $\mathcal{Q} = (C, d)$ :

$$\{Wy + b \mid y \text{ s.t. } Ay + c \leq 0\} \subseteq (C, d)$$

**Question 5.**

We are in the same situation as in Question 4, but we suppose here that  $W$  is invertible, with inverse  $W^{-1}$ . Show that the polyhedron  $\mathcal{P} = (A, c)$  determined in Question 4 is such that, for any polyhedron  $\mathcal{Q} = (C, d)$ :

$$\{Wy + b \mid y \text{ s.t. } Ay + c \leq 0\} = (C, d)$$

**Question 6.**

In the situation of Question 5, when  $W$  is invertible, do you think we can we find polyhedron  $\mathcal{P}$  with an internal representation when  $\mathcal{Q}$  is given also with an internal representation? (Give a short explanation, no need to prove anything there)

**Question 7.**

Consider the particular neural network defined in Question 3, and the polyhedron  $\mathcal{P}^1$  defined by  $y_1 \geq y_2$ . Determine a polyhedron  $\mathcal{P}^0$  such that Equation (1) is satisfied, for inputs  $x_1, x_2$  between -1 and 1. Is that the best one possible? What is missing is of course the backwards interpretation of the ReLU function, that we do not ask to treat in full generality.

## 2 Conley index

Consider the following dynamical system in  $\mathbb{R}^2$ , given in polar coordinates:

$$\dot{r} = r(1 - r), \quad \dot{\theta} = 1. \quad (3)$$

where  $r$  is the radius from point  $(0, 0)$  and  $\theta$  is the angle with respect to the abscissae. We call  $\varphi$  the flow of this ODE:  $\varphi(x, t)$  is the value at time  $t$  of the solution of Equation (3) starting at  $t = 0$  at  $x$ .

Consider the grid defined in polar coordinates, made up of five boxes in the coordinates  $(r, \theta)$ :  $A_0 = [0, 1/2] \times \mathbb{R}$ ,  $B = [1/2, 3/4] \times \mathbb{R}$ ,  $C = [3/4, 1] \times \mathbb{R}$ ,  $D = [1, 5/4] \times \mathbb{R}$  and  $E = [5/4, 3/2] \times \mathbb{R}$  and pictured in Figure ?? on the plane.

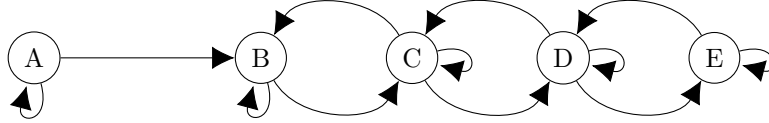


Figure 1: The graph of the multivalued map corresponding to  $\varphi_\tau$ .

We consider the discretized dynamical system  $\varphi_\tau(x, k) = \varphi(x, k\tau)$ ,  $k \in \mathbb{Z}$ , with  $\tau = 1/8$ . The objective of this exercise is to study the invariant sets of the dynamical system given by Equation (3), by finding index pairs and computing Conley indexes of the discretized system, further discretized in space on the grid  $(A, B, C, D, E)$ .

The second component  $\theta$  of Equation (3) has a trivial behavior, we concentrate on the first component  $r$  of the ODE and recall that a guaranteed outer-approximation of the first component of the image of  $R_0 \times \Theta_0$  by  $\varphi_\tau(\cdot, 1)$  is  $R$  such that:

$$R_0 + \tau R(1 - R) \subseteq R \quad (4)$$

(everything being computed using interval arithmetics).

**Question 1.**

Briefly justify Equation (4).

Hint: you can rewrite Equation (3) in integral form:  $\varphi(x, t) = \varphi(x, 0) + \int_0^t \frac{d\varphi}{dt}(x, s) ds$  for all  $t \in [0, \tau]$ .

**Question 2.**

Compute the images of each element of the grid, in terms of the elements of the grid itself.

Indications: This means you will need to find union of grid elements among  $A, B, C, D$ , and  $E$ ,  $A$  (resp.  $B, C, D$  and  $E$ ) is mapped to. You can either do the first steps of a Kleene iteration on the interval map  $F_{R_0} : R \rightarrow R_0 + \tau R(1 - R)$  to infer a postfix point and check that some unions of elements of the grid are postfix-point of  $F_{R_0}$ , or check directly for a postfix point if you have a good intuition of what it should be.

From Question 2, we get the graph pictured in Figure 1 of the multivalued map which is the representation of the discretization in space, on the grid, of the discrete dynamical system  $\varphi_\tau$ .

**Question 3.**

What is the condensation of the graph (the graph of strongly connected components) pictured in Figure 1?

**Question 4.**

Deduce the index pairs for the continuous flow of Equation (3).

**Question 5.**

What is the homological Conley index of the index pairs you found in last question? These will be computed for the geometric realization of these index pairs in the plane deduced from the respective polar coordinates.

**Question 6.**

What can you deduce about the existence of invariant sets within these index pairs, and their shape?

### 3 Euler-Poincaré characteristic and sensor networks

In the course we showed that reachable sets could be characterized geometrically, e.g. using the Euler-Poincaré characteristic, for some applications. Here we are going to show that this characteristic gives enough geometric information for being use for localizing and counting a number of targets that a network of sensors with limited visibility can spot.

For this, we start by showing that the Euler-Poincaré characteristic is some kind of measure, and are going to study an integration operator against it.

**Question 1.**

Prove  $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$  for finite simplicial complexes  $A$  and  $B$ .

We admit now that this is true also for any topological space, geometric realization of finite simplicial complexes and recall that the Euler-Poincaré characteristic of a topological space  $X$  can be defined as

$$\chi(X) = \sum_{i \in \mathbb{N}} (-1)^i rk(H_i(X)) \quad (5)$$

where  $rk(M)$  is the rank of the  $\mathbb{Z}$ -module  $M$ , which is  $m - r$  in the canonical decomposition of  $M$  as  $\mathbb{Z}^{m-r} \oplus \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_r$  (with  $d_j > 0$  and  $d_j | d_{j+1}$ ).

Question 1 implies that  $\chi$  is finitely additive, making it similar to a measure, at least in a finite context, and we wish to define an integral with respect to the *geometric measure*  $\chi$  of a class of function  $h : X \rightarrow \mathbb{Z}$  as:

$$\int_X h d\chi = \sum_{s=-\infty}^{\infty} s \chi(h^{-1}(s))$$

when this is well-defined (in particular all sums considered here and in the sequel are with finitely non-zero or non-empty summands). We will restrict to the class of  $\chi$ -integrable functions that are *simple functions*, written in *canonical form* as follows:

$$h = \sum_{s=-\infty}^{\infty} s \mathbb{1}_{X_s}$$

where  $\mathbb{1}_{X_s}$  denotes the characteristic map of  $X_s$  and where  $X_s = h^{-1}(s)$  form a finite collection of disjoint (non-empty) sets with  $X = \bigcup_{s \in \mathbb{Z}} X_s$ . In that case, the integral of  $h$  becomes trivially:

$$\int_X h d\chi = \sum_{s=-\infty}^{\infty} s \chi(X_s) \quad (6)$$

**Question 2.**

What is  $\int_X \mathbb{1}_U$  where  $U$  is any closed subset of  $X$ ? Briefly justify Equation (6).

**Question 3.**

Using additivity, proven in Question 1, show that, for a simple function not written in canonical form:

$$h = \sum_{s,t=-\infty}^{\infty} s \mathbb{1}_{X_{s,t}}$$

we still have a similar formula:

$$\int_X h d\chi = \sum_{s,t=-\infty}^{\infty} s \chi(X_{s,t})$$

with finitely many non-empty disjoint sets  $X_{s,t}$  with  $\bigcup_{s,t=-\infty}^{\infty} X_{s,t} = X$ .

**Question 4.**

Prove that  $\int_X$  is a linear form on the space of simple functions, i.e. that  $\int_X (\lambda h) d\chi = \lambda \int_X h d\chi$  for all  $\lambda \in \mathbb{Z}$ , and  $\int_X (h + g) d\chi = \int_X h d\chi + \int_X g d\chi$ .

(hint: note that writing  $X_s = h^{-1}(s)$  and  $Y_t = g^{-1}(t)$  for  $s \in \mathbb{Z}$  and  $t \in \mathbb{Z}$ ,  $h + g = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (s + t) \mathbb{1}_{X_s \cap Y_t}$ )

Because of linearity proven in last question and Question 1, the integral of functions of the form:

$$h = \sum_{\alpha \in A} c_\alpha \mathbb{1}_{U_\alpha}$$

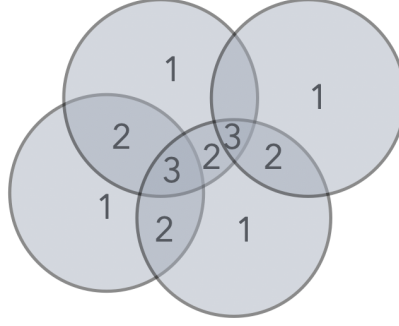


Figure 2: Regions  $U_\alpha$  for each of 4 targets, and the global counting function.

where the sets  $U_\alpha$  may not be disjoint, contrarily to the case of simple functions presented in a canonical manner, can still be written as:

$$\int_X h d\chi = \sum_{\alpha \in A} c_\alpha \chi(U_\alpha)$$

Suppose there is a certain (finite) number of *targets* in  $\mathbb{R}^n$ , that we want to count using *sensors*. To keep things simple, we suppose that we have as many sensors as we may need, and for this exam, this means we consider all points  $x \in \mathbb{R}^n$  to be a sensor. Each target  $\alpha \in A$  is going to be seen by sensor  $x \in U_\alpha$  which is a closed set of  $\mathbb{R}^n$ .

In this situation, we consider now the counting function

$$h = \sum_{\alpha \in A} \mathbf{1}_{U_\alpha} \quad (7)$$

i.e. is such that  $h(x)$  is the number of targets belonging to  $A$  that sensor  $x$  sees. An example is given with four targets at Figure 2.

We suppose that all  $U_\alpha$  have the *same shape*, i.e. in what is of concerned to us here, that there exists  $N \in \mathbb{Z}$  such that for all  $\alpha \in A$ ,  $\chi(U_\alpha) = N$ .

**Question 5.**

Suppose  $N \neq 0$ . Prove that the total number of targets is:

$$\frac{1}{N} \int_X h d\chi$$

where  $h$  is the counting function of Equation (7).

This provides us with a simple algorithm for a sensor network to determine the total number of targets: each sensor broadcasts its position and its number of targets seen, the rest is a simple algebraic calculation (generalizing the inclusion-exclusion principle).

Still, the level sets  $h^{-1}(s)$ ,  $s \in \mathbb{Z}$ , are not closed nor open sets in general, making it a bit tricky to compute their Euler-Poincaré characteristic. Instead, we can prove, but do not ask you to prove, that we can compute the Euler integral of  $h$  as:

$$\int_X h d\chi = \sum_{s=0}^{\infty} (\chi(h^{-1}((s, \infty))) - \chi(h^{-1}((-\infty, -s])))$$

where  $(a, b)$  denote the open interval, not including  $a$  nor  $b$ .

**Question 6.**

Consider the 4 targets of Figure 2 together with their respective  $U_\alpha$ , which are all closed discs in  $\mathbb{R}^2$  as pictured. Compute the Euler integral  $\int_X h d\chi$  for  $h$  the corresponding counting function  $h = \sum_{\alpha \in A} \mathbb{1}_{U_\alpha}$ . Is this consistent with the result of Question 5?

**Question 7.**

Compute the Euler-Poincaré characteristic of the  $n$ -sphere  $S^n \subseteq \mathbb{R}^{n+1}$ .

**Question 8.**

Suppose each target  $\alpha$  can be seen by sensors in  $U_\alpha$ , homotopy equivalent to  $S^n \subseteq \mathbb{R}^{n+1}$ . Is this *target counting* algorithm, consequence of the formula obtained at Question 5, applicable in all  $\mathbb{R}^{n+1}$ , whatever  $n \in \mathbb{N}$  is?