Course 5-6: From Sets to Geometry
Application to unbounded-time reachability and controllability

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MPRI

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General problematic

Pure set-based methods

- Very efficient, but ignore the underlying geometry, in general
- Either because we are looking at local solutions only
- Or because we convexify things for tractability ("outer-approximations" in particular)

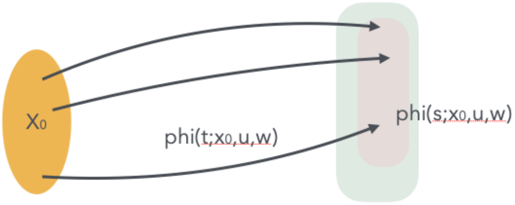
We may want more

- Just to know better what these (e.g. reachable) sets look like geometrically: connected components, holes? "homotopy type" in general?
- But also for some specific applications, some of which we will describe in the rest of the course, e.g.:
  - Going from bounded-time reachability to unbounded-time (positive invariants in particular)
  - For quantitatively determining the quality of the classification made by some neural network (wrt e.g. what we know from the training data)
  - For determining the complexity of some problems (classification, control/motion-planning)
  - For proving the impossibility for an algorithm to solve some problem (in fault-tolerant distributed computing, but also in control/motion-planning)
  - etc.

Back to reachability problems

What is missing with our approximations so far

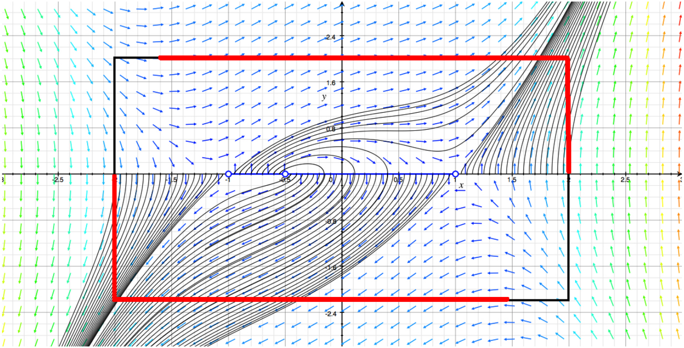
We generally use convex approximations for outer-approximations, and local inner-approximations, henceforth all information about the "topology" is lost - at least for now, we do not even know the number of connected components!



Back to reachability problems

Example: from bounded-time reachability to unbounded time reachability

For a continuous dynamical system (ODE), we will be able to "read" what happens within some compact set, from the exit set of the flow map on that compact set - we will see some applications of this (Wazewski's property, Conley index theory) to the existence of local (positive) invariants within some prescribed compact set.



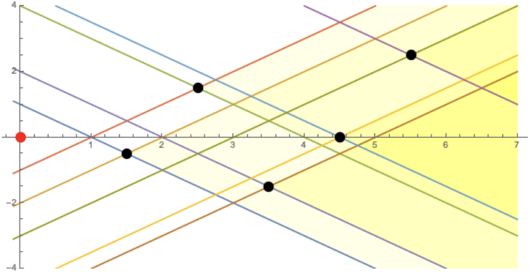
$$\dot{x} = y, \dot{y} = y + (x^2 - 1)\left(x + \frac{1}{2}\right)$$

Back to reachability problems

Example: backward reachability

Find controls that will reach some target (sub-)space after some amount of time, and will not run into some other subspaces (e.g. obstacles, as for reach-avoid problems):

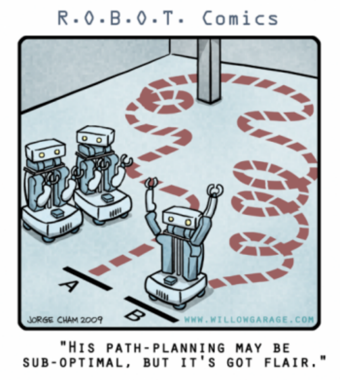
- connected components will lead to at least one discrete control ("switched controller")
- the finer topology will lead to at least know the necessary dimension, or number, of the continuous controls



(set of backward reachable states/controls with all these obstacles: 11 components)

Similarly, but for measuring the “complexity” of potential controller

Topological complexity



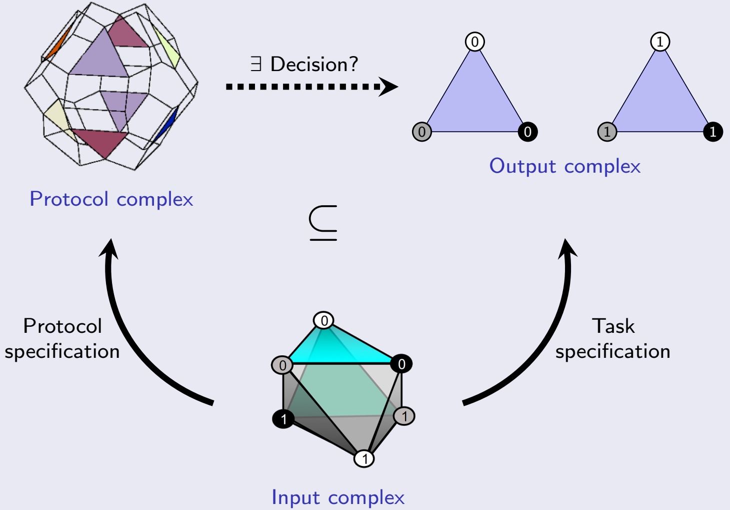
Motion planning algorithm : how do I go from point A to point B?

Its complexity : how can I partition the space of points A, B so that to apply on each part a “simple” (e.g. continuous in that case) formula?

The number of discontinuities (similarly to the last slide) is a measure of that complexity.

Example: distributed systems (see MPRI/PODC)

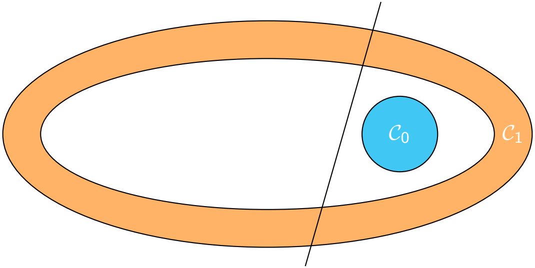
Task specification (the protocol complex is a depiction of the set of reachable states)



Example: neural network classification

The "shape of classes"

A neural network used as a classifier, trained over a set of labelled data, is going to be of quality if it "recognizes" the shapes of the different input classes.



Here: two nested classes with a linear separator; about 80% of the samples will be classified correctly, but we clearly missed the point here.

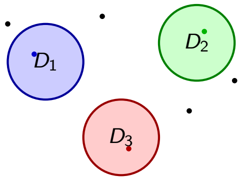
Example: complexity of classification problems

(already a glimpse on that in the last example)

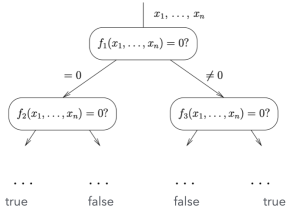
Example: complexity of data classification

When the classes labeled data has "complex" shapes, this has to reflect into the complexity of any classifier (e.g. neural network).

Suppose for now the data forms a "real" topological space (not just a discrete set of points). Then having m connected components means we need at least an algorithm with complexity $O(\log\ m)$ to classify them ("set membership problem" using algebraic decision trees)



X together with points sampled in the ambient space



Corresponding decision tree

What we are going to cover in the next two courses

These two courses are designed so that:

- I can give you the minimum algebraic topological background so that you understand what we mean by "the geometry of reachable states"
- This is applied to understanding the unbounded reachability properties of continuous and discrete dynamical systems.

More algebraic topology (but not too much) in the last two courses.

Do not worry: you come from diverse backgrounds, you will not need to become an expert algebraic topologist!

Although if you want to learn more, you can have a look at e.g.:

- Hatcher "Algebraic Topology" `https://pi.math.cornell.edu/~hatcher/AT/AT.pdf`
- Massey "A basic course in Algebraic Topology" `https://link.springer.com/book/10.1007/978-1-4939-9063-4`
- Riehl "A Leisurely introduction to simplicial sets" `https://math.jhu.edu/~eriehl/ssets.pdf`
- Even more expert: May "A concise course in algebraic topology" `https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf`

What we are going to cover in the next two courses

1. General problematic
   - Introduction to Algebraic Topology/the Classification Problem
   - A nice instance: classification of surfaces
   - Introduction: invariants and Lyapunov functions
2. From bounded to unbounded time reachability
   - Introduction: invariants and Lyapunov
   - The geometry of invariant sets: isolating blocks, isolated invariants
   - Wazewski property and the Conley index
   - Back to basic algebraic topology: the fundamental group
   - Back to basic algebraic topology: higher homotopy groups
3. From continuous dynamics to discrete dynamics
   - Back to topological spaces: combinatorial counterparts
   - From homotopy to (more computable) homology
   - Back to Conley: from continuous to discrete dynamical systems
4. Applications
   - Application 1: prove local stability of equilibrium points
   - Application 2: find periodic orbits
   - Algorithm for computing the (homological) Conley index
   - Alternate method
   - Application 3: prove controllability of a continuous system

Reminder: general topology

Topological space

A topological space X is given by a set ("of points", that we still call X) and by a set of subsets $U \subseteq X$ called "open sets", satisfying the following properties:

- X and $\emptyset$ are open sets,
- Any union of open sets $\bigcup\limits_{i \in I} U_i$ (even infinite) is an open set,
- Any finite intersection $\bigcap\limits_{i=0}^{k} U_i$ of open sets is an open set.

(models neighborhoods ("qualitatively") in sets)

Morphisms

Morphisms $f : X \rightarrow Y$ from topological space X to topological space Y are the continuous functions, i.e. set-theoretic maps from X to Y such that for all open sets V of Y, $f^{-1}(Y)$ is an open set of X.

(the category of topological space has as objects, topological spaces, and as morphisms, morphisms of topological spaces)

Closed sets

Definition

- They are the complements in X of the open sets of X (i.e. of $U \in \mathcal{O}(X)$),
- Enjoy axiomatics dual to the one of open sets.

Reminder

- There is always the smallest closed subset $cl(A)$ (or $\overline{A}$) containing A of X,
- There is alway the greatest open subset $int(A)$ within A of X.
- The frontier or boundary of A in X is $Fr(A) = cl(A) \backslash int(A)$.

Purely due to posetal arguments (see e.g. abstract interpretation!).

The classification problem in topology

Homeomorphisms

An homeomorphism is a continuous bijection, whose inverse is also continuous

(these are isomorphisms in the category of topological spaces)

Classification of topological spaces

Generally: determine at least when two topological spaces are homeomorphic. Even more dreamy: find the list of homeomorphic types of topological spaces

This way too ambitious - no way to reach this.

The classification problem in topology

Instead, we are going to use more practical ways to at least separate homeomorphism classes:

The notion of homotopy equivalence

Relaxation of homeomorphism: equivalent "modulo continuous deformations"!

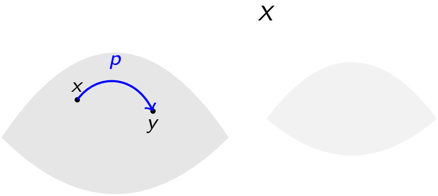
The notion of topological invariants

Any quantity associated (generally, functorially) to topological spaces, that are the space for all homeomorphic spaces.

The ones we will introduce will also be topological invariants also under homotopy equivalence.

See e.g. Hatcher "Algebraic Topology"
`https://pi.math.cornell.edu/~hatcher/AT/AT.pdf`

The classification problem in topology



Prime example: connected components [by arcs] $\pi_0(X)$

- For X a topological space, call PX the set of continuous [compact] paths in X, i.e. $PX = \{p : [0,1] \to X \mid p \text{ continuous}\}$.
- Two points x, and y in X are in the same component [by arcs] iff there exists $p \in PX$ such that $p(0) = x$ and $p(1) = y$.
- The set of connected components [by arcs] $\pi_0(X)$ is the set of classes of paths in PX modulo the equivalence relation $\sim$ "being in the same component [by arcs]".

The classification problem in topology

π_0 defines a topological invariant

- It is functorial: let $f: X \rightarrow Y$ be a continuous function, then for all $p \in PX$, $f \circ p \in PY$ and if $x \sim y$, $f(x) \sim f(y)$,
- If f is a homeomorphism between X and Y, then it induces a (set-theoretic) bijection between $\pi_0(X)$ and $\pi_0(Y)$:
  $\rightarrow$ Indeed, using f^{-1} and reasoning as above, $x' \sim y'$ in Y, and f being a bijection means there are unique x and y in X such that $x' = f(x)$, $y' = f(y)$ and, applying f^{-1}, $x \sim y$.

Homeomorphisms and homotopies

The classification problem is not feasible, but there is a “simpler” equivalence notion, “up to deformation”.

Homotopy of maps

$f_0,\ f_1:\ X \rightarrow Y$ are homotopic if there exists a continuous function $F:\ X \times [0,1] \rightarrow Y$ such that $H_{|X\times\{0\}} = f_0$ and $H_{|X\times\{1\}} = f_1$

Homotopy relative to a subspace

$f_0,\ f_1:\ X \rightarrow Y$ are homotopic relative to $A \subseteq X$ if there exists a continuous function $H:\ X \times [0,1] \rightarrow Y$ such that $H_{|X\times\{0\}} = f_0$ and $H_{|X\times\{1\}} = f_1$ and $H(x,t) = f_0(x) = f_1(x)$ for all $x \in A$ and $t \in [0,1]$.

These are equivalence relations and are noted $f_0 \simeq f_1$ and $f_0 \simeq_A f_1$ respectively.

Homotopy between spaces

Homotopy equivalence

Two topological spaces X and Y are homotopic if there exists two morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $f \circ g$ is homotopic to Id_Y and $g \circ f$ is homotopic to Id_X.

This is an equivalence relation, noted $X \simeq Y$. A space homotopic to a point is called "contractible".

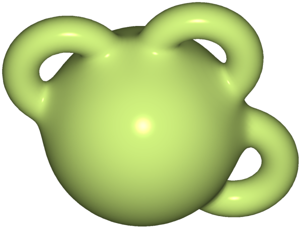
Note that X being homeomorphic to Y implies $X \simeq Y$. We may be equally interested in classification modulo homotopy (which is finer than the original classification problem).

A nice instance of classification: surfaces

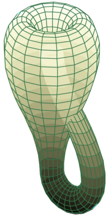
Intuition

Every compact, connected surface (2-manifold - i.e. "without boundary") is, up to homeomorphism, obtained from the sphere by "adding" either

- (1) $g \geq 0$ *handles* (orientable summands, gives connected sums of tori), or
- (2) $c \geq 0$ *crosscaps* (nonorientable summands, gives connected sums of projective planes).



Three connected tori (case (1) with $g = 3$)



Klein bottle (case (2) with $c = 2$)

Two invariants determine the type: orientability ((1): orientable, (2): non-orientable) and the Euler characteristic (or equivalently the genus, equal to g or c).

Torii? Crosscaps? Projective planes?

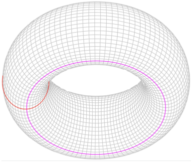
Crosscap

Intuitively, consists of piercing a hole in a surface and sewing the edge back together by identifying diametrically opposite points (see e.g. `https://www.youtube.com/watch?v=W-sKLN0VBkk`).

Projective plane

- quotienting the set of nonzero vectors of $\mathbb{R}^3$ by the equivalence relation "being collinear".
- Thus: each element of the projective plane is a line without the zero vector (we will see a "combinatorial model" for it later)

Simple: Torus=product of two circles



The Classification Theorem

More formally

Every compact, connected 2-dimensional surface without boundary is homeomorphic to exactly one of:

- the sphere S^2,
- a connected sum of g tori $T^2 \# \cdots \# T^2$ (orientable, genus $g \geq 1$),
- a connected sum of k real projective planes $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ (nonorientable, $k \geq 1$).

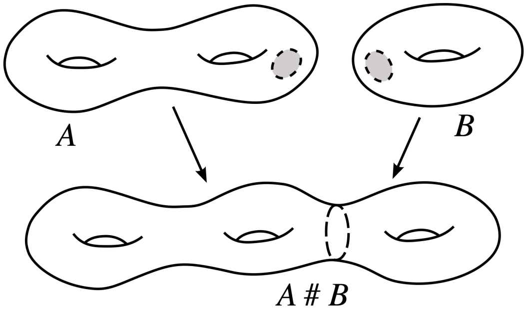
For these particular topological spaces, the classification (under homeomorphism) and the weaker classification (under homotopy equivalence) are equivalent.

(sketch of a proof later - for a modern proof, see e.g. Francis & Week 1999 "Zip proof" `https://webhomes.maths.ed.ac.uk/~v1ranick/papers/francisweeks.pdf`)

The classification Theorem

Connected sums

- A connected sum of two m-dimensional manifolds is a manifold formed by deleting a m-ball inside each manifold and gluing together the resulting boundary $(m-1)$-spheres,
- Here applied with topological manifolds of dimension $m = 2$.



The Classification Theorem

Dyck's theorem

The closed surfaces up to homeomorphism form a commutative monoid under the operation of connected sum:

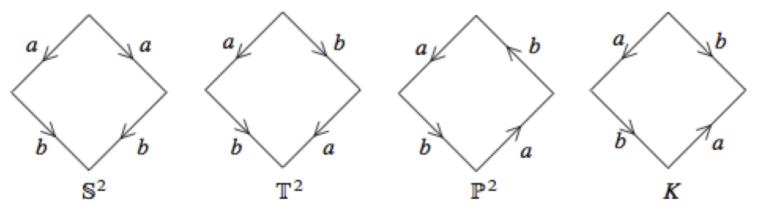
- The identity is the sphere,
- There are two generators: the real projective plane **P** and the torus **T**,
- Modulo the relation $\mathbf{P}\#\mathbf{P}\#\mathbf{P} = \mathbf{P}\#\mathbf{T}$.

(or equivalently, it is generated by the projective plane **P**, torus **T** and the Klein bottle **K** with the unique relation $\mathbf{P}\#\mathbf{K} = \mathbf{P}\#\mathbf{T}$ - since $\mathbf{K} = \mathbf{P}\#\mathbf{P}$)

The Classification Theorem

Also:

Can be understood using a combinatorial presentation of surfaces



More general combinatorial cell decomposition of a topological space than simplicial sets [to be defined later] - using polyhedric cells:

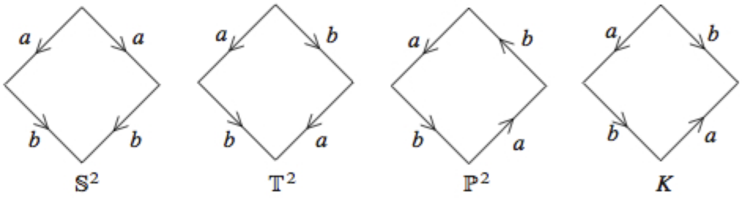
Polygonal presentation

$\mathcal{P} = \langle S \mid W_1, \ldots, W_k \rangle$ where S is a finite set and $W_1, \ldots, W_k$ are words in $S \cup S^{-1}$ of length 3 or longer.

Polygonal presentations

Geometric realization

- For each word W_i, let P_i denote the convex regular k-sided (unit sides) polygonal region in the plane centered at 0, where k is the length of W_i - have one point on the positive y axis
- Label the edges in the counterclockwise order starting at the point on the positive y axis
- Consider $\coprod_{i=1}^{k} P_i$ quotiented by identifying edges with the same label, and identifying x with x^{-1} by an homeomorphism that reverts the corresponding edges



Proof Sketch of the classification of surface theorem

Main steps:

1. Triangulate the surface (every compact surface has a finite triangulation).
2. Collapse a spanning tree in the 1-skeleton to obtain a single polygon whose boundary records edge identifications (a polygonal schema).
3. Use elementary edge moves (remove digons, flip, reorder) to simplify the boundary word to a standard normal form:

   $$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \quad \text{(orientable)}$$

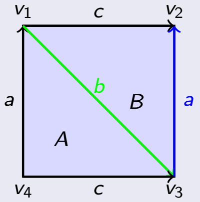
   or

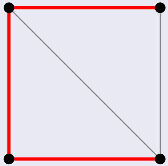
   $$a_1^2 a_2^2 \cdots a_c^2 \quad \text{(nonorientable).}$$

   (These moves preserve the homeomorphism type and eventually yield the normal forms - e.g. for the projective plane replace *baba* by $(ba)(ba)$ or AA)
4. Interpret the standard words: they correspond to sphere, sums of g tori, or c projective planes.

Example: a simple triangulated torus

Two triangular 2-cells forming a square (opposite edges later identified)





A spanning tree touches all vertices and has no cycles.

Collapse the spanning tree

Kills c, leaves generators a and b:

- We kill c and face A has boundary ab, face B has boundary ba.
- Looking at the total boundary, composed of the boundaries of the two triangles (giving the square above):

$$(ab)(ba)^{-1} = a\,b\,a^{-1}\,b^{-1}$$

Euler Characteristic and Genus

Euler characteristic of a polygonal complex

$\chi(P) = \#points - \#edges + \#surfaces$ or in short:

$$\chi(P) = V - E + F$$

Euler characteristic, genus, and the classification theorem of surfaces

For a compact connected orientable surface of genus g: $\chi = 2 - 2g$. For a compact connected nonorientable surface with k crosscaps: $\chi = 2 - k$. Hence knowledge of orientability and χ recovers the exact type (sphere, g-torus sum, or k-projective-plane sum).

Euler characteristic of a connected sum

$\chi(M\#N) = \chi(M) + \chi(N) - 2.$

Exercise

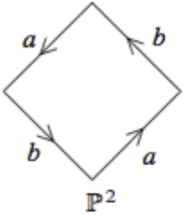
Euler characteristic of the projective plane P?

Euler characteristic and genus of torus T?

Exercise

Euler characteristic of the projective plane **P**?

By the polyhedral presentation:



2 vertices, 2 edges, 1 2 cell: $\chi(\mathbf{P}) = 2 - 2 + 1 = 1$; non orientable so $\chi = 2 - k$ so $k = 1$, this is coherent with the other way to go: $\chi = 2 - k$ and it is a sum of $k = 1$ projective planes.

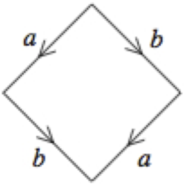
Euler characteristic and genus of torus **T**?

Exercise

Euler characteristic of the projective plane **P**?

Euler characteristic and genus of torus **T**?

By the polyhedral presentation:



$\mathbb{T}^2$

1 vertex, 2 edges, 1 2 cell: $\chi(\mathbf{P}) = 1 - 2 + 1 = 0$; orientable so $\chi = 2 - 2g$ so $g = 1$, this is coherent with the other way to go: $\chi = 2 - 2g$ and it is a sum of $g = 1$ tori.

Dyck's relations?

Check!

Recall, relation $\mathbf{P}\#\mathbf{P}\#\mathbf{P} = \mathbf{P}\#\mathbf{T}$!

Dyck's relations?

Check!

Recall, relation $\mathbf{P}\#\mathbf{P}\#\mathbf{P} = \mathbf{P}\#\mathbf{T}$!

$$\begin{aligned}
\chi(\mathbf{P}\#\mathbf{P}\#\mathbf{P}) &= \chi(\mathbf{P}\#\mathbf{P}) + \chi(\mathbf{P}) - 2 \\
&= \chi(\mathbf{P}) + \chi(\mathbf{P}) + \chi(\mathbf{P}) - 4 \\
&= 1 + 1 + 1 - 4 \\
&= -1
\end{aligned}$$

and:

$$\begin{aligned}
\chi(\mathbf{P}\#\mathbf{T}) &= \chi(\mathbf{P}) + \chi(\mathbf{T}) - 2 \\
&= 1 + 0 - 2 \\
&= -1
\end{aligned}$$

Dyck's relations?

Check!

Recall, relation $\mathbf{P}\#\mathbf{P}\#\mathbf{P} = \mathbf{P}\#\mathbf{T}$!

$$\begin{aligned}\chi(\mathbf{P}\#\mathbf{P}\#\mathbf{P}) &= \chi(\mathbf{P}\#\mathbf{P}) + \chi(\mathbf{P}) - 2 \\ &= \chi(\mathbf{P}) + \chi(\mathbf{P}) + \chi(\mathbf{P}) - 4 \\ &= 1 + 1 + 1 - 4 \\ &= -1\end{aligned}$$

and:

$$\begin{aligned}\chi(\mathbf{P}\#\mathbf{T}) &= \chi(\mathbf{P}) + \chi(\mathbf{T}) - 2 \\ &= 1 + 0 - 2 \\ &= -1\end{aligned}$$

The Euler characteristic is actually a measure of some sort ("Geometric measure") with interesting applications to e.g. localisation/detection/counting see e.g. Ghrist, "The Euler calculus", `https://arxiv.org/abs/1202.0275`.

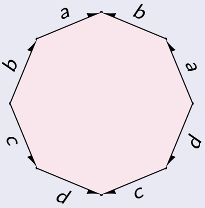
Detailed Example: octagon word

The octagon word

Consider the octagon with boundary word

$$aba^{-1}b^{-1}cdc^{-1}d^{-1}.$$

Pairing opposite edges yields two handles (one for (a, b), another one for (c, d)).



Euler characteristic:

- for the single-polygon model $F = 1$, after identifications $V = 1$, $E = 4$ (opposite edges glued in pairs) so $\chi = 1 - 4 + 1 = -2$,
- Solve $-2 = 2 - 2g$ to get $g = 2$.

Generalizations

By reduction to the case of surfaces without boundary (a boundary of a surface homeomorphic to a collection of circles - just fill them in to get a surface without boundary):

Surfaces with boundary

Two compact surfaces are homeomorphic if and only if their boundaries have the same number of connected components, they have the same orientability, and they have the same genus.

Non-compact surfaces

More complicated, have to consider the space of “ends” of the surface...

From bounded-time to unbounded-time reachability

We consider a given differential system

- $\dot{x} = f(x)$ (where x is a vector in $\mathbb{R}^n$ whose components are functions of time and $\dot{x}$ denotes its time derivative),
- Its flow function (when solutions exist!) is $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which to $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ gives $\varphi(t, x_0)$

Determine sets which are positive invariants:

Positive invariants are sets that are invariant under the flow φ, for all positive times, i.e. are sets S such that $\varphi(\mathbb{R}^+, S) \subset S$

(similar, for programs i.e. discrete dynamical systems to program invariants)

By abstract interpretation

We would normally do(?) some form of fixpoint (value) iteration:

$$S_{n+1} = \varphi([0, \tau], S_n) \cup S_n$$

with $S_0 = X_0$ the set of initial states, and for some $\tau \geq 0$ [iterating reachability]

With widenings etc. Very difficult with fine enough abstractions at hand (polynomial zonotopes maybe?)

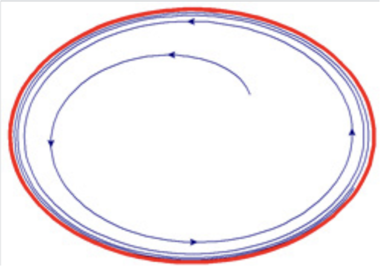
Overall picture: dynamical systems are hard to understand!

(but linear systems are simple)

Basic fact 1: Poincaré-Bendixon theorem

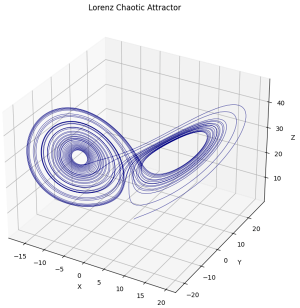
Given a differentiable real dynamical system defined on an open subset of the plane, every non-empty orbit which is included in a compact set, is either:

- converging to a fixed point,
- or converging to a periodic orbit ("limit cycle")



Overall picture: dynamical systems are hard to understand!

But in dimension 3 or higher, for non-linear dynamics, potentially chaotic dynamics



Basic fact 2: global "Morse decomposition"

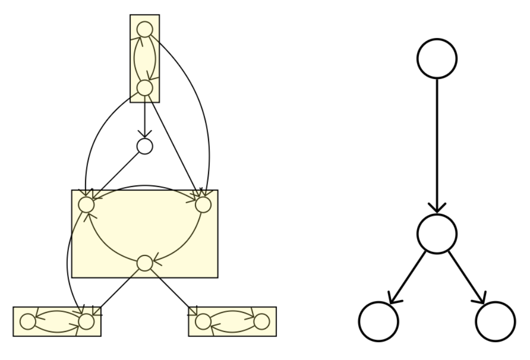
Every flow of a dynamical system with compact phase portrait admits a decomposition into a "chain-recurrent part" and a "gradient-like" flow part

(we get back to this when we talk about "complete Lyapunov functions")

Also for discrete dynamical systems!

An instance you know well

1-dimensional systems, aka (directed) graphs:



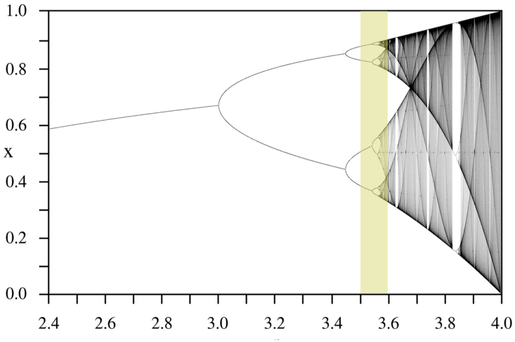
The "chain-recurrent" parts are the strongly-connected components, the "gradient-like" flow is identified (right) by the graph condensation (quotienting the SCC).

Example of chaotic discrete dynamical system

The logistic map $f(x) = rx(1-x)$

Consider the dynamical system $x_{n+1} = f(x_n)$

Fixpoints depending on r:



The logistics map

For $r = 4$

Extremely chaotic!

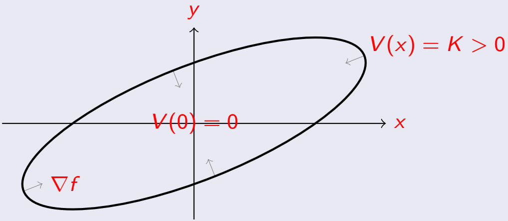
E.g. numerical iterates:

$$x_0 = 0.3333333333333333 \qquad x_0 = 0.3333333333333330$$
$$x_1 = 0.8888888888888889 \qquad x_1 = 0.8888888888888885$$
$$x_2 = 0.395061728395062 \qquad x_2 = 0.395061728395075$$
$$x_3 = 0.955951836610273 \qquad x_3 = 0.955951836610284$$
$$\vdots \qquad \vdots$$
$$x_{15} = 0.695026128241317 \qquad x_{15} = 0.695026128347429$$
$$\vdots \qquad \vdots$$
$$x_{49} = 0.071160322456580 \qquad x_{49} = 0.906436654059206$$

Finding (positive) invariants in dynamical/controlled systems

Notoriously difficult problem

- Classical: Lyapunov functions - works well for linear differential systems, much more difficult with non-linear systems, to establish (asymptotic) stability over region U:
  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that V is decreasing on the flow of f, and V positive definite over U (containing equilibrium point 0):



  Boyd, El Ghaoui, Féron, Balakrishnan "LMIs in System and Control Theory", 1994 or Henrion & Lasserre, "semi-algebraic inner-approximations of region of stability", IEEE Trans. Automatic Control 2014 etc.

- Less classical: algebraic methods (see e.g. Goubault, Jourdan, Putot, Sankaranarayanan: "Finding non-polynomial positive invariants and lyapunov functions for polynomial systems through Darboux polynomials". ACC 2014)

Here we will use the help of some basic algebraic topology.

In fact:

The fundamental theorem of dynamical systems

For an autonomous differential equation defined on a compact set X, a complete Lyapunov function V from X to $\mathbb{R}$ is such that:

- V is non-increasing along all solutions of the differential equation, and
- V is constant on the **isolated invariant sets**.

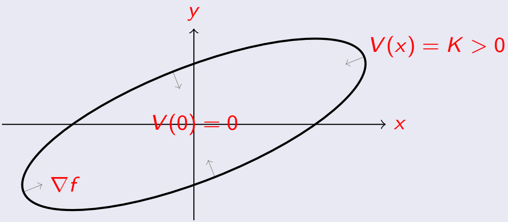
Conley's theorem states that a continuous complete Lyapunov function exists for any differential equation on a compact metric space (and similarly for discrete-time dynamical systems).

We will have to define the "isolated invariant sets"! (stronger condition than just "invariant sets").

Idea of the geometric method

Given a n-dim system of (polynomial) ODEs $\frac{d\vec{x}}{dt} = f(\vec{x})$

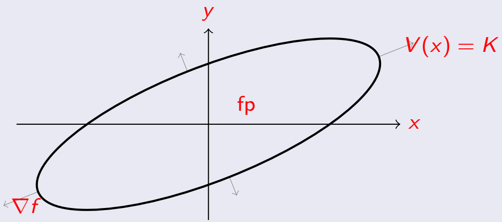
(strong) Lyapunov functions to establish (asymptotic) stability over region U:
$V : \mathbb{R}^n \to \mathbb{R}$ such that V is decreasing on the flow of f, and V positive definite over U (containing equilibrium point 0).



Idea of the geometric method

Given a n-dim system of (polynomial) ODEs $\frac{d\vec{x}}{dt} = f(\vec{x})$

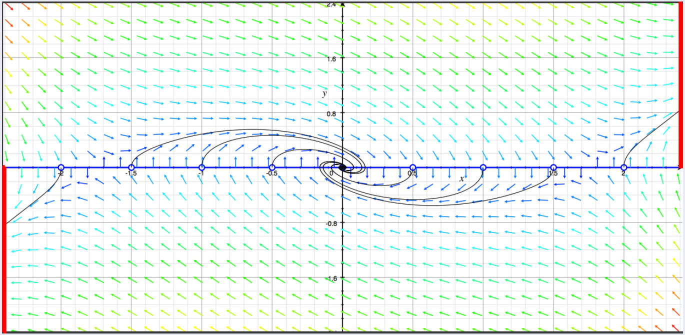
We also have information about potential invariants when the flow goes out! (at least a fixed point)



Idea of the geometric method

Given a n-dim system of (polynomial) ODEs $\frac{d\vec{x}}{dt} = f(\vec{x})$

Also potentially in mixed situations!

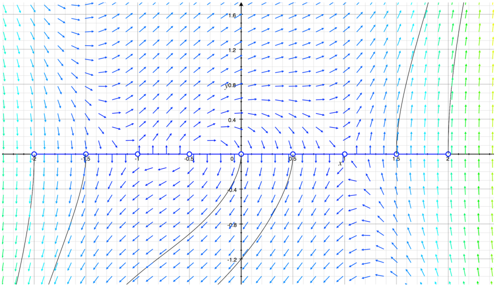


Now looking for simple "templates"

Simple geometry now, but everything up to homotopy!

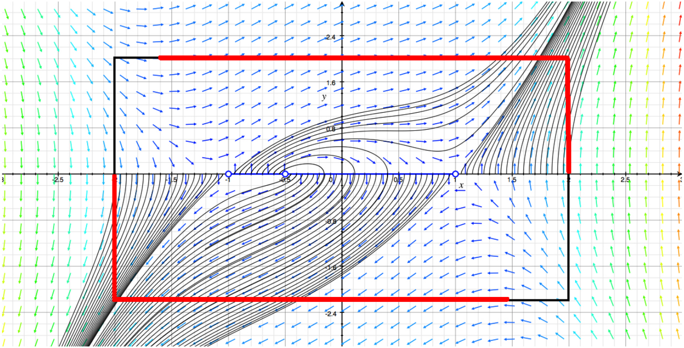
Example

$$\dot{x} = y, \dot{y} = y + (x^2 - 1)\left(x + \frac{1}{2}\right)$$



- Several invariant sets in $B = [-2,2] \times [-2,2] \subset \mathbb{R}^2$: (in particular, the “trivial” ones : fixed points $(-1,0)$, $(-\frac{1}{2},0)$ and $(1,0)$)
- Not an obvious weak Lyapunov function - no easy linear or quadratic template
- Still: $H(x,y) = \frac{y^2}{2} - \left(\frac{x^4}{4} + \frac{x^3}{6} - \frac{x^2}{2} - \frac{x}{2}\right)$ is decreasing along the flow.

We will see...



Simple linear template is enough!

To get information about what happens inside the box $B = [-2, 2] \times [2, 2]$

Will also give much more information about the dynamics within the invariant set - e.g. periodic orbits!

Some notations

Invariants

- The invariant part of a set $N \subseteq \mathbb{R}^n$, denoted as $inv(N, \varphi)$ is the largest invariant subset wrt φ: $Inv(N, \varphi) = \{x \in N \mid \varphi(x, t) \in N, \ \forall t \in \mathbb{R}\}$
- (reminder, abstract interpretation) For a discrete flow φ this is $inv(N, \varphi) = \bigcap_{i \in \mathbb{N}} \varphi(N, i)$,
- For a continuous flow φ (coming from the vector field f) we note also $inv(N, f) = \bigcap_{s \in \mathbb{R}} \varphi(N, s)$ for $inv(N, \varphi)$.

The right notions: isolating neighborhoods and isolated invariants

- A compact subset N is called an isolating neighborhood if $inv(N, \varphi) \subseteq int(N)$,
- Isolated invariants are S such that there is an isolating neighborhood N with $S = inv(N, \varphi)$.

Some notations

The right notions: isolating neighborhoods and isolated invariants

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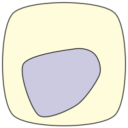
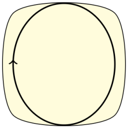




(left: isolating neighborhood; right, not isolating!)

Isolating blocks ("computational version of isolating neighborhoods")

Definition

A compact set B is an isolating block if

(a) $B^- = \{x \in B \mid \varphi([0, T), x) \not\subseteq B, \forall T > 0\}$ is closed

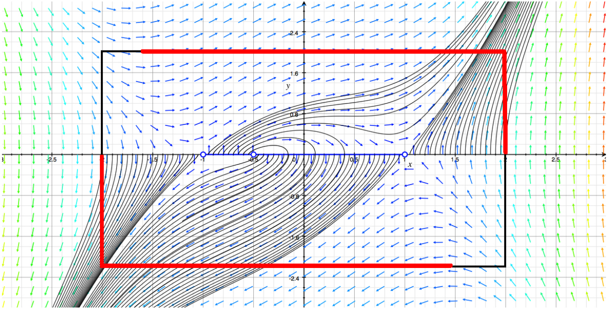
(b) $\forall T > 0, \{x \in B \mid \varphi([-T, T], x) \subseteq B\} \subseteq intB$

Condition (a) imposes that the exit set B^-, i.e. the set of states of B which leave B under flow φ, is closed in the topology of $\mathbb{R}^n$.
Condition (b) imposes that no flow is inner tangential to the boundary ∂B of B.

In particular, isolating blocks:

- their interior are "isolating neighborhoods": $inv(N, \varphi) \subseteq int(N)$. They always determine an isolated (in the sense "far enough" from others - or said differently it is maximal in $int(N)$) invariant set (possibly empty),
- robust wrt perturbations (and numerical calculations!): let B be an isolating neighborhood/block for φ on X compact metric space; there exists Φ a neighborhood of φ in the compact open topology such that B is an isolating neighborhood/block for all flows ψ in Φ.

Example



Recap:

- This is a robust notion : all $B = [-a, a] \times [-b, b]$ with $a > 1$ and $b > 1$, in particular, are isolating blocks.
- An isolating block is still isolating for "nearby flows" (in particular, robust to numerical approximations)

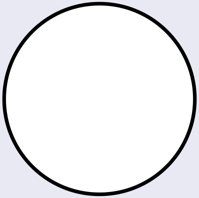
...but still an isolating block may only contain an empty invariant...needs an extra condition!

Wazewski Property and the Conley index

Wazewski property

If B is an isolating block and B^- is not a (strong) deformation retract of B then there exists a not-empty invariant set S in the interior of B.

Deformation retracts (intuition)

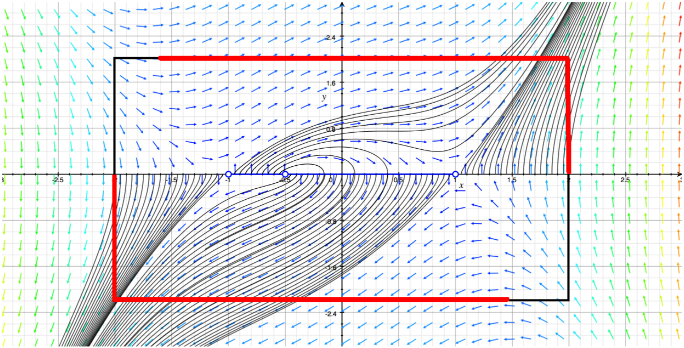


OK!

Not OK!

Higher-dimensional conditions too...and also, implies the same number of connected components!

Example



The exit set...

has two connected components, the box, just one, so it does contain a non-empty invariant! (can be characterized better though its Conley index, and further information, a bit later...)

Deformation retraction?

Retraction

A retraction is a continuous map $r: X \to A$ such that $r_{|A} = Id_A$

Deformation retraction

A subspace $A \subseteq X$ with inclusion map $i: A \hookrightarrow X$ is a retract by deformation (resp. strong deformation retract) if there exists $p: X \to A$ such that $p \circ i = Id_A$ and $i \circ p \simeq Id_X$ (resp. relatively to A).

p in the above definition will be called a retraction by deformation (resp. a strong retraction by deformation).

Properties of deformation retracts

- If A is a retract by deformation of X then A and X are homotope, $X \simeq A$,
- If A is a strong retract by deformation of X then A and X are homotope relative to A, $X \simeq_A A$.

Example/exercise

The sphere S^n is a strong retract by deformation of $\mathbb{R}^{n+1} \backslash \{0\}$

Example/exercise

The sphere S^n is a strong retract by deformation of $\mathbb{R}^{n+1} \backslash \{0\}$

- Let $r: \mathbb{R}^{n+1} \backslash \{0\} \rightarrow S^n$ with $r(x) = \frac{x}{||x||}$
- And $H: \mathbb{R}^{n+1} \backslash \{0\} \times [0,1] \rightarrow \mathbb{R}^{n+1} \backslash \{0\}$ with $H(x,t) = \frac{x}{t||x||+1-t}$

We have indeed:

- $H(x,0) = x$ and $H(x,1) = r(x)$
- $H_{|S^n} = Id_{S^n}$

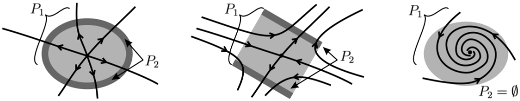
So H is a homotopy relative to S^n from r to Id and r is a strong retract by deformation.

More about the "shape" of the isolated invariant set: Conley index

Index pair

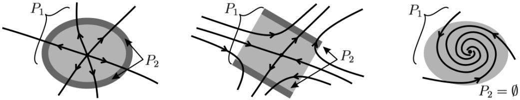
Let φ be a flow on $\mathbb{R}^n$. The pair of compact sets (P_1, P_2) such that $P_2 \subseteq P_1$ is called an index pair if the following properties hold true:

- If $x \in P_1$ and $\varphi(x, s) \notin P_1$ for some $s > 0$ then there exists $t \geq 0$ such that $\varphi(x, [0, t]) \subseteq P_1$ and $\varphi(x, t) \in P_2$,
- if $x \in P_2$ and $\varphi(x, [0, t]) \subseteq P_1$ then also $\varphi(x, [0, t]) \subseteq P_2$,
- $inv(cl(P_1 \backslash P_2), \varphi) \subseteq int(P_1 \backslash P_2)$.



All isolating neighborhoods S admit an index pair such that $S = Inv(N_1 \backslash N_2)$ and $N_1 \backslash N_2$ is a neighborhood of S. Particular case: isolating block B: (B, B^-) is an index pair!

More about the "shape" of the isolated invariant set: Conley index



(Homotopy) Conley index of an isolating neighborhood

The Conley index of the index pair (P_1, P_2) is the homotopy type of the pointed quotient space $(P_1/P_2, [P_2])$ (equivalently of the relative pair (P_1, P_2)).
This index is independent of the choice of the index pair for the isolated invariant set $S = inv(cl(P_1 \backslash P_2), \varphi)$.

This index will give us a lot of information concerning the invariant set! And will lead to a lot of knowledge e.g. about the attractivity/repulsivity, periodic aspects of it...

See e.g. `https://chomp.rutgers.edu/Project.html`.

How to recognize deformation retracts and homotopy types?

Difficult question!

But we may carry on with the idea of using topological invariants, such as $\pi_0(X)$

Introduction to higher invariants

- We have seen $\pi_0(X)$: connected components, simple instance of homotopy "structures",
- We will define $\pi_1(X)$ is the fundamental group, or Poincaré group, the first instance of general homotopy groups,
- And then higher-homotopy groups $\pi_n(X)$

In very important cases ("CW-complexes"), "having the same homotopy groups" or more exactly being "weakly homotopically equivalent" will be the same as homotopy equivalence, so these topological invariants are fairly strong.

The Fundamental Group π_1

Loops

- Fix a basepoint $x_0 \in X$. A loop is $\gamma : (I, \partial I) \rightarrow (X, x_0)$, i.e. $\gamma(0) = \gamma(1) = x_0$.
- Define a binary operation on loops ("concatenation") as follows:

$$\gamma * \tau(t) = \begin{cases} \gamma(2t) & t \leq \frac{1}{2} \\ \tau(2t-1) & t \geq \frac{1}{2} \end{cases}$$

The fundamental group

- Consider loops modulo homotopy $[\gamma]$, concatenation passes to the quotient and becomes associative, with neutral element $[x_0]$ the class of the constant loop on x_0,
- Even more, for every path γ, defining $\gamma^{-1}(t) = \gamma(1-t)$, $[\gamma * \gamma^{-1}] = [\gamma^{-1} * \gamma] = [x_0]$,
- The *fundamental group* is, the group of classes of loops $[\gamma]$ with the concatenation as group operation.

The Fundamental Group π_1

In short:

$$\pi_1(X, x_0) = [(S^1, *), (X, x_0)] \cong \{\text{loops up to based homotopy}\}.$$

This is a topological invariant (also, a "weak" topology invariant: if X and Y are homotopy equivalent, $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic groups)

(this can also be defined directly on simplicial sets)

Basic examples: S^1 and D^2 (filled in circle or disc)

$\pi_1(S^1) \cong \mathbb{Z}$ (winding number) but $\pi_1(D^2) = 0$.

π_0 does not distinguish S^1 from D^2 but the fundamental group does.

Van Kampen and Computations

Homotopy and products

$\pi_n(X \times Y)$ is isomorphic to $\pi_n(X) \times \pi_n(Y)$ for all $n \geq 0$.

van Kampen (Seifert-van Kampen):

Computes π_1 from an open cover with path-connected overlaps: e.g. U, V open cover of X such that U, V and $U \cap V$ are path-connected, induce:

$$\pi_1(U \cap V, x_0) \xrightarrow{(i_U)_*} \pi_1(U, x_0), \quad \pi_1(U \cap V, x_0) \xrightarrow{(i_V)_*} \pi_1(V, x_0)$$

and $\pi_1(X, x_0)$ as the pushout (amalgamated free product)

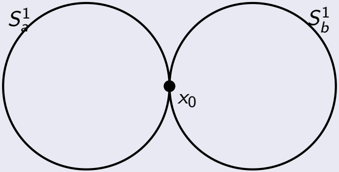
$$\pi_1(X, x_0) \cong \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0).$$

In words: $\pi_1(X)$ is obtained from the free product of $\pi_1(U)$ and $\pi_1(V)$ by identifying the images of $\pi_1(U \cap V)$.

There is no van Kampen for higher homotopy groups (or in some very specific contexts).

Application of van Kampen: Wedge of Two Circles

Let $X = S^1 \vee S^1$ be the wedge ("figure-eight"), obtained by joining two circles at a basepoint x_0.

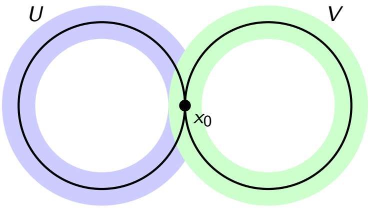


We will compute $\pi_1(X, x_0)$ via Seifert–van Kampen.

Choose an Open Cover $U \cup V$

Pick open sets U, V as follows:

- U is an open neighborhood of the left circle and the basepoint (deformation retracts to the left circle S^1_a).
- V is an open neighborhood of the right circle and the basepoint (deformation retracts to the right circle S^1_b).
- The intersection $U \cap V$ is a small neighborhood of the basepoint and is contractible (deformation retracts to $\{x_0\}$).



$U \cap V$ (contractible)

Deformation Retracts and Induced Groups

Under deformation retraction:

$$U \simeq S^1_a, \qquad V \simeq S^1_b, \qquad U \cap V \simeq \{x_0\}.$$

Therefore their fundamental groups (based at x_0) are:

$$\pi_1(U, x_0) \cong \pi_1(S^1_a, x_0) \cong \mathbb{Z} = \langle a \rangle,$$
$$\pi_1(V, x_0) \cong \pi_1(S^1_b, x_0) \cong \mathbb{Z} = \langle b \rangle,$$
$$\pi_1(U \cap V, x_0) \cong \pi_1(\{x_0\}) = 0.$$

Here a and b denote the homotopy classes of the standard loops going around the left and right circles respectively.

Fundamental group of the pieces

Apply van Kampen

We have the diagram of inclusion-induced maps:

$$0 = \pi_1(U \cap V) \xrightarrow{(i_U)_*} \pi_1(U) \quad \text{and} \quad 0 = \pi_1(U \cap V) \xrightarrow{(i_V)_*} \pi_1(V).$$

The van Kampen theorem states that

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0).$$

Since the amalgamating group is trivial, the amalgamated free product reduces to the free product:

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0) \cong \mathbb{Z} * \mathbb{Z}.$$

In generator notation: $\pi_1(X) \cong \langle a, b \mid \ \rangle$, the free group on two generators.

Back to the classification theorem!

Reminder: "combinatorial" proof of the theorem

We used a "normal form" for the boundary of the polygon that "presented" all compact surfaces without boundaries:

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \quad \text{(orientable - sum of } g \text{ torii)}$$

or

$$a_1^2 a_2^2 \cdots a_c^2 \quad \text{(nonorientable - sum of } c \text{ projective planes).}$$

This encodes directly the fundamental group of these surfaces!

- Generators are the a_i's and b_i's
- There is one relation which is given by the normal form above!

Examples/exercises

The fundamental group of a torus

The fundamental group of the projective plane

Examples/exercises

The fundamental group of a torus

Two generators a, b and relation $aba^{-1}b^{-1} = 1$ i.e. they commute: $\pi_1(\mathbf{T}) = \mathbb{Z}^2$.

The fundamental group of the projective plane

Examples/exercises

The fundamental group of a torus

The fundamental group of the projective plane

One generator a and relation $aa = 1$, i.e. $\pi_1(\mathbf{P}) = \mathbb{Z}_2$.

Higher Homotopy Groups π_n

Definition - intuition

- For $n \geq 1$, define
$$\pi_n(X, x_0) = [(S^n, *), (X, x_0)],$$
i.e. based homotopy classes of maps $S^n \to X$.
- Product on π_n is given by concatenation along a chosen equator decomposition of S^n.

(this can also be defined directly on simplicial sets - introduced later)

More formally

- Maps $f : S^n \to X$ that send some base point $*$ of S^n to x_0 are "the same" as maps $[0,1]^n \to X$ that send the boundary of the n-cube $[0,1]^n$ to x_0,
- Concatenation of such maps can be defined by:

$$f * g(t_1, t_2, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & t_1 \in \left[0, \frac{1}{2}\right] \\ g(2t_1 - 1, t_2, \ldots, t_n) & t_1 \in \left[\frac{1}{2}, 1\right] \end{cases}$$

- $\pi_n(X)$ is the group of classes $[f]$ of such maps modulo homotopy, with $*$ as group operation.

(does not depend on the choice of the base point)

Functoriality and Change of Basepoint

A continuous map $f : (X, x_0) \to (Y, y_0)$ induces

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0), \qquad [\alpha] \mapsto [f \circ \alpha].$$

For path-connected X, basepoint change gives canonical isomorphisms of π_n (well-defined up to inner automorphism for $n = 1$, canonical for $n \geq 2$).

Higher-homotopy groups

There again, $\pi_n(X)$ is a topological invariant (and a weak topological invariant).

Simple examples

- $\pi_0(X)$: path-components.
- $\pi_1(S^1) \cong \mathbb{Z}$.
- $\pi_n(S^n) \cong \mathbb{Z}$ (degree of map), generated by identity.
- $\pi_k(S^n) = 0$ for $k < n$.
- $\pi_2(S^1) = 0$; $\pi_2(S^2) \cong \mathbb{Z}$.

For $n \geq 2$ the group $\pi_n(X, x_0)$ is abelian (commutativity follows from a homotopy that swaps the order - the Eckmann-Hilton argument).

Harder...

$\pi_3(S^2) = \mathbb{Z}$ (generated by the class of the Hopf fibration)

All this is nice, but...

None of this is computable! (apart from π_0 in general)

Because of the undecidability of the word problems in groups.

Still, some can be done

Using exact sequences, when spaces can be decomposed in a way or another. Not trivial even for spheres (not all homotopy groups are known for spheres!)

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

We will use mostly a computable version of homotopy (an abelianization in some way): homology and cohomology, but still useful to know a bit on homotopy groups.

Example of an exact sequence in homotopy

Relative homotopy groups

For a pair (X, A) with basepoint $x_0 \in A$, define

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}), (X, A)]$$

classes of maps of the disk sending boundary into A.

Relative homotopy exact sequence

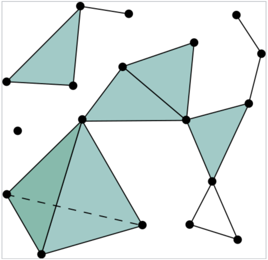
There is a long exact sequence of homotopy groups:

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots$$

Homotopy is not algorithmic, continuous systems are too hard - we go on with discretizing spaces, and get simpler topological invariants (homology).

Discrete analogues of topological spaces

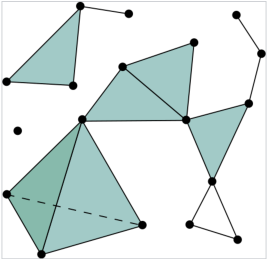
Two generalizations of graphs (1D only) more directly related to topological spaces:



(Abstract) Simplicial complexes

A simplicial complex K is a pair (S, Σ) composed of a set S of points, and Σ is a set of subsets of S such that Σ is downwards closed, i.e. every subset of Σ is in Σ.

Discrete analogues of topological spaces



Σ is the set of simplices, the dimension of a simplex $\sigma \in \Sigma$ is the cardinality of σ minus 1.

Morphisms

Simplicial morphisms $f: K = (S, \Sigma) \rightarrow L = (T, \Theta)$ are just set-theoretic maps from S to T such that it induces maps from Σ to Θ, i.e. the set-extension of f preserves simplices (of any dimension).

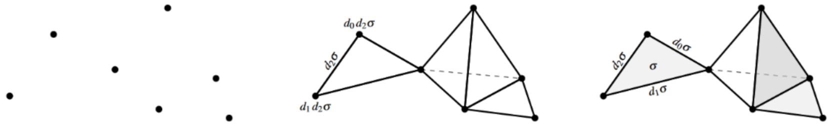
Note that π_0 can be defined directly (combinatorially) on simplicial complexes (exercice!).

Discrete analogues of topological spaces

Simplicial sets

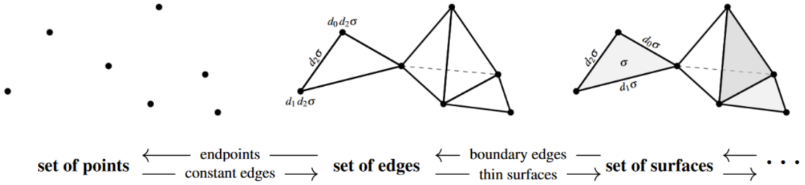
A simplicial set X is a collection of sets X_n, $n = 0, 1, 2, \ldots$, together with maps:

- $d_{n,i} : \ X_n \to X_{n-1}$ for $n = 0, 1, 2, \ldots$ and $0 \leq i \leq n$ (the face maps),
- $s_{n,i} : \ X_n \to X_{n+1}$ for $n = 0, 1, 2, \ldots$ and $0 \leq i \leq n$ (the degeneracy maps),
- Satisfying (we drop the index n when it is cumbersome):
  - $d_i d_j = d_{j-1} d_i$ if $i < j$,
  - $d_i s_j = s_{j-1} d_i$ if $i < j$,
  - $d_i s_j = Id$ if $i = j$ or $i = j + 1$,
  - $d_i s_j = s_j d_{i-1}$ if $i > j + 1$,
  - $s_i s_j = s_{j+1} s_i$ is $i \leq j$



Having only the face maps defines what is called a semi-simplicial set.

Discrete analogues of topological spaces



Face maps enumerate the face of dimension just immediately lower, and degeneracy maps identify $(n-1)$-simplices as "degenerate" or "thin" n-simplices.

Morphisms

A morphism $f : X \to Y$ of simplicial sets is a collection of set-theoretic maps $f_n : X_n \to Y_n$ such that they commute with the face and degeneracy operators.

Note that π_0 can be defined directly (combinatorially) on simplicial complexes (exercice!).

Alternate definition of simplicial sets - via category theory (for the aficionados)

Can be defined as a presheaf category over a site:

The simplex category Δ

- Objects are finite ordinals: $[n] = \{0, 1, \ldots, n\}$ (with the standard total order),
- Morphisms are order-preserving functions (increasing, but not necessarily strictly increasing).

The presheaf category $\Delta^{op} Set$

This is the category of functors from the opposite category Δ^{op} to the category *Set* of sets (with functions as morphisms).

As such it is a topos, and enjoys a wealth of interesting properties.

Exercise: show that this definition agrees with the previous one!

Relationships between all these

Simplicial complexes wrt simplicial sets?

None is more general than the other, but they are almost alike:

- Simplicial complexes form a quasi-topos, not quite a topos, so it will be a bit less practical "category-theoretically",
- Two simplices can only connected common subfaces in simplicial complexes, this is not the case for simplicial sets.

Using one or the other really depends on applications (we will use both):

- for combinatorial applications, people tend to use simplicial complexes (e.g. nerve lemma, that we will use later),
- for deeper links to topology, and category theory, people tend to use simplicial sets.

Note that for all compact surfaces without boundary, we could triangulate them! (i.e. find a "concrete simplicial complex" version of it) But not possible more generally, we need more!

Relationships between discrete "versions" of topological spaces, and topological space

Let Δ_n be the "standard n-simplex" in $\mathbb{R}^{n+1}$:

$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq 1, \sum_{i=0}^{n} x_i = 1\}$$

with maps:

$$\begin{aligned} \delta_{n+1,i} : \Delta_n &\to \Delta_{n+1} \\ (x_0, \ldots, x_n) &\longmapsto (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \end{aligned}$$

Geometric realisation of simplicial sets X (in *Top*, the category of topological spaces)

- Each abstract n-simplex in X_n will be "represented" by a copy of Δ_n, $(x, \Delta_n) = \{(x, y) \mid y \in \Delta_n\}$,
- We define an equivalence relation between all these copies:

$$(x, \delta_{n,i}(y)) \equiv (d_{n,i}(x), y) \text{ for all } y \in \Delta_{n-1}$$

The geometric realization of X is $\left(\coprod_{n \in \mathbb{N}, x \in X_n} (x, \Delta_n) \right) / \equiv$

Relationships between discrete "versions" of topological spaces, and topological space

Using category theory:

Geometric realisation of simplicial sets X (in *Top*, the category of topological spaces)

- Define a functor $F: \Delta \rightarrow Top$ with $F([n]) = \Delta_n$, $F(d_{n,i}) = \delta_{n,i}$ (and $F(s_{n,j}) = \sigma_{n,j} : \Delta_{n+1} \rightarrow \Delta_n$ with $\sigma_{n,j}(x_0, \ldots, x_{n+1}) = (x_0, \ldots, x_{j-1}, x_j + x_{j+1}, x_{j+2}, \ldots, x_n)$,
- The Kan extension of F along the Yoneda functor $\Delta \rightarrow \Delta^{op}Set$ (or Yoneda extension) gives the unique functor $|.|: \Delta^{op}Set \rightarrow \Delta^{op}Set$ such that:
  - $|[n]| = \Delta_n$,
  - $|.|$ commutes with all colimits.

Conversely

Going from topological spaces X to simplicial sets

Associate to each X, the simplicial set $S(X)$ with:

- $S(X)_n = Hom_{Top}(\Delta_n, X)$,
- $d_{n,i}(f) = f \circ \delta_{n,i}$ with $f : \ \Delta_n \rightarrow X$,
- $s_{n,i}(f) = f \circ \sigma_{n,i}$ with $f : \ \Delta_n \rightarrow X$.

$S(X)$ is called the singular set of the topological space X.

This is the only general enough way to represent combinatorially all topological spaces. Triangulations, using simplicial complexes, will not work for all.

Conversely (categorical version)

Going from topological spaces X to simplicial sets

S is actually a functor, which is right-adjoint to the geometric realization functor.

Exercise: show this!

Simplicial approximation theorem

Intuitively

- Continuous maps can be (by a slight deformation) approximated by ones that are piecewise linear (if from the geometric realization of a finite simplicial complex).
- There is a further simplicial approximation theorem for homotopies, stating that a homotopy between continuous maps can likewise be approximated by a combinatorial version.

More formally

Let K, L be two simplicial complexes. If $f : |K| \to |L|$ is a continuous map, then there are a subdivision K' of K and a simplicial mapping $s : K' \to L$ whose geometric realization $|s|$ is homotopic to f.

Let us use these ideas and try to discretize our continuous flow! (as we only need indexes "up to deformation" to determine properties we need)

But before, let us also get to simpler (computable!) topological invariants, than homotopy groups: homology!

Singular Simplexes

Reminder: Δ_n be the "standard n-simplex" in $\mathbb{R}^{n+1}$:

$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq 1, \sum_{i=0}^{n} x_i = 1\}$$

with maps:

$$\begin{aligned} \delta_{n+1,i} : \Delta_n &\rightarrow \Delta_{n+1} \\ (x_0, \ldots, x_n) &\longmapsto (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n) \end{aligned}$$

Reminder: Singular simplexes

A *singular n-simplex* in a space X is a continuous map

$$\sigma : \Delta^n \rightarrow X,$$

Constructing formal sums of singular simplexes

The free abelian group on all singular n-simplices is the group

$$C_n(X) = \mathbb{Z}\langle \text{Singular } n\text{-simplices of } X \rangle.$$

Chain complex

Boundary operator

For a singular simplex $\sigma : \Delta^n \to X$, define the boundary:

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \partial_i \sigma$$

and extend linearly.

Chain complex

This gives a chain complex

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

and satisfies $\partial_n \circ \partial_{n+1} = 0$.

Singular Homology

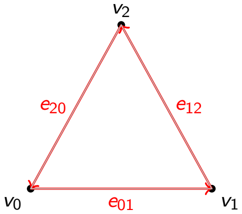
The *n*-th homology group is

$$H_n(X) = \ker(\partial_n) / \operatorname{im}(\partial_{n+1}).$$

Intuitively:

- n-cycles $= \ker \partial_n$ represent n-dimensional holes.
- Boundaries $= \operatorname{im} \partial_{n+1}$ are trivial cycles.

Example: a 1-cycle in a simplicial set



1-cycle

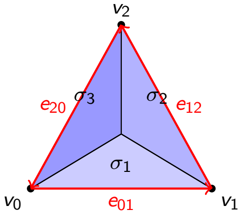
- 1-chain: $z = e_{01} + e_{12} + e_{20}$
- Boundary computation:

$$\partial_1(e_{01}) = v_1 - v_0$$
$$\partial_1(e_{12}) = v_2 - v_1$$
$$\partial_1(e_{20}) = v_0 - v_2$$

$$\partial_1(z) = (v_1 - v_0) + (v_2 - v_1) + (v_0 - v_2) = 0$$

z is a 1-cycle.

Example: 1-cycle as a boundary of 2-simplices



1-cycle as a boundary:

- 2-chain: $c = \sigma_1 + \sigma_2 + \sigma_3$
- Boundary computation:

$$\partial_2(c) = \partial_2(\sigma_1) + \partial_2(\sigma_2) + \partial_2(\sigma_3)$$

Interior edges cancel pairwise, leaving:

$$\partial_2(c) = e_{01} + e_{12} + e_{20} = z$$

z is a boundary, hence null-homologous (and null-homotopic).

Basic Properties

Hurewitz!

- For $n = 1$, isomorphism $\pi_1(X)/[\pi_1(X), \pi_1(X)] \to H_1(X)$,
- For $n \geq 2$, if X is $(n-1)-$*connected* (i.e. $\pi_i(X) = 0$ for all $i < n$), then $H_i(X) = 0$ for all $i < n$, and $\pi_n(X) \to H_n(X)$ is an isomorphism.

- Homotopy invariance: If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$.
- Excision: Removing certain subspaces does not change homology.
- Mayer–Vietoris sequence: A long exact sequence relating $H_*(A)$, $H_*(B)$, and $H_*(A \cup B)$.

In fact, homology theories satisfy the Eilenberg-Steenrod axioms, and all theories satisfying these axioms are in fact “representable” by a given spectrum of spaces.

Simplicial homology

Similarly, on a simplicial set/complex $((X_n)_n, (d_{n,i})_{n,i}, (s_{n,i})_{n,i})$

- Boundary operator For a singular simplex $\sigma : \Delta^n \to X$, define the boundary:

$$\partial_n(x) = \sum_{i=0}^{n} (-1)^i \, \mathbf{d}_{n,i}(x)$$

  and extend linearly.

- Chain complex This gives a chain complex

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

  and satisfies $\partial_n \circ \partial_{n+1} = 0$.

- The n-th homology group is

$$H_n(X) = \ker(\partial_n) / \operatorname{im}(\partial_{n+1}).$$

Feasible also for any CW-model (in the sequel, cubical complexes or other base shapes)

Simplicial homology: direct calculation

Goal: compute homology groups of cell or simplicial complexes

Given a chain complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

with $C_k \cong \mathbb{Z}^{r_k}$,

$$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Integer matrices for ∂_n allow the use of Smith Normal Form (SNF) to compute these homology groups.

Smith Normal Form: Definition

Let A be an $m \times n$ integer matrix.

Smith Normal Form (SNF)

There exist invertible integer matrices $P \in GL_m(\mathbb{Z})$, $Q \in GL_n(\mathbb{Z})$ such that

$$PAQ = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_r \end{pmatrix},$$

with $d_i > 0$ and $d_i \mid d_{i+1}$.

The diagonal matrix is the SNF of A. It describes the structure of $\mathbb{Z}^n / \operatorname{im}(A)$.

Interpretation for Homology

If $\partial : \mathbb{Z}^n \to \mathbb{Z}^m$ has SNF $\mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$, then

$$\mathbb{Z}^m / \operatorname{im}(\partial) \cong \mathbb{Z}^{m-r} \oplus \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_r.$$

Thus the SNF directly gives:

- the number of free $\mathbb{Z}$-summands,
- the torsion coefficients $\mathbb{Z}/d_i$.

(recall the fundamental theorem of finitely generated abelian groups, G is the sum of copies of $\mathbb{Z}$ and of $\mathbb{Z}/p^k\mathbb{Z}$ where p is prime)

Example: Smith Normal Form

Compute SNF of:

$$A = \begin{pmatrix} 4 & 6 \\ 2 & 8 \end{pmatrix}.$$

Step 1: Row operation $R_1 \mapsto R_1 - 2R_2$

$$\begin{pmatrix} 0 & -10 \\ 2 & 8 \end{pmatrix}.$$

Step 2: Swap rows:

$$\begin{pmatrix} 2 & 8 \\ 0 & -10 \end{pmatrix}.$$

Example: Smith Normal Form

Step 3: Column operation $C_2 \mapsto C_2 - 4C_1$:

$$\begin{pmatrix} 2 & 0 \\ 0 & -10 \end{pmatrix}.$$

Step 4: Fix signs and order:

$$\mathrm{SNF}(A) = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}.$$

Homology from SNF

Given a boundary matrix ∂_n:

Compute

$$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

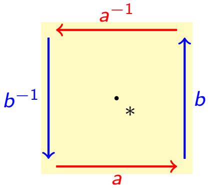
Compute:

- SNF of ∂_{n+1} to find $\operatorname{im} \partial_{n+1}$.
- SNF of ∂_n to find $\ker \partial_n$.
- Take quotient using SNFs to obtain H_n.

Example/exercise: homology of the torus

CW-model:

one 0-cell ($*$), two 1-cells (a red, b blue), one 2-cell (shaded yellow) attached along $aba^{-1}b^{-1}$.



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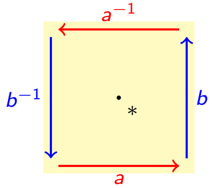


Homology calculation

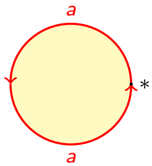
- Chain groups: $C_2 = \mathbb{Z}, C_1 = \mathbb{Z}^2, C_0 = \mathbb{Z}$
- Boundary maps: $\partial_2 = 0, \partial_1 = 0$
- Homology:

$$H_2 = \mathbb{Z}, \quad H_1 = \mathbb{Z}^2, \quad H_0 = \mathbb{Z}.$$

Example/exercise: Real Projective Plane

CW-model:

1 zero-cell, 1 one-cell a (red), 1 two-cell (shaded yellow) attached along a^2.



Example/exercise: Real Projective Plane

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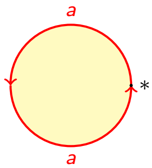


Homology calculation:

- Chain complex:

$$C_2 = \mathbb{Z} \xrightarrow{(2)} C_1 = \mathbb{Z} \xrightarrow{0} C_0 = \mathbb{Z}$$

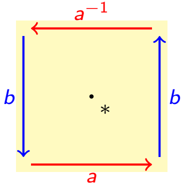
- SNF gives:

$$H_2 = 0, \quad H_1 = \mathbb{Z}/2, \quad H_0 = \mathbb{Z}.$$

Example/exercise: Klein Bottle

CW-model:

one 0-cell, two 1-cells a (red), b (blue), one 2-cell (yellow) attached along $aba^{-1}b$.



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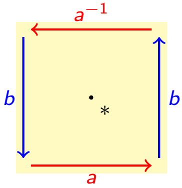

Homology calculation:

- Chain complex:

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0, \quad \partial_2(1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

- SNF computation:

$$H_2 = 0, \quad H_1 \cong \mathbb{Z} \oplus \mathbb{Z}/2, \quad H_0 = \mathbb{Z}.$$

Example/exercise: Wedge of Circles or Spheres

Wedge of 2 circles (figure eight):

$$C_1 = \mathbb{Z}^2, \quad C_0 = \mathbb{Z}, \quad \partial_1 = 0$$

$\Rightarrow H_1 = \mathbb{Z}^2, H_0 = \mathbb{Z}$.

Wedge of k spheres S^n:

$$C_n = \mathbb{Z}^k, \quad C_0 = \mathbb{Z}, \quad \text{other chains } 0$$

$\Rightarrow H_n = \mathbb{Z}^k, H_0 = \mathbb{Z}, H_i = 0$ for $i \neq 0, n$.

Discrete version of the Conley index

Discrete dynamical system

$x_{n+1} = f(x_n)$ starting from $x_0 \in X_0$. The flow of this system is the map $\varphi : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(k, x_n) = x_{n+k}$ for all $n \in \mathbb{N}$ and $-n \leq k \in \mathbb{Z}$.

Isolating neighbood and isolated invariant set for a discrete dynamical system

- N is an isolating neighborhood if it is compact and $inv(N, \varphi) \subseteq int(N)$,
- In such a case, $S = inv(N, \varphi)$ is called an isolated invariant set: "N isolates" S.

Discrete index pair

The pair of compact sets (P_1, P_2) such that $P_2 \subseteq P_1$ is called an index pair iff:

- $f(P_1 \backslash P_2) \subseteq P_1$,
- $f(P_2) \cap P_1 \subseteq P_2$,
- $inv(cl(P_1 \backslash P_2), f) \subseteq int(P_1 \backslash P_2)$.

Discretization of a continuous flow

Time-t discretization of the continuous flow φ

Is the discrete dynamical system defined as:

$$\varphi_t : \ (x, \tau) \in \mathbb{R}^n \times \mathbb{Z} \rightarrow \varphi(x, \tau t) \in \mathbb{R}^n$$

Relation between the discrete, and the continuous system

For a flow φ in $\mathbb{R}^n$, and $S \subseteq \mathbb{R}^n$, the following three conditions are equivalent:

- S is an isolated invariant set with respect to φ,
- S is an isolated invariant set with respect to φ_t for all $t > 0$,
- S is an isolated invariant set with respect to φ_t for some $t > 0$.

Proof

Relation between the discrete, and the continuous system

For a flow φ in $\mathbb{R}^n$, and $S \subseteq \mathbb{R}^n$, the following three conditions are equivalent:

- (1) S is an isolated invariant set with respect to φ,
- (2) S is an isolated invariant set with respect to φ_t for all $t > 0$,
- (3) S is an isolated invariant set with respect to φ_t for some $t > 0$.

Proof

- (1)⇒(2)⇒(3) very easy,
- Let us prove (3)⇒(1).

Proof

Relation between the discrete, and the continuous system

For a flow φ in $\mathbb{R}^n$, and $S \subseteq \mathbb{R}^n$, the following three conditions are equivalent:

- (1) S is an isolated invariant set with respect to φ,
- (2) S is an isolated invariant set with respect to φ_t for all $t > 0$,
- (3) S is an isolated invariant set with respect to φ_t for some $t > 0$.

Proof

- Suppose $S = inv(N, \phi_t)$ $t > 0$ fixed, N isolating neighborhood; consider $x \in S$,
- Discrete trajectory $(\phi_t(x, n))_{n \in \mathbb{Z}}$ in S extended to continuous $(\phi(x, s))_{s \in \mathbb{R}}$ in $\mathbb{R}^n$,
- Compactness of $S + S \subseteq int(N)$ there is an $\epsilon > 0$ s.t. $P : \phi(S, [0, \epsilon]) \subseteq int(N)$,
- Consider $t_0 = sup\{s \in [0, t] \mid \forall i \in \mathbb{Z}, \phi_t(S, i) \subseteq S\}$, prove $t_0 = t$ by reductio ad absurdum, $t_0 < t$...

Proof

Proof

- Suppose $S = inv(N, \phi_t)$ $t > 0$ fixed, N isolating neighborhood; consider $x \in S$,
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- Consider $t_0 = sup\{s \in [0, t] \mid \forall i \in \mathbb{Z}, \phi_t(S, i) \subseteq S\}$, prove $t_0 = t$ by reductio ad absurdum, $t_0 < t$...
- Continuity of φ: $\phi(S, it + t_0) \subseteq S$, $\forall i \in \mathbb{Z}$. (P) implies $\phi(S, it + t_0 + \epsilon) \subseteq int(N) \subseteq N$ contradiction,
- Finally, easy to see S is the invariant part of N wrt φ.

Discrete versus Continuous

Isolating neighborhoods

Let N be an isolating neighborhood with respect to a discretization φ_t of the flow φ. Then N is an isolating neighborhood with respect to the flow φ and $inv(N, \varphi_t) = inv(N, \varphi)$.

Theorem

The Conley index of an isolated invariant set with respect to a flow φ coincides with the corresponding index with respect to the discrete dynamical system φ_t for any $t > 0$.

First proof

Main property wrt isolating neighborhoods and isolated invariant sets

Let N be an isolating neighborhood with respect to the discrete dynamical system φ_t. Then N is an isolating neighborhood wrt the continuous dynamical system φ and the corresponding isolated invariant sets are equal: $inv(N, \varphi_t) = inv(N, \varphi)$.

Proof

- $S = inv(N, \varphi_t)$ is an isolated invariant set wrt φ_t, thus, we know now, wrt φ,
- Since $inv(N, \varphi)$ is the maximal invariant set wrt φ contained in N, $S \subseteq inv(N, \varphi)$,
- Also, $inv(N, \varphi) = \bigcap\limits_{s \in \mathbb{R}} \varphi_s(N) \subseteq \bigcap\limits_{s \in t\mathbb{Z}} \varphi_s(N) = inv(N, \varphi_t)$, so $inv(N, \varphi) \subseteq S$ and $inv(N, \varphi_t) = inv(N, \varphi)$,
- Since $inv(N, \varphi_t) \subseteq int(N)$, $inv(N, \varphi) \subseteq int(N)$: N is an isolating neighbourhood wrt φ.

Back to reachability

Cubical sets and combinatorial maps

- Simple gridding of $\mathbb{R}^n$:
  $\mathcal{H} = \{[d_1 k_1, d_1(k_1+1)] \times \ldots \times [d_n k_n, d_n(k_n+1)] \mid k_i \in \mathbb{Z}\}$, elements of which are hypercubes,
- $|A|$ is a cubical set when it denotes the union of the (finitely-many) elements of $\mathcal{A} \subseteq \mathcal{H}$,
- If $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{H}$ are finite then any multivalued map $\mathcal{F} : \mathcal{X} \multimap \mathcal{Y}$ is called a combinatorial map.

Representation of a continuous map by a combinatorial map on a grid

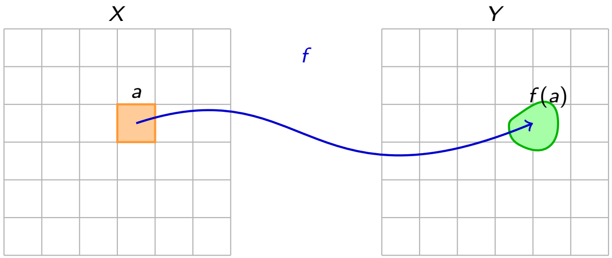
Let $f : |\mathcal{X}| \to |\mathcal{Y}|$ be a continuous map. $\mathcal{F} : \mathcal{X} \multimap \mathcal{Y}$ is called a representation of f iff for all $Q \in \mathcal{X}$:

$$f(Q) \subseteq int(|\mathcal{F}(Q)|)$$

In the case of a cubical grid, $\mathcal{F}$ is called a cubical enclosure of f.

Similar idea for practical controller synthesis from e.g. Pola, Girard, Tabuada, "Approximately bisimilar symbolic models for nonlinear control systems", 2008

Representation of a continous map by a combinatorial map on a grid



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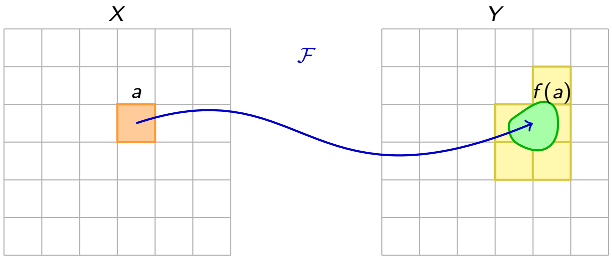


Combinatorial maps

Almost perfect combinatorial maps

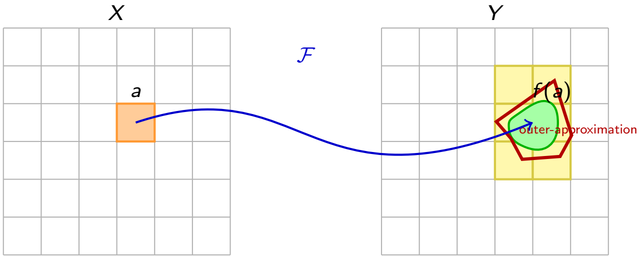
- A combinatorial map $\mathcal{F}: A \to 2^B$ is a finite combinatorial map if A is finite (as a subset of the grid $\mathcal{H}$) and if for all $a \in A$, $\mathcal{F}(a)$ is finite.
- Furthermore, $\mathcal{F}$ is an almost perfect finite combinatorial map if also, for all $a \in A$, $|\mathcal{F}(a)|$ is convex.

Using reachability methods for computing the cubical enclosure

Principle

- Compute an outer-approximation $F(a)$ of $f(a)$, for every $a \in A$,
- Define $\mathcal{F}(a) = \bigcup_{b \in B,\ F(a) \cap b \neq \emptyset} \{b\}$.

Even more: take the minimal convex cubical set containing $\mathcal{F}(a)$ and we get an almost perfect finite combinatorial map.



Finally: index pairs for representations of a flow, wrt to index pairs of the flow

Index pair for a combinatorial map

Let $\mathcal{F}$ be a combinatorial map. The pair (P_1, P_2) of finite subsets of $\mathcal{H}$ such that $P_2 \subseteq P_1$ is called a combinatorial index pair if the following properties hold true:

- $\mathcal{F}(P_1 \backslash P_2) \subseteq P_1$,
- $\mathcal{F}(P_2) \cap P_1 \subseteq P_2$.

Relationship with index pairs of a flow

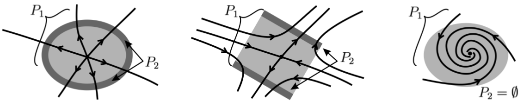
If $\mathcal{F}$ is a representation of f for some discrete dynamical system f, and (P_1, P_2) a combinatorial index pair for $\mathcal{F}$, then $(|P_1|, |P_2|)$ is an index pair for f (or φ where [reminder] $\varphi(x, n) : f^n(x)$).

Why? reminder: Conley index

Index pair (continuous case)

Let φ be a flow on $\mathbb{R}^n$. The pair of compact sets (P_1, P_2) such that $P_2 \subseteq P_1$ is called an index pair if the following properties hold true:

- If $x \in P_1$ and $\varphi(x, s) \notin P_1$ for some $s > 0$ then there exists $t \geq 0$ such that $\varphi(x, [0, t]) \subseteq P_1$ and $\varphi(x, t) \in P_2$,
- if $x \in P_2$ and $\varphi(x, [0, t]) \subseteq P_1$ then also $\varphi(x, [0, t]) \subseteq P_2$,
- $inv(cl(P_1 \backslash P_2), \varphi) \subseteq int(P_1 \backslash P_2)$.



(Homotopy) Conley index of an isolating neighborhood

The Conley index of the index pair (P_1, P_2) is the homotopy type of the pointed quotient space $(P_1/P_2, [P_2])$ (equivalently of the relative pair (P_1, P_2)).
This index is independent of the choice of the index pair for the isolated invariant set $S = inv(cl(P_1 \backslash P_2), \varphi)$.

What is next?

We can:

- Try to determine an index pair (P_1, P_2) for the cubical enclosure $\mathcal{F}$
- Compute the homology of the pair, by a simple variation of the Smith Normal Form

For this, using differential methods and set-based methods see e.g. Fribourg, Eric Goubault, Mohamed, Mrozek, Putot: A topological method for finding invariant sets of continuous systems. Inf. Comput. 2021 - more later.

And in some cases even:

- We can only deal with the Lyapunov case, $P_2 = \emptyset$ here!
- But refine our knowledge of the invariant by calculating the homology of $\mathcal{F}$: application to finding periodic orbits!

For another method for periodic orbits, using only reachability and Poincaré sections, see e.g. Bourgois, Chaabouni, Rauh and Jaulin, “Proving the stability of navigation cycles”, and Brateau, Degorre, Le Bars, Jaulin, “Proving the stability of cycle navigation using capture sets”, Mechatronics 2025.

Combinatorial Morse decomposition of a combinatorial map

Graph of a multivalued map $\mathcal{F}$

It is the graph $\mathcal{G}(\mathcal{F})$ with:

- Vertices are cubes in the grid,
- We have an arrow from cube c to cube c' iff $c' \in \mathcal{F}(c)$.

The Morse graph of a multivalued map $\mathcal{F}$

This is $\mathcal{M}(\mathcal{F})$, the condensation of $\mathcal{G}(\mathcal{F})$, i.e. vertices are SCCs in $\mathcal{G}(\mathcal{F})$ and arrows are reachability conditions between SCCs.

We denote by $|v|$ the union of the $|v'|$, the geometric realization of each cube corresponding to the vertex $v' \in \mathcal{G}(\mathcal{F})$ in the SCC v.

First method

Theorem 1

Let f be a discrete dynamical system, $\mathcal{F}$ be a cubical enclosure, and $\mathcal{M}(\mathcal{F})$ be its combinatorial Morse decomposition.
Then for all v vertex of $\mathcal{M}(\mathcal{F})$, $|v|$ is an isolating neighborhood for f.

Theorem 2

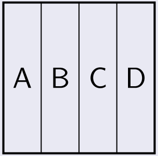
Let the pair (N, L) be defined by $N = |\mathcal{F}(v)|$ and $L = |\mathcal{F}(v)| \backslash |v|$
Then for all (N, L) is an index pair for f.

Example

Consider the discrete dynamical system on the plane:

$f(x, y) = (2x, \frac{y}{2})$

Consider the following grid on $[-1, 1] \times [-1, 1]$

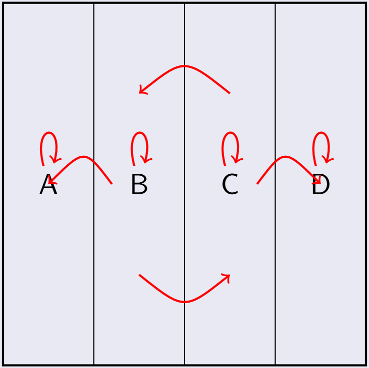


Example

Consider the discrete dynamical system on the plane:

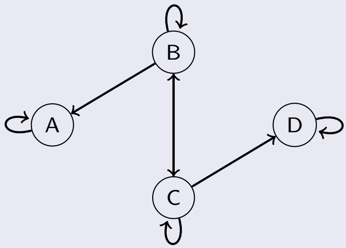
$f(x, y) = (2x, \frac{y}{2})$

The graph of the induced cubical enclosure:

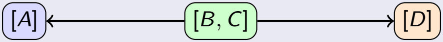


Example

Graph of the cubical enclosure



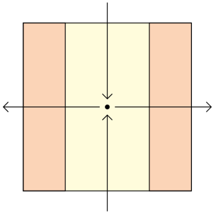
Morse decomposition



Consider the isolating neighborhood $[B, C]$, the corresponding index pair is (N, L) with $N = |A| \cup |B| \cup |C| \cup |D|$ and $L = |A| \cup |D|$.

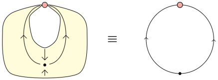
Example

Consider the isolating neighborhood $[B, C]$, the corresponding index pair is (N, L) with $N = |A| \cup |B| \cup |C| \cup |D|$ and $L = |A| \cup |D|$.



Computation of the Conley index

Quotient space is homotope to a circle! $CH_0(N) = \mathbb{Z}$, $CH_1(N) = \mathbb{Z}$ and $CH_i(N) = 0$ for all $i \geq 2$.



Proves the existence of a non-empty invariant set (here, a point!).

Cubical homology

The homological Conley index can be directly computed on the cubical grid!

Reminder: gridding of $\mathbb{R}^n$

$\mathcal{H} = \{[d_1 k_1, d_1(k_1+1)] \times \ldots \times [d_n k_n, d_n(k_n+1)] \mid k_i \in \mathbb{Z}\}$

Decomposition as a "cellular complex"

Each $e = (d_1 k_1, d_1(k_1+1)) \times \ldots \times (d_n k_n, d_n(k_n+1))$ in $\mathcal{H}$ is a n-dimension cell (denoted by $\mathcal{H}_n$) of a "cellular complex", i.e. a combinatorial complex, similar to simplicial complexes, but with cubic shapes.

Front and back faces in a cubical set

For $e = (d_1 k_1, d_1(k_1+1)) \times \ldots \times (d_n k_n, d_n(k_n+1))$ any n cell of $\mathcal{H}$, we can produce $n-1$ cells (denoted by $\mathcal{H}_{n-1}$) as their boundaries, as follows:

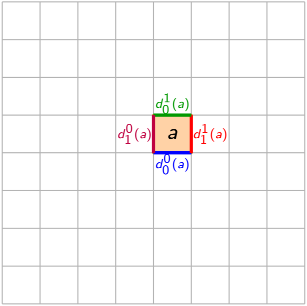
- Front i face of e in a cubical set $\mathcal{H}$ is $A_i e = \{(x_1, \ldots, x_{i-1}, d_i k_i, x_{i+1}, \ldots, x_n) \mid x \in e\}$
- Back i face of e: $B_i e = \{(x_1, \ldots, x_{i-1}, d_i(k_i+1), x_{i+1}, \ldots, x_n) \mid x \in e\}$.

Cubical homology

And this can be iterated more!

- The same boundary construction holds for any k cell e (denoted by $\mathcal{H}_k$) in $\mathcal{H}$ producing $(k-1)$ cells (denoted by $\mathcal{H}_{k-1}$).
- When e' is one of the $A_i e$ faces, we write, combinatorially, $d_i^0(e) = e'$, and when it is one of $B_i e$ faces, we write $d_i^1(e) = e'$. We have a "simplicial-like" relation:

$$d_i^k d_j^l = d_{j-1}^l d_i^k \text{ for } 0 \leq i < j \leq n,\ k = 0, 1,\ l = 0, 1.$$



Homology of a cubical set

Given the cell structure of any C subcomplex of $\mathcal{H}$

As with simplicial complexes,

- Construct $\mathbf{C}_k$ the $\mathbb{Z}$-module generated by the k cells in C_k, for all $0 \leq k \leq n$,
- Construct the boundary operator $\partial : \mathbf{C}_{k+1} \rightarrow \mathbf{C}_k$ with

$$\partial(e) = \sum_{i=0}^{k} (-1)^i (d_i^0(e) - d_i^1(e))$$

(the relations $d_i^k d_j^l = d_{j-1}^l d_i^k$ for $i < j$ imply $\partial \circ \partial = 0$ [exercise])

This gives a chain complex $(\mathbf{C}, \partial)$.

Homology of C

It is the homology of the chain complex $(\mathbf{C}, \partial)$:

$$H_i(C) = \frac{\ker \partial_{|\mathbf{C}_i}}{Im\ \partial_{|\mathbf{C}_{i-1}}}$$

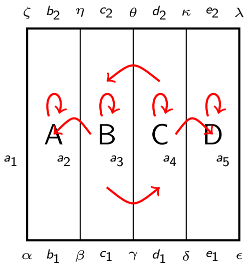
for all $i = 0, \ldots, n$.

Algorithmical computation of the Conley index: exercice!

Back to $f(x, y) = (2x, \frac{y}{2})$

- For the index pair (N, L) with $N = |A| \cup |B| \cup |C| \cup |D|$ and $L = |A| \cup |D|$,
- Write the corresponding chain complex for the quotient N/L (hint: kill cells A and D),
- Write the corresponding boundary matrices,
- Apply SNF.

Algorithmical computation of the Conley index: exercice!



Solution (quite lengthy if expanded!):

- Take SNF of $\partial_2(C) = -d_2 - a_3 + d_1$, $\partial_2(B) = a_3 - c_2 + c_1$ (we did set $A \sim 0$ and $D \sim 0$ and $a_2 \sim 0$, $a_4 \sim 0$),
- Take SNF of $\partial_1(a_3) = \theta - \gamma$, $\partial_1(c_1) = \gamma - \beta$, $\partial_1(c_2) = \theta - \eta$, $\partial(d_1) = \delta - \gamma$, $\partial(d_2) = \kappa - \theta$ (we did set $a_1 \sim 0$, $b_1 \sim 0$, $b_2 \sim 0$, $e_1 \sim 0$, $a_4 \sim 0$, $e_2 \sim 0$, $a_5 \sim 0$, $\alpha \sim 0$, $\zeta \sim 0$, $\eta \sim 0$, $\beta \sim 0$, $\delta \sim 0$, $\epsilon \sim 0$, $\kappa \sim 0$, $\lambda \sim 0$

Etc...

Application of this theory to continuous and controlled systems

Dynamical systems

Of the form $\dot{x} = f(x)$ - generally when we want to study the behavior of a controlled system with an explicit controller:

- Find invariant sets/local stability properties,
- Find the finer structure of invariant sets, such as periodic orbits
- (Study the local chaotic behaviour of non-linear dynamics - in dimension greater than 3).

See e.g. Konstantin Michaikow and Marian Mrozek, "Conley Index Theory", Chapter 9 in Handbook of Dynamical Systems, 2002, and CAPD `http://capd.ii.uj.edu.pl/index.php`.

Controlled systems

Of the form $\dot{x} = f(x, u, w)$ (with u: control, w: perturbation):

- Viability theory: existence of controllable trajectories within some set

Application to local stability properties (useful for control!)

Hyperbolic equilibrium points

$x_e \in D$ is a hyperbolic equilibrium point of $\dot{x} = f(x)$ if all the eigenvalues of the Jacobian matrix $Df(x_e)$ are not on the unit disk in $\mathbb{C}$.

Asymptotic stability $x_e \in D$ equilibrium point of $\dot{x} = f(x)$:

- is Lyapunov stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x_0 \in B(x_e, \delta)$ then $\varphi(t, x_0) \in B(x_e, \epsilon)$ for all $t \geq 0$,
- is asymptotically stable if it is Lyapunov stable and there exists $\delta > 0$ such that if $x_0 \in B(x_e, \delta)$ then $||\varphi(t, x0) - x_e|| \to 0$ as $t \to \infty$.

In control, we want to know (asymptotic stability) if we will drive the system ultimately to the target ("equilibrium") point, in a stable manner!

For more details, see e.g. "Conley index condition for asymptotic stability", Emmanuel Moulay, Qing Hui.

Main theorems for local stability

Characterization of the (potential) unstability

Let x_e be a hyperbolic fixed point with an unstable manifold of dimension n. Then

$$CH_i(x_e) = \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Characterization of stability

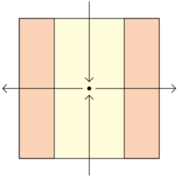
If x_e is asymptotically stable then

$$CH_i(x_e) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Back to an old example

Remember? $f(x, y) = (2x, \frac{y}{2})$

The index pair we found, below:



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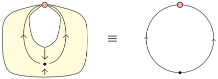


Proves the existence of a non-empty invariant set (here, point $(0, 0)$: not stable by the theorem we just saw!)

Remarks...

The proof of the 2 theorems...

- Locally to x_e, the system $\dot{x} = f(x)$ close to the linearized system $\dot{x} = Df(x_e).x$
- Conley index is robust under slight changes of systems - so ultimately, locally, will be the same as the linearized system,
- Easy to see that as in the example, what counts is the number of eigenvalues of Df of the ones strictly inside the unit disk in $\mathbb{C}$ [entrance part of the index pair], and the number n of the ones strictly outside the unit disk in $\mathbb{C}$ [exit part of the index pair],
- No eigenvalue on the unit disk, in the diagonalizable case at least, easy to see what N/L looks like [a sphere in dimension n].

Application 2: find periodic orbits

"If the Conley index looks like the one of a circle invariant, then there is a periodic invariant" (but only at some further technical condition):

Main theorem

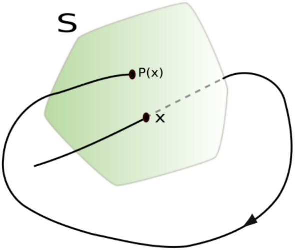
Assume N is an isolating neighbourhood for the flow φ which admits a Poincaré section Ξ.
If N has the cohomological (see later!) Conley index of a hyperbolic periodic orbit (i.e. of S^1) then $inv(N, \varphi)$ contains a periodic orbit.

See e.g. Pilarczyk, "Computer Assisted Method for Proving Existence of Periodic Orbits", Topological Methods in Nonlinear Analysis, 1999

Poincaré section?

Poincaré section:

- A compact subset S of an $(n-1)$-dimensional hyperplane H is called a local section for φ if the vector field f is transverse to H on S.
- Such a set S is called a Poincaré section for φ in an isolating neighbourhood N if $S \cap N$ is closed and for every $x \in N$ there exists $t > 0$ such that $\varphi(x,t) \in S$ (return map $P(x)$).



Application 2: find periodic orbits

"If the Conley index looks like the one of a circle invariant, then there is a periodic invariant" (but only at some further technical condition):

Main theorem (reminder)

Assume N is an isolating neighbourhood for the flow φ which admits a Poincaré section Ξ.
If N has the cohomological (see later!) Conley index of a hyperbolic periodic orbit (i.e. of S^1) then $inv(N, \varphi)$ contains a periodic orbit.

Instead of cohomological calculation, we only know homology right now, we use:

Let $\mathcal{F}$ a cubical enclosure of φ_t on the cubical grid $\mathcal{H}$. Suppose we have $C \subseteq \mathcal{H}$ a sub-cubical complex such that $(C, \emptyset)$ is an index pair for $\mathcal{F}$ (it maps C to $D \subseteq C$).

- HC or HD is iso to the homology module of S^1 (i.e. $HC_0 = \mathbb{Z}$, $HC_1 = \mathbb{Z}$ and $HC_i = 0$ for $i \geq 2$ or similarly for HD)
- $H\psi$ is an isomorphism

Then $N = |N|$ has the homology (and cohomology - see later) index of an attracting periodic orbit wrt φ.

But how do we get the map induced in (co-)homology from the cubical enclosure?

Chain maps

Let $\mathcal{F} : A \rightarrow 2^B$ be an almost perfect combinatorial map

- Where again $\mathcal{H}$ a cubical grid in $\mathbb{R}^n$, A and B two sub-cubical complexes of $\mathcal{H}$,
- For any $c = \sum_{i=0}^{k} \alpha_i a_i$ in $\mathbf{C}_q(A)$ (reminder: the q-chains in A), let $carr(c)$, the carrier of c, be $\{a_i \mid \text{for } i = 0, \ldots, k \text{ s.t. } \alpha_i \neq 0\}$,
- Let $\mathcal{A}(v) = \bigcap \{\mathcal{F}(a) \mid v \leq a\}$ where $v \leq a$ denotes the fact that v is a face of a.

Then there exists a chain map $\mathbf{F} : \mathbf{C}(A) \rightarrow \mathbf{C}(B)$ such that:

$$carr(\mathbf{F})(c) \subseteq \mathcal{A}(c)$$

for all c in $\mathbf{C}_q(A)$.

Principle of the proof/construction

Chain maps

Let $\mathcal{F}: A \to 2^B$ be an almost perfect combinatorial map

Then there exists a chain map $\mathbf{F}: \mathbf{C}(A) \to \mathbf{C}(B)$ such that:

$$carr(\mathbf{F})(c) \subseteq \mathcal{A}(c)$$

for all c in $\mathbf{C}_q(A)$.

Principle of the proof/construction

By induction on $i = 0, \ldots, n$ we construct $\mathbf{F}_i: \mathbf{C}(A)_i \to \mathbf{C}(B)_i$ such that:

- $carr(\mathbf{F}_i)(c) \subseteq \mathcal{A}(c)$
- $\mathbf{F}_{i-1} \circ \partial = \partial \circ \mathbf{F}_i$.

Chain maps

Let $\mathcal{F}: A \to 2^B$ be an almost perfect combinatorial map

Then there exists a chain map $\mathbf{F}: \mathbf{C}(A) \to \mathbf{C}(B)$ such that:

$$carr(\mathbf{F})(c) \subseteq \mathcal{A}(c)$$

for all c in $\mathbf{C}_q(A)$.

Principle of the proof/construction

For $i = 0$, and $c \in A_0$, as $\mathcal{A}(c) \neq \emptyset$, choose any vertex v in $\mathcal{A}(c)$ and set $\mathbf{F}(c) = v$

Chain maps

Let $\mathcal{F}: A \to 2^B$ be an almost perfect combinatorial map

Then there exists a chain map $\mathbf{F}: \mathbf{C}(A) \to \mathbf{C}(B)$ such that:

$$carr(\mathbf{F})(c) \subseteq \mathcal{A}(c)$$

for all c in $\mathbf{C}_q(A)$.

Principle of the proof/construction

For $i = 1$, and $c \in A_1$, we have $\partial(c) = a_1 - a_0$ and we set $z = \mathbf{F}(\partial(c)) = \mathbf{F}(a_1) - \mathbf{F}(a_0)$.

Chain maps

Let $\mathcal{F}: A \to 2^B$ be an almost perfect combinatorial map

Then there exists a chain map $\mathbf{F}: \mathbf{C}(A) \to \mathbf{C}(B)$ such that:

$$carr(\mathbf{F})(c) \subseteq \mathcal{A}(c)$$

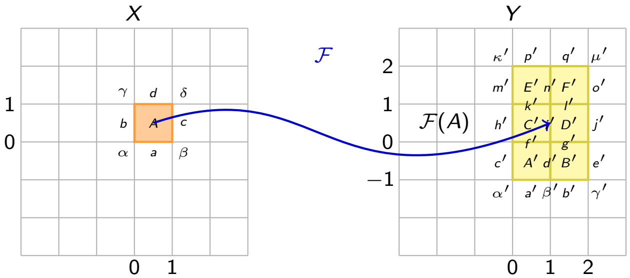
for all c in $\mathbf{C}_q(A)$.

Principle of the proof/construction

Suppose we have constructed $\mathbf{F}_i$ for $0 \leq i < q$.

- Let $c \in A_q$, we have $\partial(c) = \sum_{j=0}^{k} \alpha_j c_j$ with $c_j \in A_{q-1}$,
- We have $\mathbf{F}_{q-1}(c_j) \subseteq \mathcal{A}(c_j)$ so $\mathbf{F}_{q-1}(\partial(c)) \subseteq \mathcal{A}(c)$,
- Since $\mathcal{F}(c)$ is convex, $\mathcal{A}(c)$ is convex and as $\mathbf{F}_{q-1}(\partial(c))$ is a cycle (inductive assumption) there exists a chain $\sigma \in \mathbf{A}_q$ such that $\partial(\sigma) = \mathbf{F}_{q-1}(\partial(c))$,
- Define $\mathbf{F}_q(c) = \sigma$.

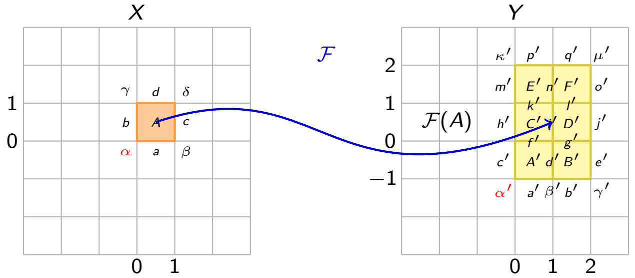
Example of the construction of the chain map



Here, as an example:

- $f(x, y) = (2x, x + 2y - 1)$ for $x \in [0, 1]$ and $y \in [0, 1]$,
- Outer-approximation computed by interval methods: $f([0, 1] \times [0, 1]) = [0, 2] \times [-1, 2]$.

Example of the construction of the chain map



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- Hence $\mathbf{F}(\alpha) = \alpha'$,

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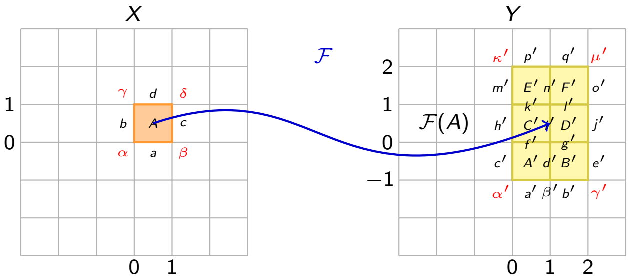

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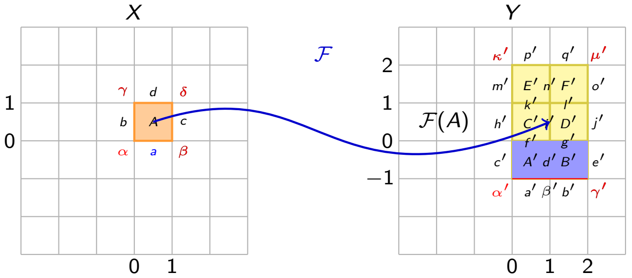
- $f(\alpha) = f(0, 0) = (0, -1) = \alpha'$ and $\mathcal{F}(\alpha) = \alpha'$,
- Hence $\mathbf{F}(\alpha) = \alpha'$,
- Similarly, $\mathbf{F}(\beta) = \gamma'$, $\mathbf{F}(\gamma) = \kappa'$, and $\mathbf{F}(\delta) = \mu'$.

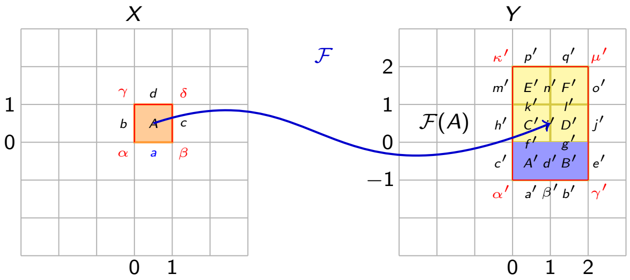
Example of the construction of the chain map



Here, as an example $f(x,y) = (2x, x+2y-1)$

- $f(a) = f([0,1],0) = [0,2] \times [-1,0]$ and $\mathcal{F}(a) = \{A', B', a', b', c', d', f', g', \alpha', \beta', \gamma', \dots\}$,
- $\partial(a) = \beta - \alpha$ hence set $\partial(\mathbf{a})(a) = \gamma' - \alpha'$ which is a 0-cycle/boundary (convexity of the outer-approximation), set $\mathbf{F}(a) = a' + b'$,

Example of the construction of the chain map



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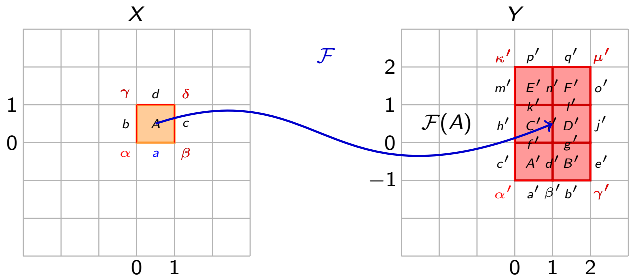
- $f(a) = f([0,1], 0) = [0,2] \times [-1,0]$ and $\mathcal{F}(a) = \{A', B', a', b', c', d', f', g', \alpha', \beta', \gamma', \ldots\}$,
- $\partial(a) = \beta - \alpha$ hence set $\partial(\mathbf{a})(a) = \gamma' - \alpha'$ which is a 0-cycle/boundary (convexity of the outer-approximation), set $\mathbf{F}(a) = a' + b'$,
- Similarly, $\mathbf{F}(b) = m' + h' + c'$, $\mathbf{F}(c) = o' + j' + e'$, and $\mathbf{F}(d) = p' + q'$.

Example of the construction of the chain map



Here, as an example $f(x, y) = (2x, x + 2y - 1)$

- $f(A) = f([0, 1], [0, 1]) = [0, 2] \times [-1, 2]$ and $\mathcal{F}(A) = \{A', B', C', D', E', F', a', b', c', d', f', g', \ldots\}$,
- $\partial(A) = d - c - a + b$ hence $\mathbf{F}(\partial(A)) = p' + q' - o' - j' - e' - a' - b' + m' + h' + c'$ which is a 1-cycle/boundary (convexity of the outer-approximation) and $\mathbf{F}(A) = A' + B' + C' + D' + E' + F'$.

From continuous flows to combinatorial maps

Overall, we have the main homological property:

If $\mathcal{F}$ is an almost perfect cubical enclosure of f, then the chain map ψ between the chain complexes $\mathcal{C}(A)$ and $\mathcal{C}(B)$ is such that ψ_* is equal to f_* (in homology).

Subtility 1

This is actually more suble than it looks: could be different ways to choose the filling of cycles in the induction step - but all give homologically equivalent maps.

Subtility 2

- May seem incredible that we did not have to subdivide (we would have had to if we would have used the simplicial approximation theorem),
- Subtility is: if we want to approximate $f: X \rightarrow Y$, then the representation of f is by star-shape valued map F (“almost perfect representation”) may not have the same range as Y.
- Can be circumvented, but not necessary here (because we are going to apply this to isolating blocks!)

More in Allili, Kaczynski “An algorithmic approach to the construction of homomorphisms induced by maps in homology”, Transactions of the American Mathematical Society, 1999. And more modern chain map calculation based on Vietoris-Begle, see Kaczynski, “Multivalued maps as a tool in modeling and rigorous numerics”. J. fixed point theory appl., 2008.

Determining isolating neighborhoods of Lyapunov type

Principle: using the last property, we only need to do this on a discrete dynamical system! (φ_t)

And we know how to do this:

- By Kleene iteration of the combinatorial map $\mathcal{F}$ (cubical enclosure of φ_t)!
- Function neighbourhood(N_1, $\mathcal{F}$): Start with N_1 a finite sub-cubical set of $\mathcal{H}$, then iterate $N_{n+1} = N_n \cup \mathcal{F}(N_n)$.
- If converges in finite time towards $N \neq \emptyset$ a sub-cubical set of $\mathcal{H}$ then N is an isolating neighborhood of $\mathcal{F}$ (then $|N|$ is an isolating neighborhood of φ_t, then of φ - with the same Conley index as for φ_t!)

Proof

- Finite termination to non-empty N implies $\exists K > 0$ s.t. $\mathcal{F}(N_K) \subseteq N_K \subseteq A$,
- $\mathcal{F}$ cubical enclosure of φ_t implies $\varphi_t(N) \subseteq int(N)$, for $N = |N_K|$ and since $inv(N, \varphi_t) \subseteq \varphi_t(N)$: $inv(N, \varphi_t) \subseteq int(N)$,
- N is compact as a finite union of hypercubes, so N is an isolating neighborhood of φ_t (the rest we proved it already)

Further improvements

Tools from abstract interpretation

I.e. other convex gridding method than by hypercubes, widening methods for the fixed-point of calculation of $\mathcal{F}$ etc.

Exercise: reduction of a cubical set

Let $a \in \mathcal{H}$ and C a sub-cubical set of $\mathcal{H}$ such that $a \in C$ and $a \cap b \neq \emptyset$ for all $b \in C$. Then $a \cap |C|$ is a strong deformation retract of $|C|$.

Can be used to reduce the size of a cubical complex C:

function *reduce*(A,D: finite subset of $\mathcal{H}$): finite subset of $\mathcal{H}$;
for each $a \in A \backslash D$:
    $C = \{b \in A \backslash \{a\}$ such that $a \cap b = \emptyset\}$;
    if ($C = \emptyset$) and (*reduce*$(C, \emptyset)$ has exactly one element) then return
reduce$(A \backslash \{a\}, D)$;
return A;

Computation of the Conley index

For Lyapunov type isolating neighborhoods, the (homological) Conley index is given by the homology of C (equivalently D):

function *computations*(N_1: finite subset of $\mathcal{H}$, $\mathcal{F}: A \rightarrow 2^{\mathcal{H}}$: finite multi-valued cubical map): (HC, HD: homology module, $H\psi: HC \rightarrow HD$);
N=neighbourhood(N_1,$\mathcal{F}$);
N'=reduce(N,$\emptyset$);
if $\mathcal{F}$ is not almost perfect on N'
    return $(\emptyset, \emptyset, \emptyset)$;
N''=reduce(N,$\mathcal{F}(N')$);
(C, D, ψ)=chainmap(N',N'', $\mathcal{F}$);
return homology(C,D,ψ)

For more on these implementations, you can have a look at CAPD/REDHOM http://capd.ii.uj.edu.pl and Kapela, Mrozek, Wilczak, Zgliczyński, "CAPD::DynSys: A flexible C++ toolbox for rigorous numerical analysis of dynamical systems", 2020.

Application 1: find periodic orbits

Main theorem

Let N_1 be a sub-cubical set of $\mathcal{H}$ and $\mathcal{F}$ a cubical enclosure of φ_t. If the previous algorithm stops and returns HC, HD and $H\psi$ (or ψ_*) such that :

- HC or HD is iso to the homology module of S^1 (i.e. $HC_0 = \mathbb{Z}$, $HC_1 = \mathbb{Z}$ and $HC_i = 0$ for $i \geq 2$ or similarly for HD)
- $H\psi$ is an isomorphism

Then $N = |N|$ has the homology (and cohomology - see later) index of an attracting periodic orbit wrt φ.

The last ingredient to determine if the isolated invariant so found contains a periodic orbit (and is homotope to a periodic orbit) is the existence of a Poincaré section.

Existence of a Poincaré section algorithmically

function *verify*(A: finite subset of $\mathcal{H}$, Q: subset of $\mathbb{R}^n$, $\tau \in \mathbb{R}_+$, $K \in \mathbb{N}$,
$\varphi_\tau : \ A \times \mathbb{N} \to R^n$): boolean;
for each $a \in A$
answer=false;
for $k = 0$ to K
    if $\mathcal{F}(a, k) \subseteq int(Q)$
        answer=true;
if answer=false
    return false
return true

Existence of a Poincaré section

Main property of the previous algorithm

Let A be a finite subset of $\mathcal{H}$. Assume $\Xi \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^n$ are such that $\Xi \cap int(Q) = \emptyset$ and $\varphi(x, \mathbb{R}_+) \cap \Xi = \emptyset$ for each $x \in int(Q)$.
Let $\tau > 0$, $K \in \mathbb{N}$, and $F : A \times \mathbb{N} \to \mathbb{R}^n$ be a map such that $\varphi(a, k\tau) \subseteq F(a, k)$ for each $a \in A$ and $k \in \mathbb{N}$.
If the previous algorithm *verify* returns true then $\psi(x, \mathbb{R}_+) \cap \Xi = \emptyset$ for every $x \in |A|$.

Proof

Given $x \in |A|$, there exists $a \in A$ such that $x \in a$. For this a there exists $k \in I_K$ such that $F(a, k\tau) \subseteq int(Q)$, because otherwise Algorithm `verify` returns false.
Denote $\varphi(x, k\tau)$ by y. Since $y \in int(Q)$, there exists $s > 0$ such that $\varphi(y, s) \in \Xi$.
Then $\varphi(x, k\tau + s) \in \Xi$, QED.

Example: Rössler equation

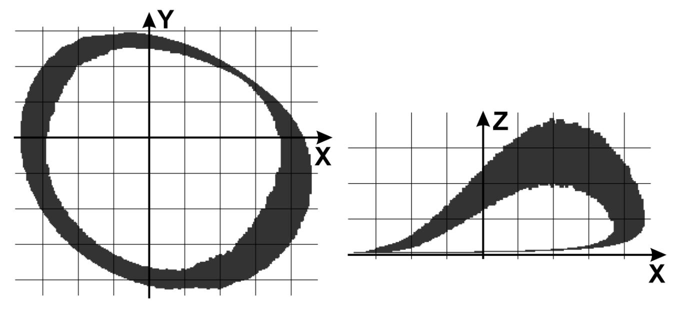
$$\begin{cases} \dot{x} = -(y+z) \\ \dot{y} = x + by \\ \dot{z} = b + z(x-a) \end{cases}$$

for $a = 5.7$ or $a = 3.1$ or $a = 2.2$ and $b = 0.2$.

Simulations seems to show $a = 5.7$: chaotic behavior, $a = 3.1$ and $a = 2.2$ existence of periodic orbits. We are going to prove it! (equivalently, use "guaranteed" estimates of the dynamics)

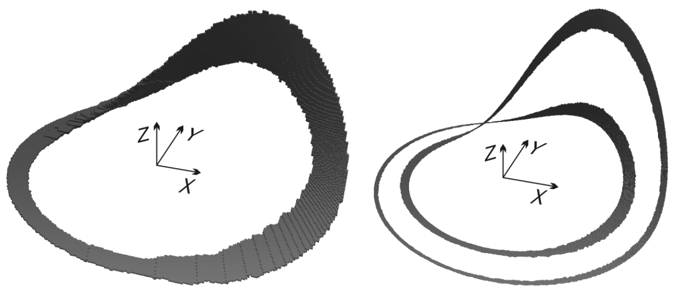
Isolated neighborhoods for Rössler dynamics

Projections on the *xy* and *zx* planes



Isolated neighborhoods for Rössler dynamics

In 3D: left for $a = 3.1$, right for $a = 5.7$



Computing particular isolating blocks given by polynomial templates

Definition

The set $B \subseteq \mathbb{R}^n$ is defined, for some vector $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$, by the m polynomial inequalities :

$$(P) \left\{ \begin{array}{rcl} p_1(x_1, \ldots, x_n) & \leq & c_1 \\ & \ldots & \\ p_m(x_1, \ldots, x_n) & \leq & c_m \end{array} \right.$$

Faces

We call P_i^c the face of template B given by $\{(x_1, \ldots, x_n) | p_i(x_1, \ldots, x_n) = c_i\} \cap B$ which might be proper (non-empty) or not. We suppose always proper in the sequel.

Minimal polynomial templates

Templates B which border ∂B is equal (and not just included as would be generally the case) to $\bigcup_{i=1}^{s} \{x | p_i(x) = c_i, p_j(x) \leq c_j \ \forall j \neq i\}$.
For $x \in \partial B$, we note $I(x)$ the non-empty and maximal set of indices in $0, \ldots, m$ such that for all $i \in I(x)$, $p_i(x) = c_i$.

Lie derivative and higher-order Lie derivatives

Lie derivative

Of $h \in \mathbb{R}[x]$ along the vector field $f = (f_1, \ldots, f_n)$ is

$$\mathcal{L}_f(h) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i = \langle f, \nabla h \rangle$$

Higher-order derivatives

Are :

$$\mathcal{L}_f^{k+1}(h) = \mathcal{L}_f(\mathcal{L}_f^{(k)}(h))$$

with $\mathcal{L}_f^0(h) = h$

Differential radical ideal

Case of polynomial dynamical systems

- By Noetherianity of the ring $\mathbb{R}[\mathbf{x}]$, only a finite number of Lie derivatives are necessary to generate all higher-order Lie derivatives
- There is a smallest N such that:

$$\mathcal{L}_f^{(N)}(h) \in < h, \mathcal{L}_f^{(1)}(h), \ldots, \mathcal{L}_f^{(N-1)}(h) >$$

(order of the "radical differential ideal", can be computed using Gröbner bases, inductively)

Example

$$\dot{x} = y, \dot{y} = y + (x^2 - 1)\left(x + \frac{1}{2}\right)$$

For the first face P_1^c

Of the template $\{p_1 = -x, p_2 = x, p_3 = -y, p_4 = y\}$ with $\{c_1 = 2, c_2 = 2, c_3 = 2, c_4 = 2\}$ i.e $P_1^c = \{-x = 2, x \leq 2, -y \leq 2, y \leq 2\}$ then

- $\mathcal{L}_f^{(1)}(p_1) = -y$,
  $\mathcal{G}(\{\mathcal{L}_f^{(0)}(p_1), \mathcal{L}_f^{(1)}(p_1)\}) = \{-x, -y\}$;
- $\mathcal{L}_f^{(2)}(p_1) = -y - (x^2 - 1)\left(x + \frac{1}{2}\right)$,
  $\mathcal{G}(\{\mathcal{L}_f^{(0)}(p_1), \mathcal{L}_f^{(1)}(p_1), \mathcal{L}_f^{(2)}(p_1)\}) = \mathbf{1}$.

So $N_1 = 3$.

Exit sets, differentially

Characterization

Let x_0 be a point on the border ∂B of a minimal polynomial template B.
Then x_0 is in the exit set B^- of B if and only if,
for some $i_0 \in I(x)$,

$$\exists k_0 > 0 \text{ such } \mathcal{L}_f^{(k_0)}(p_{i_0}) > 0 \text{ and } \forall k,\ 0 < k < k_0\ \mathcal{L}_f^{(k)}(p_{i_0}) = 0$$

Could actually use purely (continous) set-based reachability techniques here! (but not for checking closedness, see next slide)

Closedness of the exit set

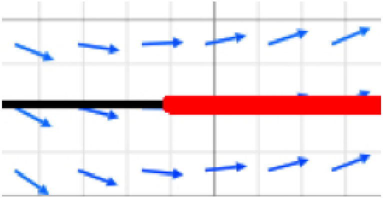
Recall condition (a) for isolating blocks
$B^- = \{x \in B \mid \varphi([0,T),x) \not\subseteq B, \forall T > 0\}$ is closed

Partial characterization

Let B be a compact minimal polynomial template defined by the set of inequalities ($P = \{p_i \leq c_i\}$) and
let N_i be the order of the differential radical ideal $\sqrt[\mathcal{L}]{< p_i >}$.
If for each face P_i^c of template B, for all $k \in \{1, \cdots, N_i - 2\}$,

$$(H_k^i) \ : \ \left\{ \begin{array}{ll} \{x \in P_i^c \mid & \mathcal{L}_f^{(1)}(p_i)(x) = 0, \ \cdots, \ \mathcal{L}_f^{(k)}(p_i)(x) = 0, \\ & \mathcal{L}_f^{(k+1)}(p_i)(x) < 0\} = \emptyset \end{array} \right.$$

then B^- is closed and is equal to $\bigcup_{i=1}^m \{x \in P_i^c \mid \mathcal{L}_f^{(1)}(p_i)(x) \geq 0\}$.



Forbid inner tangencies

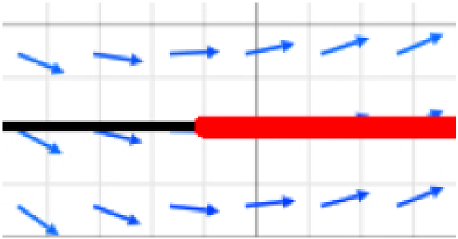
Recall condition (b) for isolating blocks :
$\forall T > 0, \{x \in B \mid \varphi([-T, T], x) \subseteq B\} \subseteq intB$

Characterization

Let B be a minimal polynomial template. It satisfies condition (b) of the definition of isolating blocks if for each of its faces P_i^c, for all $k \in \{0, \cdots, N_i - 1\}$, no $x \in P_i^c$ can satisfy the following set of equalities and inequalities :

$$\mathcal{L}_f^{(1)}(p_i)(x) = 0, \ \cdots, \ \mathcal{L}_f^{(2k-1)}(p_i)(x) = 0, \ \mathcal{L}_f^{(2k)}(p_i)(x) < 0$$

(implied by our first condition!)



Algorithmically

Using Stengle's nichtnegativstellensatz

We determine polynomials α_j ($j = 0, \ldots, k$), SoS polynomials $\beta_{S,\mu}$ ($S \subseteq \{1, \ldots, i-1, i+1, \ldots, m\}, \mu \in \{0, 1\}$) and an integer l, such that

$$\sum_{j=0}^{k} \alpha_j \mathcal{L}_f^{(j)}(p_i) \; + \sum_{S \subset \{1,\ldots,i-1,i+1,\ldots,m\}} \beta_{S,\mu} G_{S,\mu} + \left(\mathcal{L}_f^{(k+1)}(p_i)\right)^{2l} = 0$$

where $G_{S,\mu} = (-\mathcal{L}_f^{(k+1)})^{\mu} \Pi_{s \in S}(c_j - p_j)$ for any $S \subseteq \{1, \ldots, i-1, i+1, \ldots, m\}$ and $\mu \in \{0, 1\}$ (and the convention that $\mathcal{L}_f^{0}(p_i) = c_i - p_i$).
(in all our examples, we took $l = 1$).

Algorithmic improvement

Putinar's positivstellensatz

To provide for faster results, in most cases, we begin, for a given k (and i), instead of solving (H_i^k) by nichtnegativstellensatz, by solving the simpler property :

$$p_i = c \wedge (p_j \leq c_j)_{j \neq i} \wedge \mathcal{L}_f^{(1)}(p_i)(x) = 0, \; \cdots, \; \mathcal{L}_f^{(k)}(p_i)(x) = 0, \; \Rightarrow \mathcal{L}_f^{(k+1)}(p_i)(x) > 0$$

If so, we can stop testing (H_k^i) for higher values of k since they are then trivially satisfied.
Use of much less computationally demanding Putinar's positivstellensatz.

(This condition, for $k = 1$, is the one used in "Rigorous validation of isolating blocks for flows and their Conley indices", Stephens, Wanner, 2014)

Also, possible to use set-based methods with Bernstein polynomials etc.

Running example

$$\dot{x} = y, \dot{y} = y + (x^2 - 1)\left(x + \frac{1}{2}\right)$$

Putinar, $k = 2$

If we take again the face P_1^c and try to prove for example (H_1^1). A sufficient condition is to find polynomials α, γ (for equality conditions) and sum-of-squares polynomials $\beta_0, \beta_1, \beta_2, \beta_3$ (dor inequality conditions) such that

$$\mathcal{L}_f^{(2)}(p_i) = \alpha \mathcal{L}_f^{(1)}(p_i) + \beta_0 + \beta_1(c_2 - p_2) + \beta_2(c_3 - p_3) + \beta_3(c_4 - p_4) + \gamma(p_1 - c_1)$$

which is trivially satisfied with $\alpha = 1$, $\beta_0 = \frac{9}{2}$, $\beta_1 = \beta_2 = \beta_3 = 0$ and $\gamma = \left(\left(\frac{1}{2} + x\right)(2 - x) - 3\right)$.

Actually with SoSTools we found...

$\alpha = 3.187y - 0.2369x + 0.232, \gamma = -0.5842x - 0.2271y - 3.462$
$\beta_0 = 0.01433x - 0.01337y - 1.0y(0.001322x - 0.165y + 7.88e - 6xy + 2.743e - 7x^2 - 3.815e - 8y^2 + 0.01337) + 6.758e - 7xy + x(0.02629x - 0.001322y + 7.689e - 8xy + 1.009e - 6x^2 - 6.472e - 7y^2 + 0.01433) + 2.261e - 7x^2 + 3.57e - 7y^2 - 1.0y^2(6.472e - 7x - 3.815e - 8y + 6.239e - 17xy + 1.18e - 10x^2 - 2.499e - 10y^2 - 3.57e - 7) + x^2(1.009e - 6x - 2.743e - 7y + 2.104e - 15xy + 2.194e - 10x^2 - 1.18e - 10y^2 + 2.261e - 7) + xy(7.689e - 8x - 7.88e - 6y + 4.869e - 10xy + 2.104e - 15x^2 - 6.239e - 17y^2 + 6.758e - 7) + 0.04354$

A combinatorial condition

Construction of the "witness" graph G

- The exit set on P_i^c is given by those $x \in P_i^c$ such that $\mathcal{L}_f^{(1)}(p_i)(x) \geq 0$. If non empty, add node P_i^c to G
- Draw an edge in G between P_i^c and P_j^c if $P_i^c \cap P_j^c$ is non-empty by finding solutions to :

$$\mathcal{L}_f^{(1)}(p_i)(x) \geq 0 \ \wedge \ \mathcal{L}_f^{(1)}(p_j)(x) \geq 0$$

  for x in the intersection of P_i^c and P_j^c. (nichtnegativstellensatz or approximated by positvstellensatz)

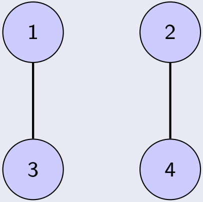
Condition

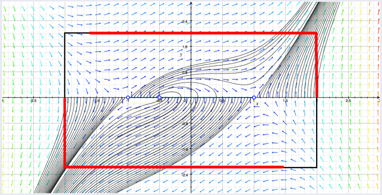
If G is not connected, then the exit set B^- is trivially not connected
(but B is connected because it is in particular contractible)
Thus B^- cannot be a deformation retract of B.
So we do have a non-empty invariant sitting inside.

Example

Graph produced

- Each of the four faces of B is a node in $G^\sharp$.
- Non-empty intersections $P_1^c \cap P_3^c = \{(-2,-2)\}$, $P_1^c \cap P_4^c = \{(-2,2)\}$, $P_2^c \cap P_3^c = \{(2,-2)\}$ and $P_3^c \cap P_4^c = \{(2,2)\}$.
- $\mathcal{L}_f^{(1)}(p_1)(-2,-2) = 2$ and $\mathcal{L}_f^{(1)}(p_3)(-2,-2) = \frac{13}{2}$ positive so edge from P_1^c to P_3^c
- $\mathcal{L}_f^{(1)}(p_1)(-2,2) = -2$ (not positive) and $\mathcal{L}_f^{(1)}(p_4)(-2,2) = \frac{5}{2}$ so no edge from P_1^c to P_4^c ; similarly no edge between P_2^c and P_3^c,
- edge between P_2^c and P_4^c.

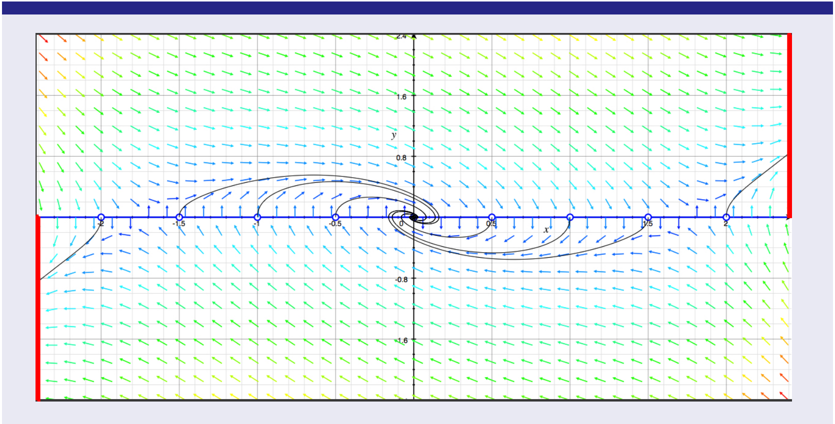




We conclude that B^- has (at least) two connected components, and that there is a non empty invariant within the square B.

Example 2

Example CDC 2014 is defined by $\left(\dot{x} = y, \dot{y} = -y - x + \frac{1}{3}x^3\right)$ with as template, the box defined by $c = (2.4, 2.4, 2.4, 2.4)^t$.



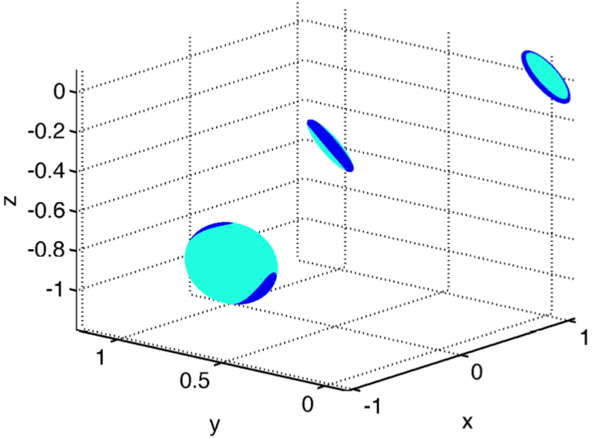
Example

A simple 3D system

$$\begin{aligned} \dot{x} &= 2x(z-y) \\ \dot{y} &= 1+z-x^2 \\ \dot{z} &= -1+y+x^2 \end{aligned}$$

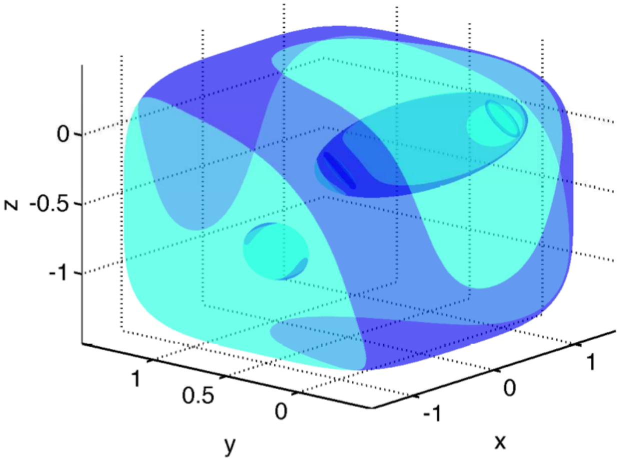
- Three equilibrium points $p_0 = (-1, 0, 0)$, $p_1 = (1, 0, 0)$ and $p_2 = (0, 1, -1)$.
- Easy to find isolating blocks for p_0 and p_1 (template $(x - p_i^x)^2 + (y - p_i^y)^2 + (z - p_i^z)^2$, $i = 0,\ 1,\ 2$; radius $\frac{3}{20}$, $\frac{1}{5}$ for instance)
- Less easy for encompassing the 3 fixed points : template $x^4 + \left(y - \frac{1}{2}\right)^4 + \left(z + \frac{1}{2}\right)$

Example - exit sets in dark blue



(from "Rigorous validation of isolating blocks for flows and their Conley indices", Stephens, Wanner, 2014)

Example - exit sets in dark blue



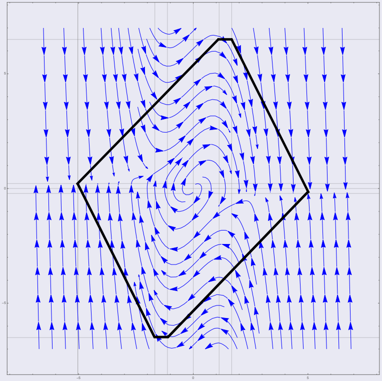
(from "Rigorous validation of isolating blocks for flows and their Conley indices", Stephens, Wanner, 2014)

Van der Pol equation

Phase space and template

$$\left\{ \begin{array}{lcl} \dot{x} & = & y \\ \dot{y} & = & \mu(1-x^2)y - x \end{array} \right.$$

where we will consider that $\mu = 1$.



Van der Pol equation

Template

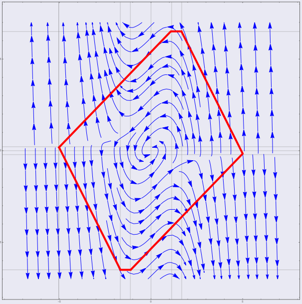
$$
\begin{aligned}
p_1(x,y) &= y \\
p_2(x,y) &= -y \\
p_3(x,y) &= y - 1.02x \\
p_4(x,y) &= -(y - 1.02x) \\
p_5(x,y) &= y + 2x \\
p_6(x,y) &= -(y + 2x)
\end{aligned}
$$

and the coefficients $c = (6.5,\ 6.5,\ 5.35,\ 5.35,\ 9.85,\ 9.85)^t$
This has an empty exit set (actually for lots of μ, e.g. from 0.95 to 1)

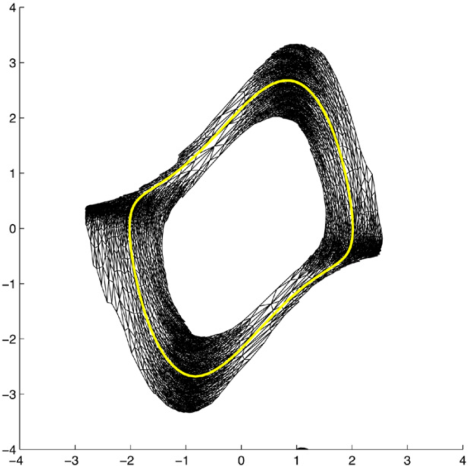
Reverse Van der Pol

Phase space

$$\begin{cases} \dot{x} &= -y \\ \dot{y} &= -\mu(1-x^2)y + x \end{cases}$$



The repulsive fixed point of the Van der Pol equation is now an attractive fixed point for reverse Van del Pol, and the maximal invariant within the template that we use has as boundaries the non-algebraic curve given by the limit cycle.



(from "Polygonal approximation of flows", Boczko, Kalies, Mischaikow, 2007)

Idea that we are going to develop

For now...

- We studied autonomous dynamical systems ($\dot{x} = f(x)$),
- this means that for application to control, we were just interesting in checking the behavior of the controlled system $\dot{x} = f(x, u)$ with u explicitly given.

And then...

- We are going to abstract the problem of finding "stabilizing" controls u as,
- Finding geometrical constraints on the "solutions" of "differential inclusions", $\dot{x} = F(x)$ where F is a map into sets; typically, $F(x) = f(x, U)$ where U is the set of potential values for control!

Differential inclusions

For that one, we need some more theory!

Recall:

We are interested in controllability of $\dot{x} = f(x, w, u)$, so at the existence of u such that we keep all trajectories within some fixed set K.

Differential inclusions

When w and u are only known as sets, the differential system of interest becomes a differential inclusion:

$$\dot{x} \in F(x) \tag{1}$$

where F is a map from $\mathbb{R}^n$ to $\wp(R^n)$

Solutions

A function $x(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is a solution of the differential inclusion if :

- x is an absolutely continuous function
- and satisfies for almost all $t \in \mathbb{R}$, $\dot{x}(t) \in F(x(t))$

$S_F(x_0)$ denotes the set of all solutions starting from x_0 at time 0.

Existence of solutions

Marchaud maps

The set-valued map $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Marchaud map if F is upper semicontinuous (in short: u.s.c.) with compact convex values and linear growth (that is, there is a constant $c > 0$ such that $|F(x)| := sup\{|y| \mid y \in F(x)\} \leq c(1 + |x|)$, for every x).

Existence of solutions to differential inclusions

When F is a Marchaud map, then the inclusion

$$\dot{x}(t) \in F(x(t))$$

has a solution such that $x(t_0) = x_0$ (for all x_0) and for a sufficiently small time interval $[t_0, t_0 + \varepsilon)$, $\varepsilon > 0$.

Viability

Viability kernel

Let K be a closed subset of $\mathbb{R}^n$.

- A trajectory of the differential inclusion

$$\dot{x}(t) \in F(x(t))$$

  $t \rightarrow x(t)$, is said to be viable (in K) when for all t, $x(t) \in K$.
- The viability kernel of this differential inclusion in K is $Viab_K(F)$, the set of initial conditions $x_0 \in K$ such that there exists a solution of $S_F(x_0)$ staying forever in K.

A closed set $K \subset \mathbb{R}^n$ being given, we study the following problem of the existence of trajectories remaining in K:

$$\text{is } Viab_F(K) \text{ not empty?}$$

Viability theorem

Bouligant contingent cone

Let us denote by $C_K(x)$ the Bouligant contingent cone of K at x, which, in the case where K is a closed convex subset of $\mathbb{R}^n$ is :

$$\bigcup_{h>0} \left\{ \frac{k-x}{h} | k \in K \right\}$$

Viability theorem

Consider a Marchaud map $F : \mathbb{R}^n \to \wp(\mathbb{R}^n)$ and a closed convex $K \subseteq \mathbb{R}^n$. Suppose that

$$\forall x \in K, \;\; F(x) \cap C_K(x) \neq \emptyset$$

then $Viab_K(F) = K$, i.e. there always exists a trajectory for the differential inclusion $\dot{x}(t) \in F(x(t))$ from any point of K, staying in K.

The idea behind this theorem is that if there is always a vector field which points inside K in $F(x)$, for all $x \in K$, then there is a way to follow it to stay inside K.

A Ważewski property for differential inclusions

Exit sets

$K^S(F) = \{x_0 \in Fr(K) \mid \forall x \in S_F(x0) : \ x$ leaves K immediately$\}$, be the exit set for the differential inclusion F.
("immediately" means that for every $\epsilon > 0$ there is $0 < t < \epsilon$ such that $x(t) \notin K$)

Wazewski property for differential inclusions

Let K be a closed convex subset of $\mathbb{R}^n$ and F a Marchaud map.
If the set $K^S(F)$ is closed and not connected then

$$Viab_F(K) \neq \emptyset$$

(convention : the empty set is not connected)

Switched systems

Modes and solutions

- Suppose that we are given a family f_i, $i \in Q = \{1, \ldots, q\}$ of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. The set Q is called the set of modes. We still assume here that the functions are polynomials (hence locally Lipschitz).
- Let G defined on every point x of $\mathbb{R}^n$ by $G(x) = \{f_1(x), \ldots, f_q(x)\}$. It has closed, non-empty values, and is locally-Lipschitz, hence, the corresponding differential inclusion

$$\dot{x} \in G(x) \tag{2}$$

  has solutions over finite time intervals.
- A solution of such a differential inclusion is any absolutely continuous functions satisfying

$$\dot{x}(t) \in G(x(t))$$

  almost everywhere.
- Such functions define time-dependent trajectories of the switched systems with the q modes $f_1, \ldots, f_q$.

Filippov-Ważewski

Relation between solutions of switched systems with solutions to some differential inclusions

- Consider instead the differential inclusion equation $\dot{x} \in F(x)$ where F is defined by :

$$F(x) = \overline{co(f_1, \ldots, f_q)} \tag{3}$$

- The Filippov-Ważewski theorem states that :
  all solutions of the convexified equation can be approximated by solutions of the switched system with the same initial value, at least over a compact time interval, and under some simple hypotheses.

From there on...

As what we did for differential equations

Through multivalued maps and reachability methods...

See e.g. Thieme, "Isolating Neighborhoods and Filippov Systems: Extending Conley Theory to Differential Inclusions" 2020, Górniewicz, "Differential inclusions - the theory initiated by Cracow Mathematical School" 2011

Templates

As with the former "alternate" approach.

see e.g. Fribourg, Goubault, Mohamed, Putot, "A Topological Method for Finding Invariant Sets of Switched Systems", HSCC 2016

Convex polynomial inclusions

- We consider differential inclusions given as the closed convexification of a finite set of polynomial vector fields $f_1, \ldots, f_q$:

$$F(x) = \overline{co}(f_1, \ldots, f_q) \tag{4}$$

  where $co(y_1, \ldots, y_q)$ is the convex combination of the q vectors $y_1, \ldots, y_q$ in $\mathbb{R}^n$ and $\overline{A}$ is the topological closure of A in $\mathbb{R}^n$.
- F is a Marchaud map

For every such $\lambda = (\lambda_1, \ldots, \lambda_q)$ we will write $f_\lambda = \sum_{i=1}^{q} \lambda_i f_i$ so that $F(x)$ can be identified with the set of all such $f_{\lambda(x)}$, where λ is a continuous function.

Polynomial templates

$K \subset \mathbb{R}^n$, which are defined, for some vector $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$, by the m polynomials inequalities:

$$(P) \begin{cases} p_1(x_1, \ldots, x_n) & \leq & c_1 \\ & \ldots & \\ p_m(x_1, \ldots, x_n) & \leq & c_m \end{cases}$$

Assumption

We call *minimal polynomial templates*, the templates K which border ∂K is equal (and not just included as would be generally the case) to

$$\bigcup_{i=1}^{m} \{x \mid p_i(x) = c_i, p_j(x) \leq c_j \ \forall j \neq i\}$$

Template with one face

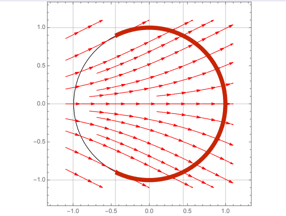
Computing the exit set

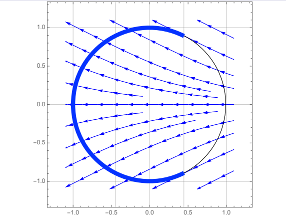
If the template K is defined by a unique polynomial p_1,

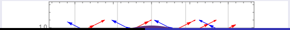
$$K^S(F) = \bigcap_{i=1}^{q} K^S(\{f_i\})$$

Example

$$f_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1+y^2\\y\end{pmatrix} \text{ and } f_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1-y^2\\y\end{pmatrix} \text{ and } p = x^2+y^2 = 1$$







Example

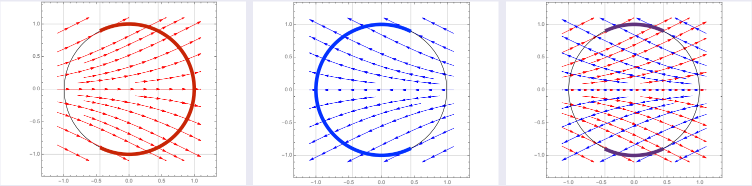
Ball template in dim 2

Let us now take a unit ball template: $p = x^2 + y^2$ and $c = 1$.

$$K^S(F) = \{(x, y) \in [-0.445042, 0.445042] \times [-1, 1] \mid x^2 + y^2 = 1\}$$

which is closed and disconnected

Picture



Hence the ball contains a viable trajectory for some time-dependent switching strategy.

Example

Ball template in dim 3

Consider now the following generalization of the previous system, to dimension 3 :

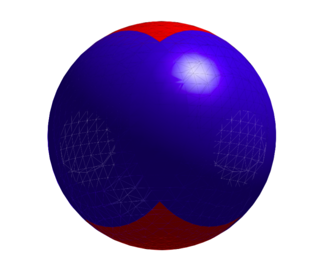
$$f_1\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+y^2+z^2\\y\\z\end{pmatrix},f_2\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1-y^2-z^2\\y\\z\end{pmatrix},$$

$$f_3\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\1+x^2+z^2\\z\end{pmatrix},f_4\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\-1-x^2-z^2\\z\end{pmatrix}$$

An example in dimension 3

Template

We use a ball template defined by $p = x^2 + y^2 + z^2$ and $c = 1$.
$K^S(F) = \{(x, y, z) \in$
$[-0.445042, 0.445042] \times [-0.445042, 0.445042] \times [-1, 1] \mid x^2 + y^2 + z^2 = 1\}$.



There exists a time-dependent switch stabilizing this system of four non-linear ODEs in the unit ball of dimension 3.

Filippov-Ważewksi revisited

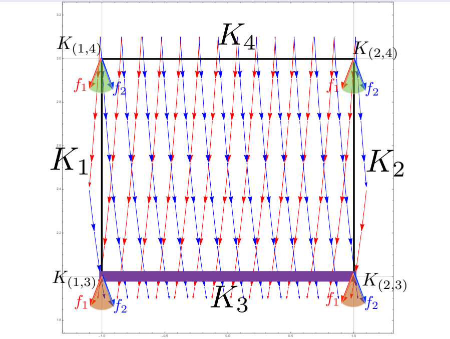
Infinite time horizon

Let $0 < T \leq \infty$.

- Suppose the set-valued map $G : \mathbb{R}^n \to \wp(\mathbb{R}^n)$ is measurable with respect to the Borel subsets of $\mathbb{R}^n$.
- Suppose also that for all $R > 0$ there exists $k_R \in \mathbb{R}$ such that for any $\xi, \eta \in B(0, R)$, $d_H(G(\xi), G(\eta)) \leq k_R|\xi - \eta|$
- and that there exists $\alpha_R \in \mathbb{R}$ such that for each $\xi \in B(0, R)$, $\sup\{|\zeta| : \zeta \in G(\xi)\} \leq \alpha_R$

Fix $\xi \in X$ and let $z \in [0, T) \to X$ be a solution of $\dot{x} \in \overline{co}(G(x)),\ x(0) = \xi$.
Let $r = [0, T) \to \mathbb{R}$ be a continuous function satisfying $r(t) > 0$ for all $t \in [0, T]$.
Then there exists $\eta^0 \in B(\xi, r(0))$ and a solution $x = [0, T) \to X$ of
$\dot{x} \in G(x), \quad x(0) = \eta^0$ which satisfies $|z(t) - x(t)| \leq r(t)\ \forall t \in [0, T)$.

Example



- We know that for template $K' = [-1,1] \times [2,3]$, $K^S(F) = [-1,1] \times \{2\}$
- The exit set is connected (so we cannot conclude).
- It is actually clear that there is no switching that can stabilize F within K'.

Viability and invariants of state-dependent switched systems

Consider a state-dependent switched system, described by a $\mathcal{C}^1$ switching surface $\mathcal{S}$, given by equation $s(x) = 0$ separating $\mathbb{R}^n$ into two open components $\mathcal{S}_+ = \{x \in \mathbb{R}^n \mid s(x) > 0\}$ and $\mathcal{S}_- = \{x \in \mathbb{R}^n \mid s(x) < 0\}$, and two subsystems $\dot{x} = f_i(x)$, $i = +, -$, one on each side of each element of $\mathcal{S}$:

$$\dot{x} = \begin{cases} f_+(x) & \text{if } s(x) > 0 \\ f_-(x) & \text{if } s(x) < 0 \end{cases} \tag{5}$$

Solutions of state-dependent switched systems

Given a state-dependent switched system $\mathcal{H}$ defined by Equation 5, a function $x(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is a solution of $\mathcal{H}$ if it is absolutely continuous and satisfies the differential inclusion $\dot{x}(t) \in F(x(t))$ for almost all $t \in \mathbb{R}^+$, where F is a multi-valued function defined as follows:

$$F(x) = \begin{cases} \{f_-(x)\} & \text{if } x \in \mathcal{S}_- \\ \{f_+(x)\} & \text{if } x \in \mathcal{S}_+ \\ \overline{co}\{f_+(x), f_-(x)\} & \text{if } x \in \mathcal{S} \end{cases}$$

An uncertain differential system

- We consider :

$$f_\epsilon \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^3 + y - x/10 + \varepsilon \\ -x - y/10 + \varepsilon \\ 5z + \varepsilon \end{pmatrix}$$

  and we take as differential inclusion $F(x, y, z) = \{f_\epsilon \mid -0.05 \leq \varepsilon \leq 0.05\}$.

- We consider the template given by the unique face

$$p_1 = x_1^2 + (x_2 - 1)^2 + (x_3 + 1)^2$$

  and $c_1 = 1/25$ the exit set for the differential inclusion is proved closed in 348 seconds.

- The exit set has two connected components hence there is a viable trajectory within the template considered.

Boost DC-DC Converter

The boost DC-DC converter is an example from power electronic, where the state of the system is $x(t) = [i_l(t)\ v_c(t)]^T$ with i_l the current intensity in an inductor, and $v_c(t)$ the voltage of a capacitor.

- The aim of the control is to maintain the system inside a given zone K (while the output voltage stabilizes around a desired value).
- The dynamics associated with mode u is given by $\dot{x}(t) = f_u(x) = A_u x(t) + b$ $(u = 1, 2)$ with $b = (\frac{v_s}{x_l}\ 0)^T$. We use in the experiments the numerical values for A_1 and A_2,

$$A_1 = \begin{pmatrix} -0.0167 & 0 \\ 0 & -0.0142 \end{pmatrix}, A_2 = \begin{pmatrix} -0.0183 & -0.3317 \\ 0.0142 & -0.0142 \end{pmatrix}$$

- and study the system in the rectangle $K = [1.55, 2.15] \times [1.0, 1.4]$.

Time-dependent switching

Lie derivatives on faces

The Lie derivatives for each dynamic and each face, instantiated for the chosen parameters and template K, are given below

$$\begin{array}{ll} \mathcal{L}_{f_1}(p_1) = -0.017x + .3 & \mathcal{L}_{f_2}(p_1) = -0.018x - 0.33y + .3 \\ > 0 & < 0 \\ \mathcal{L}_{f_1}(p_2) = -L_{f_1}(p_1) < 0 & \mathcal{L}_{f_2}(p_2) = -\mathcal{L}_{f_2}(p_1) > 0 \\ \mathcal{L}_{f_1}(p_3) = -0.014y < 0 & \mathcal{L}_{f_2}(p_3) = 0.014x - 0.014y > 0 \\ \mathcal{L}_{f_1}(p_4) = -L_{f_1}(p_3) > 0 & \mathcal{L}_{f_2}(p_4) = -\mathcal{L}_{f_2}(p_3) < 0 \end{array}$$

Viability

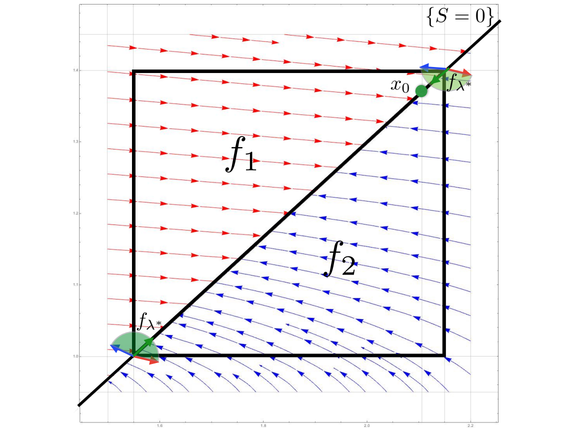
All first-order Lie derivatives are strictly positive on the faces.
This is checked in 12.22 seconds using our Matlab implementation.
Even points $K_{(1,3)} = (1.55, 1.0)$ and $K_{(2,4)} = (2.15, 1.4)$ are not part of the exit set

Therefore $K^S(F) = \emptyset$ and there exists a viable solution in any open set containing K.

State-dependent switching

Switching surface

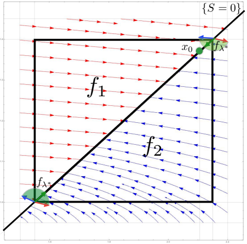
Same system with a switching surface S given by the affine function going through the corners (1.55, 1.0) and (2.15, 1.4)
I.e. $\{s(x) = y - ax - b = 0\}$ where $a = 2/3$ and $b = -1/30$, and f_1 is applied in $\{s(x) > 0\}$ and f_2 in $\{s(x) < 0\}$



State-dependent switching : example

Viability

- $K^S(F)$ is empty
- All the points of $\mathcal{S}_- \cap \partial K$ and $\mathcal{S}_+ \cap \partial K$ are entrant.
- For the corners located on $\mathcal{S}$, this is less trivial since f_1 and f_2 are both exiting. But there are values of λ for which flow f_λ is entrant.
- Actually, the surface $\mathcal{S}$ can be seen as a sliding surface, i.e. there exists a solution which stays indefinitely on it:
- the sliding strategy makes all the trajectories starting from K converge to an equilibrium point $(2.10648, 1.37098)$



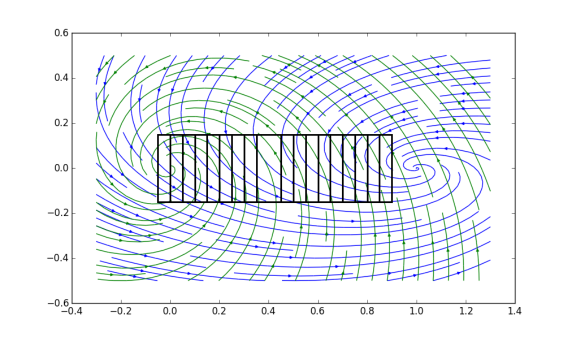
Defocused switched systems

System

$$f_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-\rho_A & -1/E\\ E & -\rho_A\end{pmatrix}\begin{pmatrix}x-x_c\\y-y_c\end{pmatrix} \quad (6)$$

$$f_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-\rho_B & -1\\ 1 & -\rho_B\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad (7)$$

We consider here $(x_c, y_c) = (cos(\phi), sin(\phi))$, $\rho_A = 0.5, \rho_B = 0.4, E = 0.5, \phi = 0$.



Defocused system : example

Template

We choose box templates centered on the x axis, defined by

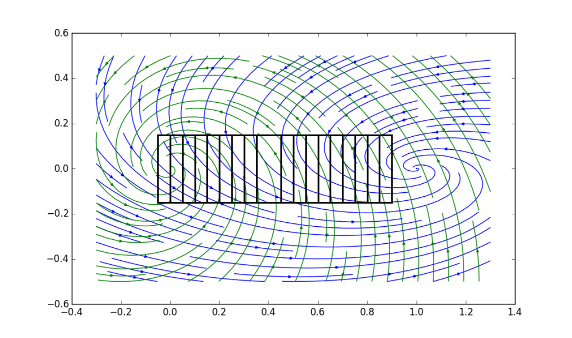
$$\begin{aligned} p_1 &= x \\ p_2 &= -x \\ p_3 &= y \\ p_4 &= -y \end{aligned}$$

and $c_1 = c_0 + \delta_1$, $c_2 = -c_0 + \delta_1$, $c_3 = c_4 = \delta_2$

Defocused system : example

Lie derivatives on the faces

Calculating the first-order Lie derivative for each dynamic and face, we see that the boxes have an empty exit set $K_S(F)$ the state of the system can be maintained inside any box using an appropriate switching law.

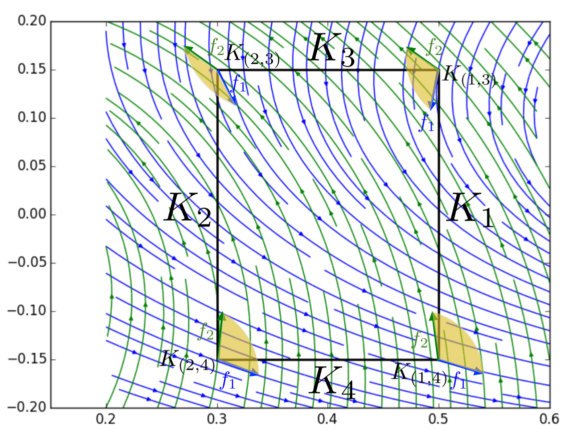


Box templates between the equilibrium points of the defocused switched system.

Defocused system : example

Controllability

For example the system can be controlled inside the box defined by $(c_0 = 0.4, \delta_1 = 0.1, \delta_2 = 0.15)$ and $(c_0 = 0.55, \delta_1 = 0.05, \delta_2 = 0.15)$, hence fairly accurately.



Disconnected exit sets

A 2D switched system

We consider the switched system defined by

$$f_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1+y^2\\y\end{pmatrix}\text{ and }f_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1-y^2\\y\end{pmatrix}$$

Template and Lie derivatives

We consider the template $p_1=-x$, $p_2=x$, $p_3=-y$, $p_4=y$ and $c_1=c_2=c_3=c_4=1$.
We deduce the closedness of the exit set, in 22 seconds in Matlab : $K^S(F)=[-1,1]\times\{-1\}\cup[-1,1]\times\{1\}$, which is disconnected.