Course 7: Homology, Co-homology and Motion Planning

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MPRI

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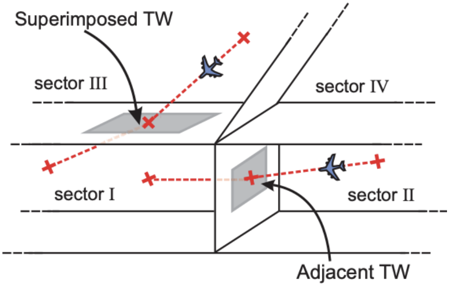
Context of this course

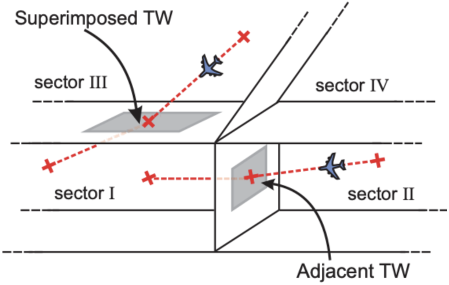
Motivation

- After having looked at unbounded-time (forward) properties of controlled systems, we turn to bounded-time backward (constrained) reachability
- With the essential (backward) "reach-avoid" properties



What does the space of controls/parameters that allow for reaching a target state without going through dangerous states look like?

Context of this course



Implications

- May be able to understand how to encode the relevant controls: only connected components? [switched control] "Dimension" of the control space? [how many discrete/continuous control parameters]
- Gives a natural notion of complexity of a reach-avoid task

In the second part of the course, we will develop another complexity measure: [directed] "topological complexity", also introducing some ideas from directed topology.

Method

- Can be done by backward set-based semantics, or pavings using forward set-based semantics (will get back to that in course 8, for neural networks!)
- Here we want to determine also the cohomological type of the (backward) reach-avoid set!
- Possible in some interesting cases using only forward set-based reachability.

We need first to introduce some cohomology theory!

To go further: practical algorithms for finding controls with reach-avoid properties in the case of NPC (non-positively curved) configuration spaces [distributed coordination],

See e.g. Ghrist, O'Kane, Lavalle, "Computing Pareto Optimal Coordinations on roadmaps", Springer Tracts in Advanced Robotics, 2005

Contents of the course

Cohomology

Singular Cochains

For a space X, define the cochain groups as

$$C^n(X) = \mathrm{Hom}(C_n(X), \mathbb{Z}).$$

Elements of $C^n(X)$ are called singular n-cochains.

Cochains behave like functions detecting how chains pass through regions of X.

Coboundary Operator

The coboundary $\delta^n : C^n(X) \to C^{n+1}(X)$ is defined by

$$(\delta^n \varphi)(\sigma) = \varphi(\partial_{n+1}\sigma).$$

This gives a cochain complex

$$0 \to C^0(X) \xrightarrow{\delta^0} C^1(X) \xrightarrow{\delta^1} C^2(X) \to \cdots$$

and satisfies $\delta^{n+1} \circ \delta^n = 0$.

Singular Cohomology

The n-th singular cohomology group is

$$H^n(X) = \ker(\delta^n)/\operatorname{im}(\delta^{n-1}).$$

Cohomology groups are contravariant: a map $f : X \to Y$ induces

$$f^* : H^n(Y) \to H^n(X).$$

Also, notion of relative cohomology groups $H^n(X; A)$ where $A \subseteq X$ (kill the cochains in A in the cochain complex)

There again cohomology groups are topological invariants (and weak topological invariants).

Motivation: Why More Than Homology?

Homology groups $H_*(X)$ measure holes, but:

- Many spaces have the same homology but are not homotopy equivalent.
- Homology groups forget how cycles interact.

Cohomology $H^*(X)$ has extra structure:

Key idea

The cup product equips cohomology with a graded ring structure.

This ring captures geometric intersection information invisible to homology.

Cup Product

Co-homology is not just a dual of homology: it has a finer structure:

Cohomology ring

Cohomology has a natural ring structure from the cup product:

$$\smile : H^p(X) \times H^q(X) \to H^{p+q}(X).$$

For cochains $\varphi \in C^p(X)$ and $\psi \in C^q(X)$,

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|[v_0, \ldots, v_p]) \cdot \psi(\sigma|[v_p, \ldots, v_{p+q}]).$$

This makes $H^*(X)$ a graded-commutative ring.

Basic examples

- S^n: $H^0(S^n) \cong \mathbb{Z}$ and $H^n(S^n) \cong \mathbb{Z}$ - trivial cup-product.
- Torus T^2: $H^*(T^2) \cong \Lambda(u, v)$ with u, v in degree 1 (free exterior algebra, $u \smile v = -v \smile u$ the only generator in degree 2, $u \smile u = 0$, $v \smile v = 0$).

Algebraic Properties

The cup product makes $H^*(X)$ into a graded ring:

- Graded commutativity:

$$\alpha \smile \beta = (-1)^{pq} \, \beta \smile \alpha$$

- Associativity
- Unit: $1 \in H^0(X)$

Thus $H^*(X)$ contains more information than the graded groups alone.

Cup-product in relative cohomology

Note that:

$$\smile : H^p(X; A) \times H^q(X; B) \to H^{p+q}(X; A \cup B).$$

Example: The Circle S^1

Cohomology groups:

$$H^0(S^1) = \mathbb{Z}, \quad H^1(S^1) = \mathbb{Z}$$

Cup product

Let $a \in H^1(S^1)$ be a generator.

Degree reasons force:

$$a \smile a = 0 \quad (\text{since } H^2(S^1) = 0)$$

So the cohomology ring is:

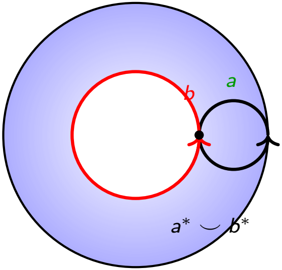
$$H^*(S^1) \cong \mathbb{Z}[a]/(a^2)$$

Example: Torus $T^2 = S^1 \times S^1$

Cohomology ring

Generators:

$$a, b \in H^1(T^2) \quad ; \quad H^*(T^2) = \mathbb{Z}[a,b]/(a^2, b^2, ab + ba)$$



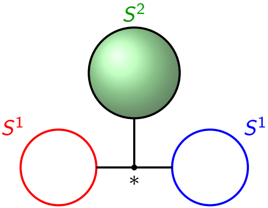
$$a^* \in H^1(T^2), \quad b^* \in H^1(T^2), \quad a^* \smile b^* \neq 0 \in H^2(T^2)$$

The product $a \smile b$ generates $H^2(T^2)$ and represents the blue surface of the torus (or intersection shown - by Poincaré duality).

Cup Product vs Homology

The torus and wedge $S^1 \vee S^1 \vee S^2$ have:

$$H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z}^2, \quad H_2 = \mathbb{Z}$$



But:

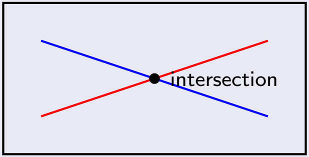
- Torus: $a \smile b \neq 0$
- Wedge: all cup products vanish

Same homology, different cohomology rings!

Intersection Interpretation

Via Poincaré duality (for manifolds):

$$\alpha \smile \beta \leftrightarrow [\alpha] \cap [\beta]$$



Cup products encode intersection numbers.

Topological manifolds

Definition

A topological manifold of dimension n is a Hausdorff topological space M such that every point $x \in M$ has an open neighborhood homeomorphic to $\mathbb{R}^n$.

Key features:

- Locally Euclidean
- No boundary (unless explicitly allowed)
- Global topology can be highly nontrivial

Examples: circles, surfaces, tori, projective spaces (all surfaces without boundaries we have been classifying in particular!).

Triangulated manifolds

Definition

A triangulated manifold is a topological manifold M together with a simplicial complex K and a homeomorphism

$$|K| \xrightarrow{\cong} M,$$

where the link of every simplex is homeomorphic to a sphere of the appropriate dimension.

Intuition:

- M is built from simplices glued along faces
- Local combinatorics model $\mathbb{R}^n$
- Enables algebraic tools (chains, cochains)

Triangulations allow homology, cohomology, and duality to be computed combinatorially.

Poincaré duality for closed manifolds

Homology measures holes. Cohomology measures functions on holes.

On a closed manifold, these two viewpoints are deeply related.

Poincaré Duality

On an oriented closed n-manifold, k-dimensional holes correspond to $(n-k)$-dimensional holes.

This correspondence is geometric: intersection.

Oriented Manifolds

Let M be a smooth or triangulated manifold of dimension n.

Orientation

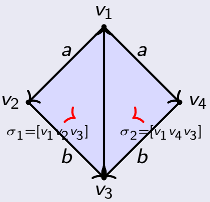
- M is oriented if it has a consistent choice of local orientation.
- Orientation determines a fundamental class

$$[M] \in H_n(M; \mathbb{Z}).$$

Intuitively, $[M]$ is the sum of all top-dimensional simplices with coherent signs.

Orientation class via a triangulation

Reminder: model for S^2



Orientation class

$$[M] \;=\; \sigma_1 - \sigma_2 \;\in\; C_2(M)$$

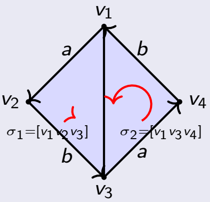
All top-dimensional simplices are given coherent orientations, so

$$\partial[M] = (b - [v1v3] + a) - (b + a - [v1v3]) = 0, \quad \text{and} \quad [M] \in H_2(M;\mathbb{Z})$$

is the fundamental (orientation) class.

Failure of a global orientation: sign mismatch

Reminder: model for $\mathbb{R}P^2$



$$\sigma_1 + \sigma_2 \;\notin\; Z_2(M), \qquad \partial(\sigma_1 + \sigma_2) = (b - [v1v3] + a) - (a + b + [v1v3]) \neq 0$$

The shared edge appears twice with the same sign.

Statement of Poincaré Duality

Theorem

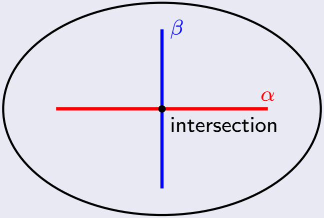
Let M be a closed, connected, oriented n-manifold. Then for every k:

$$H^k(M; R) \cong H_{n-k}(M; R)$$

for any coefficient ring R.

Geometric Intuition

Cohomology classes can be represented by *dual submanifolds*



Intersection numbers encode duality.

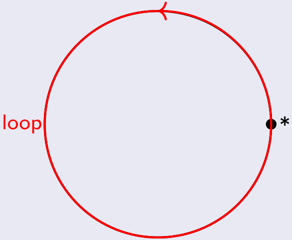
Example: The Circle S^1

S^1 is a closed oriented 1-manifold

$$H^0(S^1) = \mathbb{Z}, \quad H^1(S^1) = \mathbb{Z}$$

Poincaré duality gives:

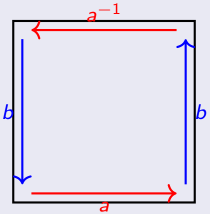
$$H^0 \cong H_1, \quad H^1 \cong H_0$$



Cup product on the Klein bottle

Cohomology groups: $H^0 = \mathbb{Z}, \quad H^1 = \mathbb{Z}, \quad H^2 = \mathbb{Z}/2$

(recall, homology groups: $H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z} \oplus \mathbb{Z}/2, \quad H_2 = 0$)
Let $a, b \in H^1$ correspond to the horizontal and vertical loops.



Degree reasons force all cup products in degree 2 to vanish. However, the torsion class in H^1 has nontrivial interactions mod 2: α generator of H^1, β generator of H^2 then $\alpha \cup \alpha = \beta$ (and the rest of cup products is 0).

A remark on practical computation of cohomology

Similar to homology, except for the ring structure, much more complex to determine.

Through linear algebra again

Also exact sequences

- Excision and long exact sequences hold as in homology,
- Universal Coefficient Theorem: relates $H^n(X)$ to $H_n(X)$,
- Relative exact sequence etc.

Setting up the scene: LTI control systems and obstacles

Consider a linear (possibly switched) control system first

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{1}$$

with $\mathbf{x}(0) = \mathbf{x}_*$, $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{u} \in P$ (P is a convex open subset of $U = \mathbb{R}^m$).

Solutions

For integrable $u : [0, T] \to \mathbb{R}^d$, trajectories $x = \varphi(x_*, u; .) : [0, T] \to \mathbb{R}^d$ such that Equation (1) holds almost everywhere.

Controllability

General controllability

The system given by Equation (1) is controllable if the evaluation map taking trajectories and sending them to its end points has an open range.

Necessary and sufficient condition for linear systems

$[B\ AB\ A^2B\ \dots\ A^{d-1}B]$ has rank d

Obstacles

Trajectories avoiding instantaneous obstacles

- Obstacles in $\mathbb{R}^d$, $\mathbf{O}_k$, $k = 1, \ldots, K$ "realized" at times $0 < t_1 < t_2 < \ldots < t_k < \ldots < t_K < T$,
- Notation: $\mathcal{O} = \{(\mathbf{O}_k, t_k) \mid k = 1, \ldots, K\}$.

We are interested in trajectories of the system avoiding the obstacles:

$$D_{\mathcal{O}} = \{(\mathbf{x}(.) : \ \mathbf{x}(t_k) \notin \mathbf{O}_k \text{ for } k = 1, \ldots, K\}$$

A similar (backwards) reach-avoid set

The set of initial conditions and controls avoiding obstacles is the solution to a backward reachability problem, the (backwards) reach-avoid set $R_{\mathcal{O}}$:

$$R_{\mathcal{O}} = \{(x_*, u) \in \mathbb{R}^d \times ([0, T] \to \mathbb{R}^d) \mid \varphi(x_*, u; t_k) \notin \mathbf{O}_k, \ \text{for } k = 1, \ldots, K\}$$

Control avoiding obstacles

Equivalently

This set of trajectories is homeomorphic to (using the flow map of the system of Equation (1)) to the set of corresponding initial conditions and controls.

As a consequence

- The homotopy type of the obstacle avoiding trajectories $D_{\mathcal{O}}$ is equal to the homotopy type of the set $R_{\mathcal{O}}$,
- Too complex: we are going to compute the cohomology of $D_{\mathcal{O}}$.

Applications:

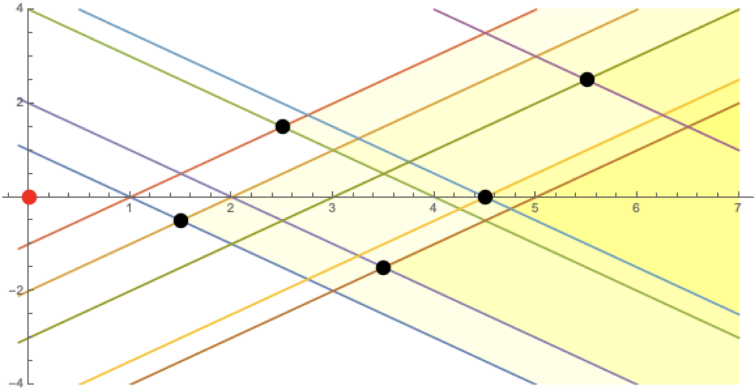
- understanding the structure of potential controllers (switched, number of parameters etc)
- complexity of controllers for a given motion planning problem,
- criteria for finding the space of controllers (subdivision of paving until correct homology is found - we will do something like it with neural networks).

For more details see e.g. Baryshnikov, "Linear obstacles in linear systems, and ways to avoid them", 2023

Example: a simple integrator

Simple integrator

$\dot{x} = u$ for $u \in \mathbb{R}$, $x(0) = 0$ and $|u| < 1$



Example: a simple integrator

Chains of obstacles

- Augment the collection of obstacles with $\mathbf{O}_0 = (0, x_*)$. A chain $k_0, \ldots, k_K$ is a sequence of (indexes of) obstacles (starting with $\mathbf{O}$) such that any two of them can be connected by a straight segment with slope between -1 and 1.
- These are the chains in the partial order of the obstacles with $k_i < k_{i+1}$ iff $|x_{k_{i+1}} - x_{k_i}| < t_{k_{i+1}} - t_{k_i}$.

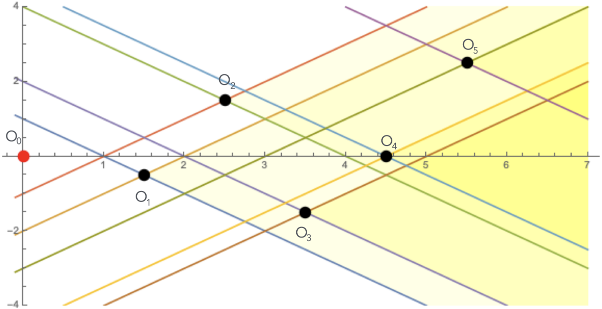
First result

The space of obstacle avoiding trajectories in this system is a disjoint union of finitely many contractible components, which are in one-to-one correspondence with the chains of the ordering $<$.

Example: a simple integrator

Simple integrator

$\dot{x} = u$ for $u \in \mathbb{R}$, $x(0) = 0$ and $|u| < 1$



Indeed in that example, we have 11 different chains.

$\mathbf{O}_0$, $\mathbf{O}_0 < \mathbf{O}_1$, $\mathbf{O}_0 < \mathbf{O}_2$, $\mathbf{O}_0 < \mathbf{O}_3$, $\mathbf{O}_0 < \mathbf{O}_4$, $\mathbf{O}_0 < \mathbf{O}_5$, $\mathbf{O}_0 < \mathbf{O}_1 < \mathbf{O}_3$, $\mathbf{O}_0 < \mathbf{O}_1 < \mathbf{O}_4$, $\mathbf{O}_0 < \mathbf{O}_1 < \mathbf{O}_5$, $\mathbf{O}_0 < \mathbf{O}_2 < \mathbf{O}_4$, $\mathbf{O}_0 < \mathbf{O}_2 < \mathbf{O}_5$.

Proof - for the example

Obstacle avoiding trajectories for the system form a finite collection of disjoint open convex subsets

- if two functions $x_1, x_2 \in D_{\mathcal{O}}$ are in the same path-connected component, their values at each of the obstacle times t_l are in the same path-connected component of the real line from which the obstacles at that time are removed,
- As these components are convex, the linear homotopy between the functions x_1 and x_2 will avoid the obstacles

Proposition

The space of obstacle avoiding trajectories in this system is a disjoint union of finitely many contractible components, which are in one-to-one correspondence with the chains of the ordering described above starting with $(0, x_*)$.

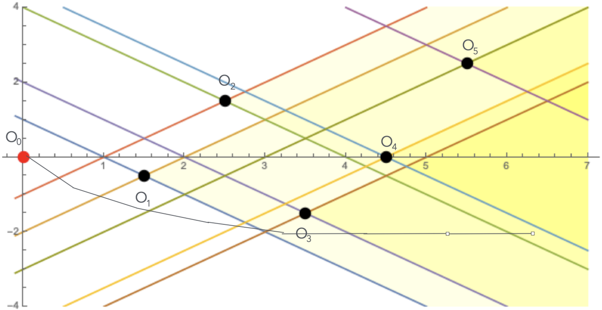
Proof - for the example

Proposition

The space of obstacle avoiding trajectories in this system is a disjoint union of finitely many contractible components, which are in one-to-one correspondence with the chains of the ordering described above starting with $(0, x_*)$.

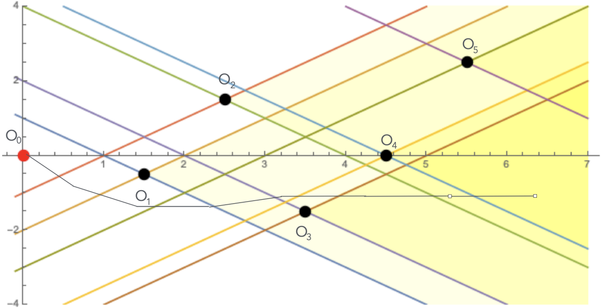
Proof

- Consider the decomposition of $D_{\mathcal{O}} = \coprod_{\gamma} D_{\mathcal{O}}(\gamma)$ into the disjoint union of open convex components indexed by γ,
- For each such component, the pointwise infimum of all trajectories in it is a piece-wise linear Lipschitz function $x_\gamma(t) = inf\ x \in D_{\mathcal{O}}(\gamma)\ x(t)$ passing through some collection of obstacles (including, necessarily, $O_0 = (0, x_*)$),
- Those of the obstacles near which x_γ is not locally a linear function, form, necessarily, a chain, which we will refer to as the marker of the component γ,
- This gives a mapping from the components of $D_{\mathcal{O}}$ to the chains.
- To associate to a chain a component, pick a small slack ϵ, and for a chain γ consider the function $x_\gamma(t) := \max_{k \in \gamma}(x_k + \epsilon - (1 - \epsilon)|t - t_k|)$.
- For small enough ϵ, this function will be in the component of $D_{\mathcal{O}}$ whose marker coincides with the chosen chain.



Indeed in that example, we have 11 different (maximal) chains.

O_0, $O_0 < O_1$, $O_0 < O_2$, $O_0 < O_3$, $O_0 < O_4$, $O_0 < O_5$, $O_0 < O_1 < O_3$, $O_0 < O_1 < O_4$, $O_0 < O_1 < O_5$, $O_0 < O_2 < O_4$, $O_0 < O_2 < O_5$.



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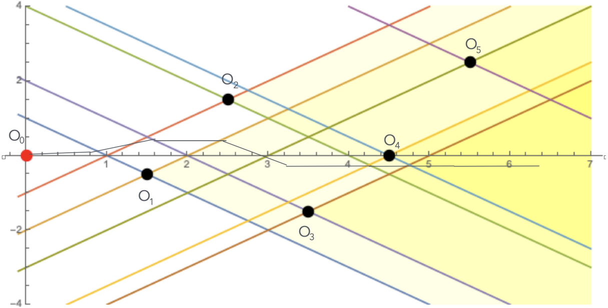

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More generally

For obstacles that are affine linear subspaces of $\mathbb{R}^d$

Avoidance classes

- For $\mathbf{O}$ an affine linear subspace of $\mathbb{R}^d$ of dimension l, $\mathbb{R}^d \backslash \mathbf{O}$ is homotopy equivalent to the $c = d - l - 1$-dimensional sphere S^c.
- The avoidance class $a(\mathbf{O})$ is the generator of $H^c(\mathbb{R}^d \backslash \mathbf{O})$.

(because of dimensionality issues, $a(\mathbf{O}) \smile a(\mathbf{O}) = 0$ unless $c = 0$ and then $a(\mathbf{O}) \smile a(\mathbf{O}) = a(\mathbf{O})$)

Notations

- Evaluating a trajectory in $D_{\mathcal{O}}$ at the obstacle instances t_k generates the evaluation mapping $e_k : \ D_{\mathcal{O}} \rightarrow V \backslash O_k$,
- Pull-backs of the avoidance classes to the space $D_{\mathcal{O}}$ result in the cohomology classes $\alpha_k = e_k^*(a(O_k)) \in H^{c_k}(D_{\mathcal{O}})$ (here $c_k = d - (dim\ O_k + 1)$ is the dimension of the avoidance class $a(O_k)$) - that we call also “avoidance classes”.

As in the linear example...

Chains of obstacles

An (ordered) sequence of obstacles $0 < O_{k_1} < O_{k_2} < \ldots < O_{k_L}$ forms a chain if there exists a trajectory of (1) starting at x_* and passing through all of the obstacles in the chain, for some admissible control $u(.)$ in P.

This is entirely determined by (forward) reachability properties of the differential system (1)!

Lemma

The collections of trajectories passing through a chain of obstacles c is a convex subset of the space of trajectories.

Viewed as a system of subsets of the collection of obstacles, the chains form, clearly, a simplicial complex (subset of a chain is a chain), which we will denote as $C_{\mathcal{O}}$.

Main result

Definition

Consider the graded ring generated by elements o_k of degree c_k (one generator for each obstacle; the degree equals the dimension of the corresponding avoidance class) and subject to the relations

- $o_k^2 = 0$ unless $c_k = 0$, in which case $o_k^2 = o_k$;
- $o_{k_1} \smile \ldots \smile o_{k_L} = 0$ unless the sequence of obstacles $\{O_{k_l}\}, k_1 < \ldots < k_L$ forms a chain.

This ring R_I depends only on the combinatorial data: the simplicial complex of chains $C_{\mathcal{O}}$ and dimensions $c_k = d - (dim\ O_k + 1)$.

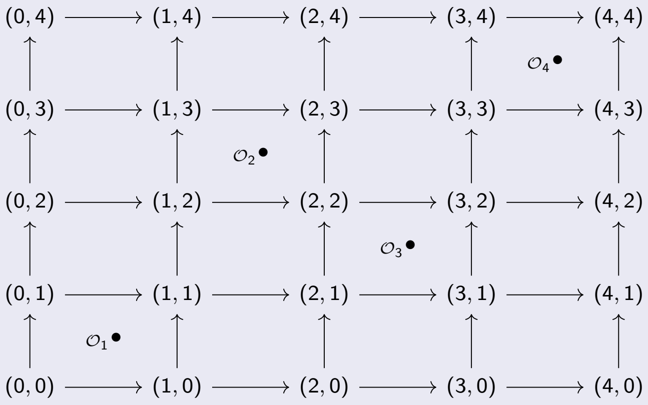
The ring R_O as an abelian group is freely generated by the products $o_{k_1} \ldots o_{k_L}$, where $O_{k_1} < \ldots < O_{k_L}$ is a chain.

Main theorem

The cohomology ring $H^*(D_{\mathcal{O}})$ is isomorphic to $R_{\mathcal{O}}$ under the homomorphism sending each o_k to the corresponding avoidance class.

A simple example in 2D

Missing points $\mathcal{O}_1$ to $\mathcal{O}_4$ in $[0,4] \times [0,4]$

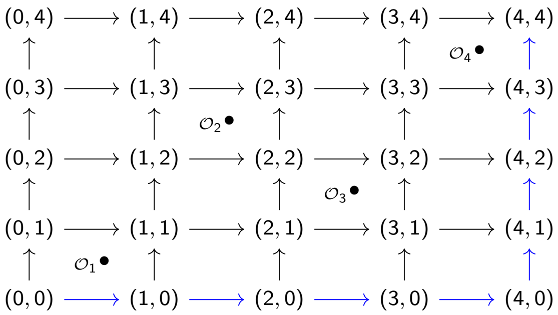


A simple example in 2D

0th cohomology classes of the backwards reach-avoid set between u, v in bijection with chains of obstacle $\mathcal{O}_i$ from u to v.

12 0th cohomology classes from $u = (0, 0)$ to $v = (5, 5)$

- $(u < v)$



A simple example in 2D

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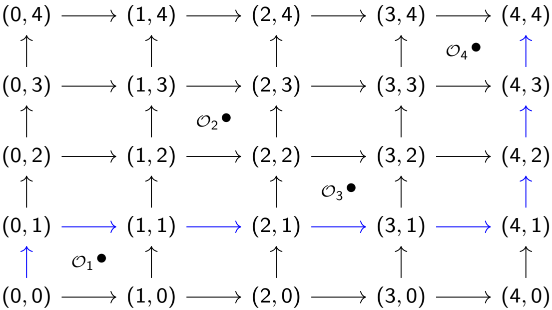

A simple example in 2D

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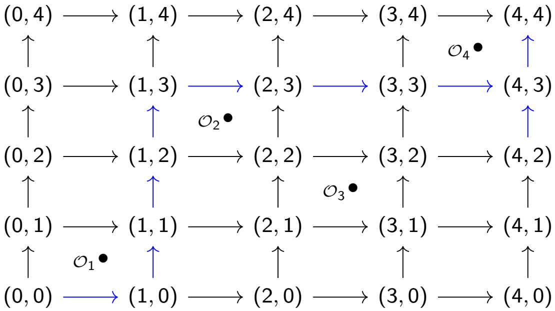

A simple example in 2D

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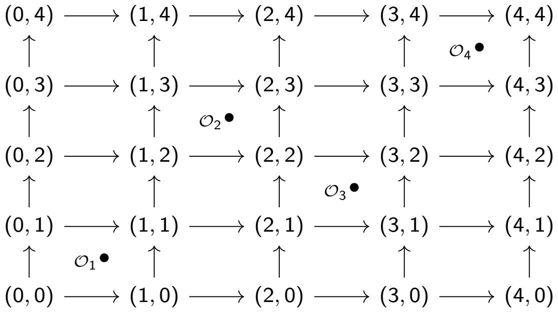
- $(u < v)$ $(u < \mathcal{O}_1 < v)$, $(u < \mathcal{O}_2 < v)$, $(u < \mathcal{O}_3 < v)$, $(u < \mathcal{O}_4 < v)$
- $(u < \mathcal{O}_1 < \mathcal{O}_2 < v)$, $(u < \mathcal{O}_1 < \mathcal{O}_3 < v)$, $(u < \mathcal{O}_1 < \mathcal{O}_4 < v)$, $(u < \mathcal{O}_2 < \mathcal{O}_4 < v)$, $(u < \mathcal{O}_3 < \mathcal{O}_4 < v)$,
- $(u < \mathcal{O}_1 < \mathcal{O}_2 < \mathcal{O}_4 < v)$, $(u < \mathcal{O}_1 < \mathcal{O}_3 < \mathcal{O}_4 < v)$

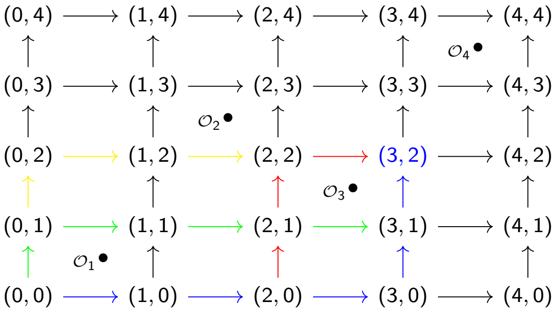


A simple example in 2D

The 0th cohomology classes between different pairs of points (u, v) are a subset of these classes, given by the same chain where they make sense.

E.g. 4 cohomology classes from $u = (0, 0)$ and $v = (3, 2)$

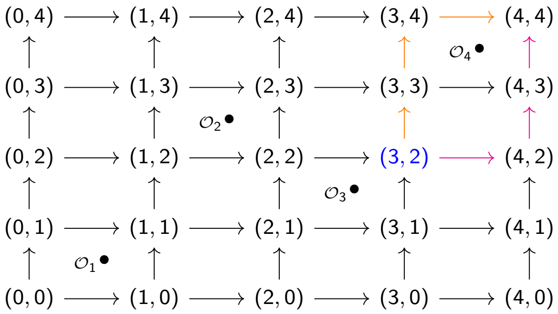
- $(u < v)$, $(u < \mathcal{O}_1 < v)$, $(u < \mathcal{O}_3 < v)$,
- $(u < \mathcal{O}_1 < \mathcal{O}_3 < v)$



A simple example in 2D

E.g. 3 cohomology classes from $u = (3, 2)$ to $v = (4, 4)$

- $u < v$,
- $u < \mathcal{O}_4 < v$



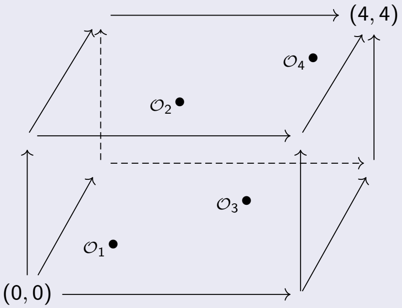
Cohomologically

As an application of our main result

- o_1, o_2, o_3 and o_4 generators of $H^0(D_{\mathcal{O}})$ ($d = 1$, $l = 0$, so $c = 1 - 0 - 1 = 0$), with $o_i^2 = o_i$, $i = 1, \ldots, 4$,
- extra generators in H^0, $o_1 \smile o_2$, $o_1 \smile o_3$, $o_1 \smile o_4$, $o_2 \smile o_4$ and $o_3 \smile o_4 \ldots$

A simple example in 3D

$\mathcal{O}_1 < \mathcal{O}_2, \mathcal{O}_3 < \mathcal{O}_4$ as before but in 3D



with the four points having coordinates $(1/2, 1/2, 1/2)$, $(3/2, 5/2, 3/2)$, $(5/2, 3/2, 5/2)$ and $(7/2, 7/2, 7/2)$.

A simple example in 3D

Avoidance classes (Barychnikov)

For each obstacle $\mathcal{O}_i$, $i = 1, \ldots, 4$, take "avoidance class" generator c_i of $H^1(\mathbb{R}^2 \backslash \mathcal{O}_i)$ ($\mathbb{R}^2 \backslash \mathcal{O}_i$ is homotopy equivalent to the 1-sphere S^1).

Cohomology ring of the trace space between u and v

Given by generators $c_1, \ldots, c_4$ with (Theorem 3.7 of Barychnikov):

- In dimension 0: 1 generator (corresponding to the empty chain of obstacles)
- $c_1, \ldots, c_4$ have degree 1
- $c_i \cup c_i = 0$ for $i = 1, \ldots, 4$
- $c_{i_1} \cup c_{i_2} \cup \ldots \cup c_{i_l} = 0$ unless the sequence $\mathcal{O}_{i_k}$ $k = 1, \ldots, l$ is a chain in the reachability ordering.

A simple example in 3D

Avoidance classes (Barychnikov)

For each obstacle $\mathcal{O}_i$, $i = 1, \ldots, 4$, take "avoidance class" generator c_i of $H^1(\mathbb{R}^2 \backslash \mathcal{O}_i)$ ($\mathbb{R}^2 \backslash \mathcal{O}_i$ is homotopy equivalent to the 1-sphere S^1).

In our case, cohomology ring for traces between $(0,0)$ and $(4,4)$

- In dimension 0: 1 generator (corresponding to the empty chain of obstacles)
- In dimension 1: 4 generators c_1, c_2, c_3, c_4,
- In dimension 2: 5 generators $c_1 \cup c_2$, $c_1 \cup c_3$, $c_1 \cup c_4$, $c_2 \cup c_4$, $c_3 \cup c_4$,
- In dimension 3: 2 generators $c_1 \cup c_2 \cup c_4$, $c_1 \cup c_3 \cup c_4$
- All the other cohomology rings are trivial.

A simple example in 3D

Avoidance classes (Barychnikov)

For each obstacle $\mathcal{O}_i$, $i = 1, \ldots, 4$, take "avoidance class" generator c_i of $H^1(\mathbb{R}^2 \backslash \mathcal{O}_i)$ ($\mathbb{R}^2 \backslash \mathcal{O}_i$ is homotopy equivalent to the 1-sphere S^1).

Changing the end points to $u = (0, 0)$ and $v = (2, 3, 2)$

We get only the cohomology classes which contain "reachable" generators $\mathcal{O}_1$ and $\mathcal{O}_3$:

- In dimension 0: 1 generator (corresponding to the empty chain of obstacles)
- In dimension 1: 2 generators c_1, c_3,
- In dimension 2: 1 generator $c_1 \cup c_3$

Indeed, the trace space between u and v can be shown to be homotopy equivalent to $S^1 \times S^1$, hence the resulting cohomology ring is the tensor product of (R, R) with itself, giving (R, R^2, R).

A simple example in 3D

Avoidance classes (Barychnikov)

For each obstacle $\mathcal{O}_i$, $i = 1, \ldots, 4$, take "avoidance class" generator c_i of $H^1(\mathbb{R}^2 \backslash \mathcal{O}_i)$ ($\mathbb{R}^2 \backslash \mathcal{O}_i$ is homotopy equivalent to the 1-sphere S^1).

Exercise: Changing the end points to $u = (2, 3, 2)$ and $v = (4, 4, 4)$

A simple example in 3D

Avoidance classes (Barychnikov)

For each obstacle $\mathcal{O}_i$, $i = 1, \ldots, 4$, take "avoidance class" generator c_i of $H^1(\mathbb{R}^2 \backslash \mathcal{O}_i)$ ($\mathbb{R}^2 \backslash \mathcal{O}_i$ is homotopy equivalent to the 1-sphere S^1).

Exercise: Changing the end points to $u = (2, 3, 2)$ and $v = (4, 4, 4)$

- In dimension 0: 1 generator (corresponding to the empty chain of obstacles)
- In dimension 1: 1 generator c_4,

A simple example in 3D

Avoidance classes (Barychnikov)

For each obstacle $\mathcal{O}_i$, $i = 1, \ldots, 4$, take "avoidance class" generator c_i of $H^1(\mathbb{R}^2 \backslash \mathcal{O}_i)$ ($\mathbb{R}^2 \backslash \mathcal{O}_i$ is homotopy equivalent to the 1-sphere S^1).

Exercise: from $u = (1, 1, 1)$ to $v = (2, 2, 2)$

A simple example in 3D

Avoidance classes (Barychnikov)

For each obstacle $\mathcal{O}_i$, $i = 1, \ldots, 4$, take "avoidance class" generator c_i of $H^1(\mathbb{R}^2 \backslash \mathcal{O}_i)$ ($\mathbb{R}^2 \backslash \mathcal{O}_i$ is homotopy equivalent to the 1-sphere S^1).

Exercise: from $u = (1, 1, 1)$ to $v = (2, 2, 2)$

- In dimension 0: 1 generator (the empty chain of obstacles)
- 2 generators (o_2 and o_3)

This corresponds to a wedge of circles S^1 (cohomology sees the difference from a torus)

Effect of inner and outer approximations of reachability properties

Outer-approximation of reachability

- We may have more reachability relations, hence more generators of the form $o_{k_1} \smile \ldots \smile o_{k_L}$ with $\{O_{k_l}\}$, $k_1 \overline{<} \ldots \overline{<} k_L$ ($\overline{<}$ is the reachability relation deduced from the outer-approximation)
- This gives an epi-morphism from $\overline{R}_{\mathcal{O}}$ (the ring $R_{\mathcal{O}}$ deduced from $\overline{<}$) to $R_{\mathcal{O}}$.

Inner-approximation of reachability

- We may have less reachability relations, hence less generators of the form $o_{k_1} \smile \ldots \smile o_{k_L}$ with $\{O_{k_l}\}$, $k_1 \underline{<} \ldots \underline{<} k_L$ ($\underline{<}$ is the reachability relation deduced from the inner-approximation)
- This gives an epi-morphism from $R_{\mathcal{O}}$ to $\underline{R}_{\mathcal{O}}$ (the ring $R_{\mathcal{O}}$ deduced from $\underline{<}$).

Overall:

$$\underline{R}_{\mathcal{O}} \twoheadleftarrow R_{\mathcal{O}} \twoheadleftarrow \overline{R}_{\mathcal{O}}$$

Consequences

Structure of the control

- Connected components of $R_{\mathcal{O}}$ may imply one switch control,
- The finer structure of components may imply some particular parametrization of the continuous part of the controller: i.e. if the cohomology is that of S^1, one variable in $\mathbb{R}$ (or even in S^1) is OK etc.

Complexity of controls for reach-avoid tasks

We will see in Course 8 a measure of complexity (decision trees) for deciding membership in a space X, where a lower bound is given by $\sum_{i\in\mathbb{N}} \log(\beta_i(X))$ ($\beta_i(X)$ is the ith Betti number of X.

in our 3D vs 2D ex: I $log(1) + log(5) + log(4) + log(2) = log(40) \gg log(1) + log(12) = log(12)$

But we are going to see a slightly different way of measuring control/motion planning complexity.

Extensions

Of this particular work

- The main theorem remains true for time-dependent linear systems $x = A(t)x + B(t)u$, as long as they remain controllable over the any time sub-interval containing more than one obstacle time.
- Similarly, can be generalized to a significantly broader class of the obstacles (including the non- instantenous ones).
- Well-know case of still obstacles for systems with orthant future cone (directed topology).

  See e.g. Raussen, Ziemiański " Homology of spaces of directed paths on Euclidean cubical complexes", Journal of Homotopy and Related Structures, 2013

Extensions

Of other instances in control

- Explicit computation of the backward reach-avoid set
  feasible by extensions of our set-based methods, e.g. Goubault, Mudiyanselage, Putot, "Solving quantified constraint satisfaction problems with applications to control", submitted 2025

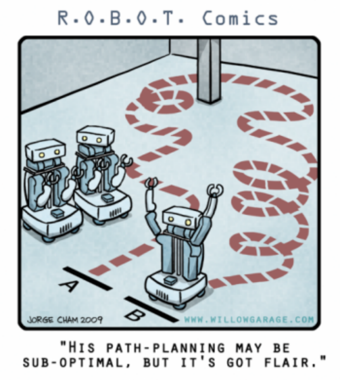
  and by Jacobi-Hamilton-Bellman methods, e.g. Fisac, Chen, Tomlin, Shankar Sastry, "Reach-avoid problems with time-varying dynamics, targets and constraints", HSCC, 2015
- Combinatorial search of the controller space

  see e.g. Pola, Girard, Tabuada, "Approximately bisimilar symbolic models for nonlinear control systems", 2008
- Much simple "coordination problems": several robots must coordinate their speed along predefined trajectories, so that they do not collide. Lots of non-positively curved geometry with simple algebraic topological contents (concentrated in π_0 and π_1), allows for finding practical controllers!

  see e.g. Ghrist, O'Kane, Lavalle, "Computing Pareto Optimal Coordinations on roadmaps", Springer Tracts in Advanced Robotics, 2005

Motivation for a measure of “topological” complexity

Characterize complexity of motion planning



Motion planning algorithm : how do I go from point A to point B?

Its complexity : how can I partition the space of points A, B so that to apply on each part a “simple” formula?

Classical case, as in e.g. M. Farber’s work

The configuration space is totally undirected (e.g. move an arm forward/backward)

Strong relations between topology and the “complexity” of motion planning algorithms.

Classical topological complexity

Path fibration map

$\chi : PX = X^{[0,1]} \to X \times X$ which associates to every continuous path p on X, its pair of start and end points. [this is a Serre fibration]

Topological complexity : definition

The topological complexity $TC(X)$ of a topological space X is [the Schwartz genus of fibration χ] the minimum number n (or ∞ if no such n exists) such that $X \times X$ can be covered by n open [resp. closed] subsets $F_1, \ldots, F_n$ such that there exists a map $s : X \times X \to PX$ (not necessarily continuous) with :

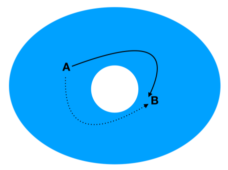
- $\chi \circ s = Id$ (s is a, non-necessarily continuous, section of χ)
- $s_{|F_i} : F_i \to PX$ is continuous

Why?

Topological complexity as a tool to understand the underlying geometry

Intuition

Somehow, $TC(X)$ is the minimal size of a partition of $X \times X$ such that on each part, the motion planning is "straightforward" (continuous)



Solid or dashed path?

Spaces of complexity 1

Are exactly the topological spaces which are contractible!

Topological complexity is an invariant of homotopy equivalence

More in e.g. Farber, "Invitation to topological robotics", 2008.

Exercise

Let X be a topological space. X is of complexity one iff X is contractible (i.e. homotope to a point).

Exercise

Let X be a topological space. X is of complexity one iff X is contractible (i.e. homotope to a point).

Hint:

Use $s \ : X \times X \rightarrow X^{[0,1]}$, continuous, to construct paths from all points B to some point $x_0 \in X$, continuous in B. This is a contracting homotopy.

Exercise

Let X be a topological space. X is of complexity one iff X is contractible (i.e. homotope to a point).

Proof

Suppose X is of complexity one. Then there exists $s \ : X \times X \rightarrow X^{[0,1]}$ continuous such that $\chi \circ s = Id$. Let $x_0 \in X$, we construct $H : X \times [0,1] \rightarrow X$ a homotopy between $f = Id : \ X \rightarrow X$ and $g = x_0 : \ X \rightarrow X$ the constant map $x_0(x) = x_0$:

$$H(x,t) = s(x,x_0)(t)$$

Conversely, suppose X is homotope to a point x_0 with H as a homotopy as before. Define $s : \ X \times X \rightarrow X^{[0,1]}$ to be $s(x,y) = H(x,.) * H(y,.)^{-1}$.

Alternate definition with ENRs

ENRs

A topological space X is a Euclidean Neighborhood Retract (ENR) if it can be embedded into a Euclidean space $X \subseteq \mathbb{R}^N$ such that for some open neighborhood $X \subseteq U \subseteq \mathbb{R}^N$ there exists a retraction $r: \; U \to X$, $r_{|X} = Id_X$.

A subset $X \subseteq \mathbb{R}^N$ is an ENR if and only if it is locally compact (in particular, all finite-dimensional cell complexes and all manifolds are ENR.

Topological complexity using ENRs

A motion planning algorithm $s: \; X \times X \to X^{[0,1]}$ is called tame if $X \times X$ can be split into finitely many sets $X \times X = F_1 \cup F_2 \cup F_3 \cup \ldots \cup F_k$ such that:

- The restriction $s_{|F_i}: \; F_i \to X^{[0,1]}$ is continuous for $i = 1, \ldots, k$,
- $F_i \cap F_j = \emptyset$ for $i \neq j$,
- Each F_i is an ENR.

The index k above coincides with the topological complexity defined before (when X is a smooth manifold, or more generally when X is homeomorphic to the geometric realization of a finite-dimensional simplicial complex).

Relative topological complexity

Definition

Let X be a topological space and $A \subseteq X \times X$ be a subspace.
Then the number $TC_X(A)$ is the smallest integer k such that:

- there is an open cover $U_1 \cup U_2 \cup \ldots \cup U_k = A$,
- with the property that each $U_i \subseteq A$ is open,
- and the projections $X \leftarrow U_i \rightarrow X$ on the first and the second factors are homotopic to each other, for each $i = 1, \ldots, k$.

Clearly $TC(X) = TC_X(X \times X)$.

Weights

Weight of a cohomology class

Let $u \in H^*(X \times X)$ be a cohomology class of $X \times X$.
We say that u has weight $(wgt(u))$ $k \geq 0$ if k is the largest integer with the property that for any open subset

$$A \subseteq X \times X \text{ with } TC_X(A) \leq k \text{ one has } u_{|A} = 0.$$

Then, obviously:

Proposition

If there exists a nonzero cohomology class $u \in H^*(X \times X)$ with $wgt(u) \geq k$, then $TC(X) > k$.

A useful lemma

Lemma

For $u \in H^*(X \times X)$ one has $wgt(u) \geq 1$ if and only if $u_{|\Delta X} = 0 \in H^*(X)$.
Here $\Delta X \subseteq X \times X$ denotes the diagonal.

Such cohomology classes u are called zero-divisors.

Proof

Comes directly from the following obvious characterization of relative topological complexity:
For a subset $A \subseteq X \times X$ the following properties are equivalent:

1. $TC_X(A) = 1$,
2. the projections $X \leftarrow A \rightarrow X$ are homotopic,
3. the inclusion $A \rightarrow X \times X$ is homotopic to a map $A \rightarrow X \times X$ with values in the diagonal $\Delta X \subseteq X \times X$.

Weight and cup-products

Lemma

Let $u \in H^n(X \times X)$ and $v \in H^m(X \times X)$ be two cohomology classes. Then the weight of their cup product $u \smile v \in H^{n+m}(X \times X)$ satisfies

$$wgt(u \smile v) \geq wgt(u) + wgt(v).$$

Proof

- $r = wgt(u)$, $s = wgt(v)$,
- Any open $A \subseteq X \times X$ with $TC_X(A) \leq r + s$ is $A = B \cup C$ where $B, C \subseteq X \times X$ are open subsets with $TC_X(B) \leq r$ and $TC_X(C) \leq s$,
- Then $u_{|B} = 0$ and hence there exists $u' \in H^n(X \times X, B)$ with $u'_{|(X \times X)} = u$,
- Similarly, there exists $v' \in H^m(X \times X, C)$ with $v'_{|(X \times X)} = v$,
- Then $u' \smile v' \in H^{n+m}(X \times X, A)$ satisfies $(u' \smile v')_{|A} = 0 = (u \smile v)_{|A}$.

Zero-divisors and cup-product

If the cup-product of k zero-divisors $u_i \in H^*(X \times X; R_i)$, $i = 1, \ldots, k$ is nonzero, then

$$TC(X) > k$$

.

Proof

- Lemma before and def of zero-divisors: $wgt(u_1 \smile \ldots \smile u_k) \geq \sum_{i=1}^{k} wgt(u_i) \geq k$,
- By Proposition before, implies $TC(X) > 1$.

Applications

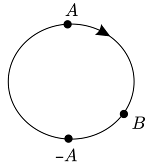
To the spheres

One has $TC(S^n) = 2$ for n odd and $TC(S^n) = 3$ for n even.

Proof

First, observe:

- $TC(S^n) \leq 2$ for n odd:
  - $F_1 \subseteq S^n \times S^n$ be the set of all pairs (a, b) such that $a \neq -b$
  - local section $s_1 : \ F_1 \to (S^n)^{[0,1]}$ by moving a towards b along the shortest geodesic arc.
  - $F_2 \subseteq S^n \times S^n$ of antipodal points $(a, -a)$.
  - $s_2 : \ F_2 \to (S^n)^{[0,1]}$ as, given non-vanishing tangent vector field v on S^n; Move a towards $-a$ along the semi-circle tangent to $v(a)$.



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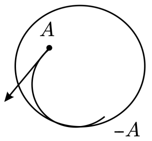


Applications

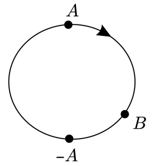
To the spheres

One has $TC(S^n) = 2$ for n odd and $TC(S^n) = 3$ for n even.

Proof

First, observe:

- $TC(S^n) \leq 3$ for n even:
  - F_1 and s_1 as before,
  - F_2 and s_2 as before except for $(a_0, -a_0)$ (a_0 the only point at which v vanishes), then set $F_3 = \{(a_0, -a_0)\}$ with s_3 arbitrary path.



Applications

To the spheres

One has $TC(S^n) = 2$ for n odd and $TC(S^n) = 3$ for n even.

Proof

Then:

- Let $u \in H^n(S^n)$ denote the fundamental [orientation] class,
- Then $\overline{u} = 1 \times u - u \times 1 \in H^n(S^n \times S^n)$ is a nonzero zero-divisor and $\overline{u}^2 = u \smile u = -[1 + (-1)^n].u \times u$.
- Thus $\overline{u}^2$ nonzero for n even; by Lemma above, $TC(S^n) \geq 3$ for n even.
- Similarly, Corollary above implies that $TC(S^n) \geq 2$ for n odd.

The case of digraphs

Proposition

If X is a connected finite graph then

$$TC(X) = \begin{cases} 1 & \text{if } b_1(X) = 0, \\ 2 & \text{if } b_1(X) = 1, \\ 3 & \text{if } b_1(X) > 1. \end{cases}$$

Proof

- If $b_1(X) = 0$, then X is contractible and hence $TC(X) = 1$,
- If $b_1(X) = 1$, then X is homotopy equivalent to the circle and therefore $TC(X) = TC(S^1) = 2$ (see above),
- Assume $b_1(X) > 1$. Then there exist two linearly independent classes $u_1, u_2 \in H^1(X)$. Thus $\overline{u_i} = 1 \times u_i - u_i \times 1$, $i = 1, 2$ are zero-divisors and their product $u_2 \times u_1 - u_1 \times u_2 \neq 0$ is nonzero.
- This implies $TC(X) \geq 3$ by Corollary above,
- On the other hand, we know that $TC(X) \leq 3$ (special case of the fact that for any polyhedral complex, $TC(X) \leq 2dim(X) + 1$ - exercise), therefore, $TC(X) = 3$.

Let Σ_g denote a closed orientable surface of genus g. Then

$$TC(\Sigma_g) = \begin{cases} 3 & \text{for } g = 0, \text{ or } g = 1, \\ 5 & \text{for } g \geq 2 \end{cases}$$

Proof

- $g = 0$: already seen (sphere S^2),
- $g = 1$: torus - exercise!
- $g \geq 2$: we have cohomology classes $u_1, v_1, u_2, v_2 \in H^1(\Sigma_g)$ such that $u_i \smile u_j = u_i \smile v_j = v_i \smile v_j = 0$ for $i \neq j$ and $u_i^2 = 0$, $v_i^2 = 0$, and, $u_1 \smile v_1 = u_2 \smile v_2 = a \in H^2(\Sigma_g)$ is the fundamental [orientation] class,
- Then, $\overline{u}_1 \smile \overline{u}_2 \smile \overline{v}_1 \smile \overline{v}_2 = -2a \times a \neq 0$.

Hence, the product of four zero-divisors is nonzero and $TC(\Sigma_g) \geq 5$

(inverse inequality $TC(\Sigma_g) \leq 5$ - exercise)

Motivation 1

Our aim : a generalization to controlled systems

In general, movement is constrained by physically feasible controls. E.g. the dynamics is given by ODEs/switched system/... of the form :

$$\frac{dx}{dt}(t) = f(x(t), u(t), t)$$

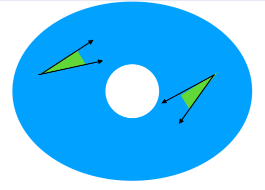
where $t \to u(t)$ is the control (thrust, orientation etc.) which is constrained (in magnitude, gradient etc.).

Intrinsic complexity of the motion planning algorithm under constraints!

The set of constraints on controls makes us consider general differential inclusions :

$$\frac{dx}{dt}(t) \in F(x(t), t)$$

where F is a set valued function.



Long term goal : should measure the complexity gap between local (Pontryagin) and global (HJB) methods in optimal control.

Motivation 2

Take another view on directed topological invariants

- Lots of ideas out there, but none definitive
- In particular concerning the "algebraic contents" of dihomotopy (in the sense of e.g. closed model categories etc.)

Directed topological complexity...

...seen here as a good litmus test for dihomotopy equivalences, (in the classical case, topological complexity is a homotopy invariant) and as a good application for directed homological theories.

Also, this applies to concurrency and distributed systems theory (sections are schedulers)!

Quick recap on directed algebraic topology

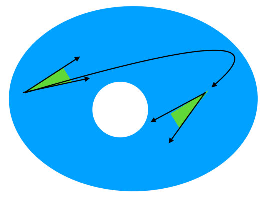
Definition of general directed spaces [M. Grandis]

A directed topological space, or d-space $X = (X, PX)$ is a topological space equipped with a set PX of continuous maps ("directed paths") $p : I \to X$ ($I = [0, 1]$)

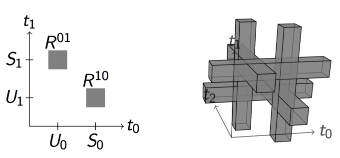
- every constant map $I \to X$ is directed
- PX closed under composition with continuous and non-decreasing maps from I to I
- PX closed under concatenation

(others from Fajstrup-Goubault-Raussen, Krishnan etc.)

Examples



From differential inclusions



From concurrency/distributed systems (complexity of scheduling/task solvability), graphs$\cdots$

From differential inclusions to d-spaces

Solutions of differential inclusions

- Differential inclusions we consider are

$$\dot{x} \in F(x) \tag{2}$$

  where F is a map from $\mathbb{R}^n$ to $\wp(R^n)$, the set of subsets of $\mathbb{R}^n$.
- A function $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is a solution of Equation (2) if x is an absolutely continuous function and satisfies for almost all $t \in \mathbb{R}$, $\dot{x}(t) \in F(x(t))$
- In general, there can be many solutions to a differential inclusion.

From differential inclusions to d-spaces

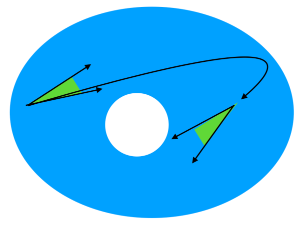
Existence of (local) solutions

When e.g. the set-valued map $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is an upper hemicontinuous function of x, measurable in t, and such that $F(t,x)$ is a closed, convex set for all t and x.

Classically, F is the closed convex hull of continuous Lipschitz maps.

Directed space produced

- $S(F,X)$ = set (p,t) of solutions p over $[0,t]$
- Concatenation operation *, for (p,u) and (q,v) in $S(F,X)$
- For $(p,u) \in S(F,X)$, $\tilde{p}$ is the map from I to X defined by $\tilde{p}(t) = p(ut)$.
- Define now dX to be the set $\{\tilde{p} \circ \phi \mid (p,u) \in S(F,X)$, $\phi : I \to I$ non-decreasing$\}$.



Directed topological complexity

Dipath space map (replacing the path fibration map)

We define the dipath space map
$\chi : PX \to \Gamma_X = \{(x, y) \mid \exists p \in PX,\ p(0) = x,\ p(1) = y\ \} \subseteq X \times X$ of X by
$\chi(p) = (p(0), p(1))$ for $p \in PX$.

Directed topological complexity

The directed topological complexity $\overrightarrow{TC}(X)$ of a d-space X is the minimum number n (or ∞ if no such n exists) such that Γ_X can be partitioned into n sets (ENRs in our cases) $F_1, \ldots, F_n$ such that there exists a map $s : \Gamma_X \to PX$ (not necessarily continuous) with :

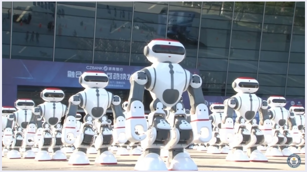
- $\chi \circ s = Id$
- $s_{|F_i} : F_i \to PX$ is continuous

A collection of such ENRs $F_1, \ldots, F_n$ with n equal to the directed topological complexity of X is called a patchwork.

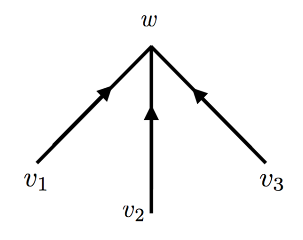
Variations (outside the scope of this course)

"Consensus" directed topological complexity

Gathering robots [link to distributed computing]



Replace the space of dipaths $PX = X^{\overrightarrow{[0,1]}}$ by X^K with K being :

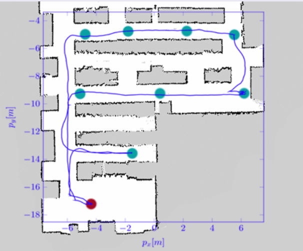


and Γ_X by some subspace of $X \times X \cdots \times X$.

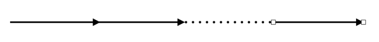
Variations (outside the scope of this course)

Higher directed topological complexity

Motion planning with a certain number of waypoints.



Replace the space of dipaths $PX = X^{\overrightarrow{[0,1]}}$ by X^K with K being :



and Γ_X by some subspace of $X \times X \cdots \times X$.

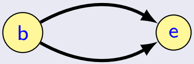
Comparison with classical topological complexity

Other equivalent definitions, in the classical case

In the directed case, it is different to consider partitions into ENRs and coverings by open sets.

Example : the directed circle

$\overrightarrow{\mathbb{O}^1}$ the directed circle :



- Consider a covering C by open sets of the pairs of reachable points of $\overrightarrow{\mathbb{O}^1}$ s. t. there exists a continuous section of χ on each open set.

Comparison with classical topological complexity

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In the directed case, it is different to consider partitions into ENRs and coverings by open sets.

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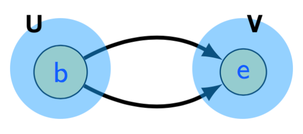

- Consider a covering C by open sets of the pairs of reachable points of $\overrightarrow{\mathbb{O}^1}$ s. t. there exists a continuous section of χ on each open set.
- Let O an open subset of $\Gamma\overrightarrow{\mathbb{O}^1}$ in C, containing (b, e) : $U \ni b$, $V \ni e$ open sets of $\mathbb{O}^1$ s. t. $U \times V \in O$.
- Impossible to find a continuous section of χ on $U \times V$ (we will come back to this a bit later).

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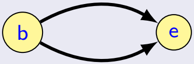


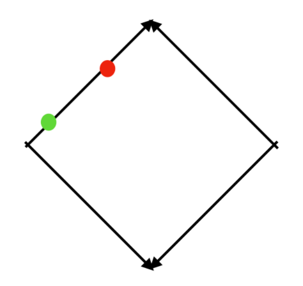
$\chi \; : P(\mathbb{O}^1) \rightarrow \Gamma_{\mathbb{O}^1}$ is not a fibration ($\chi^{-1}(b, e)$ is S^0, χ^{-1} of all other pairs of points is one point).

Also, in general, one cannot compare $TC(X)$ and $\overrightarrow{TC}(X)$, except in some cases...

Comparison with classical topological complexity

The problem is that we cannot in general use a partition for Γ_X as a basis for a partition for $X \times X$ (and vice-versa)

Ex. : a non trivial graph with directed complexity 1



$\overrightarrow{TC}(G) = 1$! but classical TC is 2.

Comparison with classical topological complexity

In some cases though, we can!

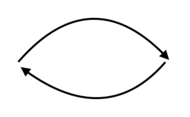
Strongly-connected directed spaces

We will say that a d-space X is strongly connected if for all (x, y) in $X \times X$, there exists $\gamma \in PX$ with $\gamma(0) = x$, $\gamma(1) = y$, and there exists $\delta \in PX$ with $\delta(0) = y$, $\delta(1) = x$.

For any strongly connected d-space X, $TC(X) \leq \overrightarrow{TC}(X)$

- Strong connectedness implies that $\Gamma_X = X \times X$
- Given a set of patchworks for X, this provides a covering by ENRs with a local section into the (di)path space.

Example : case of $\overrightarrow{\mathbb{S}^1}$



Ordinary topological complexity is 2, so $\overrightarrow{TC}(\overrightarrow{\mathbb{S}^1}) \leq 2$ (get back to that later)

Directed spaces with a distinguished point

E.g. with initial point

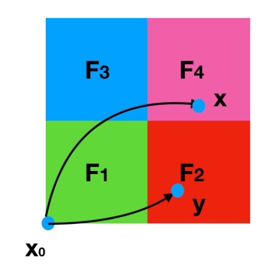
$x_O \in X$ is an initial point if for all $x \in X$, $(x_0, x) \in \Gamma_X$

Lower bound for directed topological complexity

If X [is regular] and has an initial point,

$$TC(X) \leq 2\overrightarrow{TC}(X) - 1$$

- Consider the map $\chi' : PX(x_0, -) \to X$ taking the end of dipaths and $X = F_1 \cup \cdots \cup F_k$ with continuous sections $s_i : F_i \to PX(x_0, -)$. Necessarily $k \leq \overrightarrow{TC}(X)$ (as if $\Gamma_X = G_1 \cup \cdots \cup G_l$, for all (x_0, x), there is some G_i which contains it).
- Consider $G_{i,j} = F_i \times F_j$ and section $t_{i,j} : G_{i,j} \to X^{[0,1]}$ of the path space fibration, by $t_{i,j}(x, y) = s_i(x)^{-1} * s_j(y)$
- Take as partition of ENRs for $X \times X$, $G_m = \bigcup_{i+j=m+1} G_{i,j}$ [use regularity]



Classical topological complexity of graphs

[Farber]

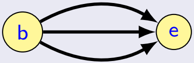
There are three cases :

- $b_1(G) = 0$ (contractible) then G is a point and $TC(G) = 1$
- $b_1(G) = 1$ (one cycle) then $TC(G) = 2$
- $b_1(G) = 2$ (two independent cycles) then $TC(G) = 3$

(pretty weak invariant of homotopy equivalence : graphs are all homotopy equivalent to a bouquet of circles)

Comparison with classical topological complexity

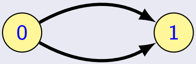
Example with $\overrightarrow{TC}(G) < TC(G)$



- Patchwork for Γ_G : $E_1 = (a, b)$ and $E_2 = \Gamma_G \backslash E_1$. We thus have $\overrightarrow{TC}(G) = 2$ (there again, easy to prove there is no global section)
- But $TC(G) = 3$ [Farber]

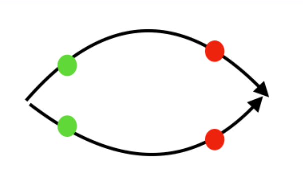
Example of directed complexity of graphs

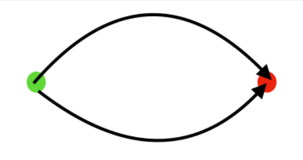
Case of $\overrightarrow{\mathbb{O}^1}$



$\overrightarrow{TC}(\overrightarrow{\mathbb{O}^1}) = 2$ (why is it impossible to find a global section?)

Partition, for instance,

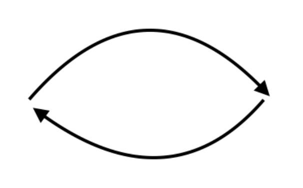




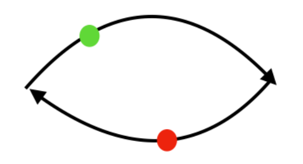
[Other partitions possible]
Can use $TC(\overrightarrow{\mathbb{O}^1}) = 2 \leq 2\overrightarrow{TC}(\overrightarrow{\mathbb{O}^1}) - 1$ for instance - $\overrightarrow{\mathbb{O}^1}$ has an initial point

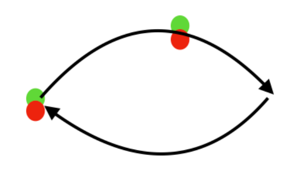
Example of directed complexity of graphs

Case of $\overrightarrow{\mathbb{S}^1}$



Directed TC is 2





[At least this shows again ≤ 2, we will see later [directed homology] that it is not 1]

Directed complexity of directed graphs

Claim : for general graphs

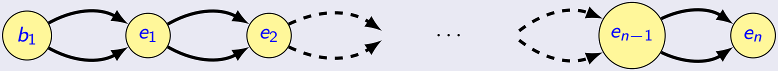
We have $\overrightarrow{TC}(G) \leq 3$.

Principle

Take as a partition in ENRs the following :

- $F_1 = \{(x, y) \in \Gamma_G | x, y \text{ vertices }\}$
- $F_2 = \{(x, e), (e, x) \in \Gamma_G | x \text{ vertex and } e \text{ within an edge}\}$
- $F_3 = \{(e, f) \in \Gamma_G | e, f \text{ within edges}\}$

Example : $X = \overrightarrow{\mathbb{S}^{1 \star n}}$ - $\overrightarrow{TC}(X) = 2$



Directed complexity of directed graphs

Claim : for general graphs

We have $\overrightarrow{TC}(G) \leq 3$.

Principle

Take as a partition in ENRs the following :

- $F_1 = \{(x,y) \in \Gamma_G | x, y \text{ vertices } \}$
- $F_2 = \{(x,e),(e,x) \in \Gamma_G | x \text{ vertex and } e \text{ within an edge}\}$
- $F_3 = \{(e,f) \in \Gamma_G | e, f \text{ within edges}\}$

Example : $X = \overrightarrow{\mathbb{S}}^{1 \star n}$ - $\overrightarrow{TC}(X) = 2$

Section : always try to go by the upper arrows first

The case of strongly connected directed graphs

Same as for classical topological complexity!

Let G be a strongly connected directed graph. Then

$$\overrightarrow{TC}(G) = TC(G) = \min(b_1(G), 2) + 1.$$

($b_1(G)$ is the first Betti number of the graph G).

Sketch of proof

By [M. Farber], $TC(G) = \min(b_1(G), 2) + 1$. As G is strongly connected, $\overrightarrow{TC}(G) \geq TC(G) = \min(b_1(G), 2) + 1$, and we have 3 cases :

- $b_1(G) = 0$. Since G is contractible and strongly connected, G must be a single point : $\overrightarrow{TC}(G) = 1$.
- $b_1(G) = 1$. G must be a cycle. Already seen that $\overrightarrow{TC}(G) = 2$ in our examples before.
- $b_1(G) \geq 2$. Then $TC(G) = 3$ and hence $\overrightarrow{TC}(G) \geq 3$. On the other hand, $\overrightarrow{TC}(G) \leq 3$ for general graphs. Thus $\overrightarrow{TC}(G) = 3$.

Product spaces

Or, can we go to higher directed topological complexities?

Product of d-spaces : $(X, PX) \times (Y, PY) = (X \times Y, PX \times PY)$.

Similarly to the classical case

$\overrightarrow{TC}(X \times Y) \leq \overrightarrow{TC}(X) + \overrightarrow{TC}(Y) - 1$ (for "regular d-spaces X and Y - details omitted)

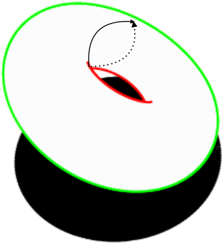
Sketch of proof

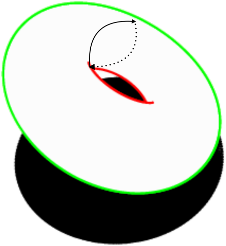
- $\Gamma_{X \times Y} = \Gamma_X \times \Gamma_Y$
- If $\Gamma_X = F_1 \cup \cdots \cup F_k$ and $\Gamma_Y = G_1 \cup \cdots \cup G_l$, with continuous sections $s_i \ : F_i \rightarrow PX$, $t_j \ : G_j \rightarrow PY$, then

$$\Gamma_{X \times Y} = \bigcup_{i=1\cdots k, j=1\cdots l} F_i \times G_j$$

with $s_i \times t_j \ : F_i \times G_j \rightarrow P(X \times Y) = PX \times PY$ continuous
- Take as final partition of ENRs for $\Gamma_{X \times Y}$, $H_m = \bigcup_{i+j=m+1} F_i \times G_j$ ("regularity" is used here)

Application





N-tori $\overrightarrow{\mathbb{O}^1}^n$

$\frac{n}{2} + 1 \leq \overrightarrow{TC}(\overrightarrow{\mathbb{O}^1}^n) \leq n + 1$ (use distinguished point)

N-tori $\overrightarrow{\mathbb{S}^1}^n$

$\overrightarrow{TC}(\overrightarrow{\mathbb{S}^1}^n) = n + 1$ (use strong-directness)

(two essentially different d-spaces)

A general upper bound

Classical case

For a cellular complex, $TC(X) \leq 2\ dim(X) + 1$

Claim : in the directed case

- Not clear in general, except for some restricted class of cubical complexes (for which reachability within closure of cells is determined locally)
- This seems to be the case for "geometric" precubical sets (Fajstrup), or cubical complexes in the sense of (Goubault and Mimram 2016).

The classical topological complexity of spheres

Result (M. Farber)

The topological complexity of $\mathbb{S}^n$ is :

- 2 if n is odd
- 3 if n is even

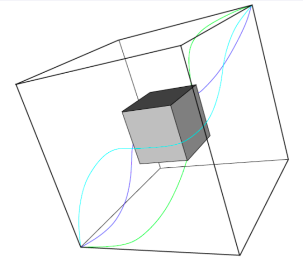
Principle of the proof

- No $\mathbb{S}^n$ is contractible : $TC(\mathbb{S}^n) \geq 2$.
- Suppose n is odd, and take a never vanishing vector field f on $\mathbb{S}^n$. Set $F_1 = \{(x, y) | x, y \in \mathbb{S}^n, \text{ not antipodal}\}$ and $F_2 = \{(x, y) | x, y \in \mathbb{S}^n \text{ antipodal}\}$. Take sections from F_1 to $X \times X$ to be the shortest path linking x with y. For linking points x to y such that $(x, y) \in F_2$, take the unique geodesic tangent at x to f.
- Suppose n even ; take f which has just one zero. Take F_1, the set of non antipodal points on $\mathbb{S}^n$, F_2 the set of antipodal points, except for the pair containing (x_0, y_0) with y_0 antipodal to x_0, and $F_3 = \{(x_0, y_0)\}$.

Directed spheres

"The" directed sphere of dimension n

- Let $\square^n$ = d-space, cartesian product of n copies of the unit segment with the d-structure generated by the standard ordering on $[0, 1]$.
- Directed sphere of dimension n the d-space $\overrightarrow{\mathbb{O}^n}$ = boundary $\partial\square^{n+1}$.



No global section

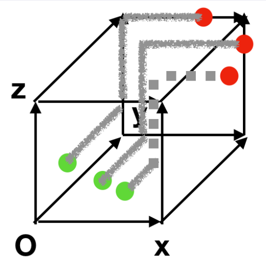
Same type of argument as for $\overrightarrow{\mathbb{O}^1}$: $\Gamma_{\overrightarrow{\mathbb{O}^n}}$ is locally contractible, $PX(U, V)$ for U convex neighborhood of 0, V convex neighborhood of 1 is homotopy equivalent to S^{n-1}.

But $\overrightarrow{TC}(\overrightarrow{\mathbb{O}^n})$ is always 2 for $n \geq 1$! (M. Grant and A. Borat)

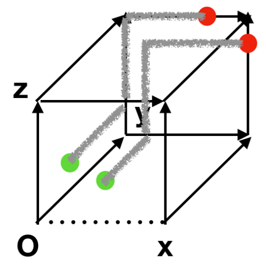
Main idea

Patchwork of cardinality 2?

Idea : correct difference between x, y, z coordinates from start to end, in some predefinite order : here, correct y before z before x, this is continuous



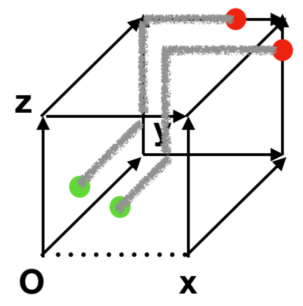
But not defined on the whole of $\overrightarrow{\mathbb{O}}^2$!



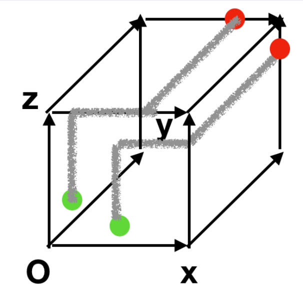
Leave out 1 face (front one)

A patchwork of cardinality 2 for $\overrightarrow{\mathbb{O}^2}$

Idea of the construction



=space X_1 (section “first y then z then x”)



=space X_2 (section “first z then x then y”)

For $(x, y) \in \Gamma_{\overrightarrow{\mathbb{O}^2}}$, either $(x, y) \in \Gamma_{X_1}$ or $(x, y) \in \Gamma_{X_2}$ - get a partition of ENRs out of it.

(slightly different motion planner and partition in [A. Borat & M. Grant, 2019], but equivalent in spirit)