

MPRI
PRFSYS

Oct. 14th

Proofs as Objects of the Formalism
or
Dependent Types

Adding sum and product types to $\lambda \rightarrow$

$$T ::= a \mid T \rightarrow T \mid T \times T \mid T + T$$

$$t ::= x \mid \lambda x.t \mid t t \mid (t, t) \mid \pi_1(t) \mid \pi_2(t) \mid i(t) \mid j(t) \mid \delta(t, x.t, x.t)$$

$$\lambda x.t \ u \triangleright_{\beta} t[x \setminus u]$$

$$\pi_1(t, u) \triangleright_{\beta} t$$

$$\pi_2(t, u) \triangleright_{\beta} u$$

$$\frac{\vdash t : A \quad \vdash u : B}{\vdash (t, u) : A \times B}$$

$$\frac{\vdash t : A \times B}{\vdash \pi_1(t) : A}$$

$$\frac{\vdash t : A \times B}{\vdash \pi_2(t) : B}$$

Adding sum and product types to $\lambda \rightarrow$ (2)

$$T ::= a \mid T \rightarrow T \mid T \times T \mid T + T$$

$$t ::= x \mid \lambda x.t \mid t t \mid (t, t) \mid \pi_1(t) \mid \pi_2(t) \mid i(t) \mid j(t) \mid \delta(t, x.t, x.t)$$

$$\lambda x.t u \triangleright_{\beta} t[x \setminus u]$$

$$\pi_1(t, u) \triangleright_{\beta} t$$

$$\pi_2(t, u) \triangleright_{\beta} u$$

$$\frac{\vdash t : A}{\vdash i(t) : A + B}$$

$$\frac{\vdash u : B}{\vdash j(u) : A + B}$$

$$\delta(i(t), x^A.u, y^B.v) \triangleright_{\beta} u[x^A \setminus t]$$

$$\delta(j(t), x^A.u, y^B.v) \triangleright_{\beta} v[y^B \setminus t]$$

$$\frac{\vdash t : A + B \quad \vdash u : C \quad \vdash v : C}{\vdash \delta(t, x^A.u, y^B.v) : C}$$

Normalization of $\lambda \rightarrow x +$

$$| A \times B | \equiv \{ t \in SN \mid t \triangleright^* (u, v) \Rightarrow u \in |A| \wedge v \in |B| \}$$

$$| A + B | \equiv \{ t \in SN \mid (t \triangleright^* i(u) \Rightarrow u \in |A|) \wedge (t \triangleright^* j(v) \Rightarrow v \in |B|) \}$$

$\mathcal{N}$ = terms which are not of the form: $\lambda x.t$, (u,v) , $i(u)$ or $j(v)$

In general, the metatheory follows the same schemes: inversion lemmas, type inference, subject reduction, normalization...

Time to switch to contexts for typing

$$\begin{array}{c}
 \frac{}{\Gamma \vdash x : A} \text{ if } (x:A) \in \Gamma \\
 \frac{\Gamma (x : A) \vdash t : B}{\Gamma \vdash \lambda x:A. t : A \rightarrow B} \\
 \frac{\Gamma \vdash t : A \rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash t u : B} \\
 \frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash (t,u) : A \times B} \\
 \frac{\Gamma \vdash t : A}{\Gamma \vdash i_B(t) : A + B} \quad \frac{\Gamma \vdash u : B}{\Gamma \vdash j_A(t) : A + B} \\
 \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_1(t) : A} \quad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_2(t) : B} \\
 \frac{\Gamma \vdash t : A + B \quad \Gamma(x:A) \vdash u:C \quad \Gamma(y:B) \vdash v:C}{\Gamma \vdash \delta(t, x.u, y.v) : C}
 \end{array}$$

Annotations to keep type uniqueness / type inference

Given Γ and t , one can decide whether there exists T s.t. $\Gamma \vdash t:T$

A (canonical)
proof of...

... is:

- ▶ $A \wedge B$ (a, b) where a is a proof of A , b a proof of B
- ▶ $A \vee B$ (ε, c) with $\varepsilon=0$ and c proof of A or $\varepsilon=1$ and c proof of B
- ▶ $A \Rightarrow B$ f , with $f(a)$ proof of B for any proof a of A
- ▶ $\forall x . A$ f , with $f(t) : A[x \backslash t]$ for any object t
- ▶ $\exists x . A$ (t, a) with a a proof of $A[x \backslash t]$

A (canonical)
proof of...

... is:

- ▶ $A \wedge B$ (a, b) where $a : A, b : B$
- ▶ $A \vee B$ (ε, c) with $\varepsilon=0$ and $c : A$ or $\varepsilon=1$ and $c : B$
- ▶ $A \Rightarrow B$ f , with $f(a) : B$ for any $a : A$
- ▶ $\forall x . A$ f , with $f(t) : A[x \backslash t]$ for any object t
- ▶ $\exists x . A$ (t, a) with $a : A[x \backslash t]$

Implicational Fragment

$$\frac{}{\Gamma \vdash x : A} \text{ if } (x:A) \in \Gamma$$

$$\frac{\Gamma (x : A) \vdash t : B}{\Gamma \vdash \lambda x:A. t : A \rightarrow B}$$

$$\frac{\Gamma \vdash t : A \rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash t u : B}$$

$$\text{Axiom } \frac{}{\Gamma \vdash x : A} \text{ if } (x:A) \in \Gamma$$

$$\Rightarrow -i \quad \frac{\Gamma (x : A) \vdash t : B}{\Gamma \vdash \lambda x:A. t : A \Rightarrow B}$$

$$\Rightarrow -e \quad \frac{\Gamma \vdash t : A \Rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash t u : B}$$

LF aka λP

$$\begin{array}{c} [] \text{ wf} \quad \frac{\Gamma \vdash T : s}{\Gamma (x:T) \text{ wf}} \quad \frac{\Gamma \text{ wf}}{\Gamma \vdash x : T} \quad \frac{\Gamma \text{ wf}}{\Gamma \vdash \text{Type} : \text{Kind}}\end{array}$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma (x:A) \vdash B : s}{\Gamma \vdash \Pi x : A . B : s}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash A : s}{\Gamma \vdash t : B} A =_{\beta} B$$

$$\frac{\Gamma \vdash \Pi x : A . B : \text{Type} \quad \Gamma(x:A) \vdash t : B}{\Gamma \vdash \lambda x : A . t : \Pi x : A . B}$$

$$\frac{\Gamma \vdash t : \Pi x : A . B \quad \Gamma \vdash u : A}{\Gamma \vdash (t \ u) : B [x \setminus u]}$$

No unicity of typing ! (modulo only beta-conversion)

$$1. \Gamma \Delta \text{ wf} \Rightarrow \Gamma \text{ wf} \quad \text{and} \quad \Gamma \vdash t:T \Rightarrow \Gamma \Delta \vdash t:T$$

$$2. \Gamma(x:T)\Delta \vdash u:U \wedge \Gamma \vdash t:T \Rightarrow \Gamma \Delta[x \setminus t] \vdash u[x \setminus t] : U[x \setminus t]$$
$$\text{and} \quad \Gamma(x:T)\Delta \text{ wf} \wedge \Gamma \vdash t:T \Rightarrow \Gamma \Delta[x \setminus t] \text{ wf}$$

3. Inversion lemma :

- ▶ If $\Gamma \vdash x : T$ then $(x:T') \in \Gamma$, $T =_{\beta} T'$ and $\Gamma \vdash T' : s$
- ▶ If $\Gamma \vdash \lambda x:A.t : B$ then $B =_{\beta} \Pi x:A.C$ and $\Gamma(x:A) \vdash t : C$
- ▶ If $\Gamma \vdash (t \ u) : C$ then $\Gamma \vdash t : \Pi x:A.B$, $\Gamma \vdash u : A$ and $C =_{\beta} B[x \setminus u]$
- ▶ If $\Gamma \vdash \Pi x:A.B : T$ then $\Gamma \vdash A : \text{Type}$, $\Gamma(x:A) \vdash B : s$ and $T =_s$

If $\Gamma \vdash t : T$ and $\Gamma \vdash t : U$ then $T =_{\beta} U$

Adding disjoint sum types

$$t ::= x \mid \lambda x : t . t \mid (t \ t) \mid \Pi x : t . t \mid t + t \mid \underset{t}{i}(t) \mid \underset{t}{j}(t) \mid \delta(t, t, t)$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash B : \text{Type}}{\Gamma \vdash A+B : \text{Type}}$$

$$\frac{\Gamma \vdash A+B : \text{Type} \quad \Gamma \vdash t : A}{\Gamma \vdash \underset{B}{i}(t) : A+B}$$

$$\frac{\Gamma \vdash A+B : \text{Type} \quad \Gamma \vdash t : B}{\Gamma \vdash \underset{A}{j}(t) : A+B}$$

$$\frac{\Gamma \vdash t : A+B \quad \Gamma \vdash C : \text{Type} \quad \Gamma \vdash u : A \rightarrow C \quad \Gamma \vdash v : B \rightarrow C}{\Gamma \vdash \delta(t, u, v) : C}$$

for decidability of type checking / inference

Adding Sigma types

$$t ::= x \mid \lambda x : t . t \mid (t \ t) \mid \Pi x : t . t \mid t + t \mid i(t) \mid j(t) \mid \delta(t, t, t) \\ \mid \Sigma x : t . t \mid \underbrace{(t, t)}_{\Sigma x : t . t} \mid \pi_1(t) \mid \pi_2(t)$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma(x:A) \vdash B : \text{Type}}{\Gamma \vdash \Sigma x : A . B : \text{Type}}$$

$\Sigma x : A . B$ written $A \times B$
when $x \notin \text{FV}(B)$

$$\frac{\Gamma \vdash \Sigma x : A . B : \text{Type} \quad \Gamma \vdash t : A \quad \Gamma \vdash u : B[x \setminus t]}{\Gamma \vdash \underbrace{(t, u)}_{\Sigma x : A . B} : \Sigma x : A . B}$$

$$\frac{\Gamma \vdash t : \Sigma x : A . B}{\Gamma \vdash \pi_1(t) : A} \qquad \frac{\Gamma \vdash t : \Sigma x : A . B}{\Gamma \vdash \pi_2(t) : B[x \setminus \pi_1(t)]}$$

for decidability of type checking / inference

1. $\Gamma \Delta \text{ wf} \Rightarrow \Gamma \text{ wf}$ and $\Gamma \vdash t:T \Rightarrow \Gamma \Delta \vdash t:T$
2. $\Gamma(x:T)\Delta \vdash u:U \wedge \Gamma \vdash t:T \Rightarrow \Gamma \Delta[x \backslash t] \vdash u[x \backslash t] : U[x \backslash t]$
and $\Gamma(x:T)\Delta \text{ wf} \wedge \Gamma \vdash t:T \Rightarrow \Gamma \Delta[x \backslash t] \text{ wf}$
3. Inversion lemma :
 - ▶ If $\Gamma \vdash x : T$ then $(x:T') \in \Gamma, T =_{\beta} T'$ and $\Gamma \vdash T' : s$
 - ▶ If $\Gamma \vdash \lambda x:A.t : B$ then $B =_{\beta} \Pi x:A.C$ and $\Gamma(x:A) \vdash t : C$
 - ▶ If $\Gamma \vdash (t \ u) : C$ then $\Gamma \vdash t : \Pi x:A.B \ \Gamma \vdash u : A$ and $C =_{\beta} B[x \backslash u]$
 - ▶ If $\Gamma \vdash \Pi x:A.B : T$ then $\Gamma \vdash A : \text{Type}, \Gamma(x:A) \vdash B : s$ and $T = s$
 - ▶ If $\Gamma \vdash (t,u)_{\Sigma x:A.B} : C$ then $\Gamma \vdash t:A, \Gamma \vdash u:B[x \backslash u], C =_{\beta} \Sigma x:A.B$
 - ▶ If $\Gamma \vdash \Sigma x:A.B : T$ then $\Gamma \vdash A : \text{Type}, \Gamma(x:A) \vdash B : s$ and $T = \text{Type}$

If $\Gamma \vdash t : T$ and $\Gamma \vdash t : U$ then $T =_{\beta} U$ (same for $A+B$)

$$\begin{array}{c} [] \text{ wf} \quad \frac{\Gamma \vdash T : s}{\Gamma (x:T) \text{ wf}} \quad \frac{\Gamma \text{ wf}}{\Gamma \vdash x : T} \quad \frac{\Gamma \text{ wf}}{\Gamma \vdash \text{Type} : \text{Kind}}\end{array}$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma (x:A) \vdash B : s}{\Gamma \vdash \Pi x : A . B : s}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash A : s}{\Gamma \vdash t : B} A =_{\beta} B$$

$$\frac{\Gamma \vdash \Pi x : A . B : \text{Type} \quad \Gamma(x:A) \vdash t : B}{\Gamma \vdash \lambda x : A . t : \Pi x : A . B}$$

$$\frac{\Gamma \vdash t : \Pi x : A . B \quad \Gamma \vdash u : A}{\Gamma \vdash (t \ u) : B [x \setminus u]}$$

If Γ wf and $(x:T) \in \Gamma$, then $\Gamma \vdash T:s$

Let us write $(\alpha:K)$ when $K:Kind$ and $(x:T)$ when $T:Type$

If $\Gamma \vdash t:T$, then $\Gamma \vdash T:s$ (or $T=kind$)

Let us write $T:K$ when $K:Kind$ and $t:T$ when $T:Type$

We have a stratification of (typed) terms:

- | | |
|------------------------|--|
| – Kind | Kind |
| – $K : Kind$ (kinds) | Type, $\Pi x : T. K$, $\Pi x_1:T_1. \dots \Pi x_n:T_n.Type$ |
| – $T : K$ (predicates) | α , $\Pi x:T_1.T_2$ ($T \ t$), $\lambda x:T_1.T_2$ |
| – $t : T$ (proofs) | x , $(t_1 \ t_2)$, $\lambda x:T.t$ |

Erasing type dependencies

– Kind	Kind
– $K : \text{Kind}$ (kinds)	Type, $\Pi x:T_1.T_2$ $(T \ t)$, $\lambda x:T_1.T_2$
– $T : K$ (predicates)	α , $(T \ t)$, $\lambda x:T_1.T_2$
– $t : T$ (proofs)	x , $(t_1 \ t_2)$, $\lambda x:T.t$

We map predicates to simple types:

$$\overline{\alpha} \equiv \alpha$$

$$\overline{\Pi x:T_1.T_2} \equiv \overline{T_1} \rightarrow \overline{T_2}$$

$$\overline{T \ t} \equiv \overline{T}$$

$$\overline{\lambda x:T_1.T_2} \equiv \overline{T_2}$$

$$|x| \equiv x$$

$$|\lambda x:T.t| \equiv \lambda x:\overline{T}.|t|$$

$$|t_1 \ t_2| \equiv |t_1| \ |t_2|$$

$$\text{If } t:T \text{ then } |t| : \overline{T}$$

$$\text{If } t \triangleright_{\beta} t' \text{ then } |t| \triangleright_{\beta} |t'| \text{ or } |t|=|t'|$$

Strong Normalization

$$\overline{\alpha} \equiv \alpha$$

$$\overline{\Pi x:T_1.T_2} \equiv \overline{T_1} \rightarrow \overline{T_2}$$

$$\overline{T \ t} \equiv \overline{T} \ \overline{t}$$

$$\overline{\lambda x:T_1.T_2} \equiv \overline{T_2}$$

$$|x| \equiv x$$

$$|\lambda x:T.t| \equiv \lambda x:\overline{T}.(\lambda _:\beta.|t| \ |T|)$$

$$|t_1 \ t_2| \equiv |t_1| \ |t_2|$$

$$[] \equiv (\beta:\text{Type})$$

$$\Gamma(x:T) \equiv \overline{\Gamma} \ (x:\overline{T})$$

$$\Gamma(\alpha:K) \equiv \Gamma(\alpha:\text{Type})(\underline{\alpha}:\overline{K})$$

Predicates: $\alpha, \ \Pi x:T_1.T_2, \ (T \ t), \ \lambda x:T_1.T_2$

$$|\alpha| \equiv \underline{\alpha}$$

$$|T \ t| \equiv |T| \ |t|$$

$$|\lambda x:T_1.T_2| \equiv \lambda x:\overline{T_1}.|T_2|$$

$$|\Pi x:T_1.T_2| \equiv \lambda x:\overline{T_1}.|T_2| \ |T_1|$$

If $t \triangleright_{\beta} t'$, then $|t| \triangleright_{\beta^+} |t'|$

consequence:

SN for simple types entails SN for LF