MPRI

PRFSYS

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Cuts in Heyting Arithmetic

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A presentation of Heyting Arithmetic

Axioms

$\forall x.\ x=x$

$\forall x. \forall y.\ x=y \wedge P(x) \Rightarrow P(y)$

$P(0) \wedge (\forall x. P(x) \Rightarrow P(S(x)) \quad \Rightarrow \quad \forall y.\ P(y)$

closed normal object:

$0, S(0), S(S(O)), \ldots$

closed normal atomic proposition

$n=m \quad (\top$ and $\perp$ are not atomic)

Rewrite rules

$0 + x \quad \rhd \quad x$

$S(x) + y \quad \rhd \quad S(x+y)$

$0 \times x \quad \rhd \quad 0$

$S(x) \times y \quad \rhd \quad x \times y + y$

$pred(S(x)) \quad \rhd \quad x \qquad pred(0) \rhd 0$

$EQZ(S(x)) \rhd \perp \qquad EQZ(0) \quad \rhd \quad \top$

Cuts in deduction modulo

Previous presentation: new additional rule

$$(\text{conv})\ \frac{\Gamma \vdash A}{\Gamma \vdash B} \quad \text{if } A =_R B$$

we do not want it to interfere with cuts.

should be a cut

$$\wedge\text{-e}\ \dfrac{(\text{conv})\ \dfrac{\wedge\text{-i}\ \dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}}{\Gamma \vdash A' \wedge B}}{\Gamma \vdash A'}$$

We can rather reformulate the rules:

$$\wedge\text{-i}\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash C} \quad \text{if } C =_R A \wedge B$$

is now a cut

$$\wedge\text{-e}\ \dfrac{\wedge\text{-i}\ \dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A' \wedge B}}{\Gamma \vdash A'}$$

(we do the same for all rules)

Axiomatic Cuts

Equality Cut

"introduction"

"elimination"

$$\dfrac{\dfrac{\dfrac{\forall x\,.\,x=x}{t=t} \qquad \dfrac{\sigma_P}{P(t)}}{t=t \wedge P(t)} \qquad \dfrac{\forall x\,y.\ x=y \wedge P(x) \Rightarrow P(y)}{t=t \wedge P(t) \Rightarrow P(t)}}{P(t)}$$

$$\dfrac{\sigma_P}{P(t)}$$

Induction Cut (1)

$$\dfrac{\dfrac{(P(0) \land \forall x\, P(x)\Rightarrow P(S(x))) \Rightarrow \forall y.P(y) \qquad \dfrac{\sigma_0}{P(0)} \qquad \dfrac{\sigma_S}{\forall x\, P(x)\Rightarrow P(S(x))}}{\forall y.P(y)}}{P(0)}$$

$$\dfrac{\sigma_0}{P(0)}$$

Induction cut (2)

$$\dfrac{\dfrac{(\, P(0) \wedge \forall\, x\, P(x) \Rightarrow P(S(x))\,) \Rightarrow \forall\, y.P(y) \qquad \dfrac{\sigma_0}{P(0)} \qquad \dfrac{\sigma_S}{\forall\, x\, P(x) \Rightarrow P(S(x))}}{\forall\, y.P(y)}}{P(S(t))}$$

$$\dfrac{\dfrac{\dfrac{\sigma_S}{\forall\, x\, P(x) \Rightarrow P(S(x))}}{P(t) \Rightarrow P(S(t))} \qquad \dfrac{\dfrac{(\, P(0) \wedge \forall\, x\, P(x) \Rightarrow P(S(x))\,) \Rightarrow \forall\, y.P(y) \qquad \dfrac{\sigma_0}{P(0)} \qquad \dfrac{\sigma_S}{\forall\, x\, P(x) \Rightarrow P(S(x))}}{\forall\, y.P(y)}}{P(t)}}{P(S(t))}$$

Cut Free Proofs

Properties

easy:

If t is a term without free variables, then $t \rhd^* S^n(0)$

Cut free proofs:

Take A without free variables. Any cut-free proof of A in HA either :

- ends with an introduction
- is refl or t=t (from refl)
- is Leibniz or partial application of L : $\forall y.\ t=y \wedge P(t) \Rightarrow P(y)$, $u=t \wedge P(t) \Rightarrow P(u)$
- Is Induction or a partial application of it: $\forall y.\ P(y)$

by induction over the structure of the proof (somewhat tedious)

A without free variables. A cut-free proof of A in HA is either :

- ends with an introduction
- is refl or t=t (from refl)
- is Leibniz or partial application of L : $\forall$ y. t=y$\wedge$ P(t) $\Rightarrow$ P(y), u=y$\wedge$ P(t) $\Rightarrow$ P(u)
- Is Induction of proof partial application: $\forall$ y. P(y)

Constructivity :

- If $\vdash_{HA}$ A$\vee$ B, then either $\vdash_{HA}$ A or $\vdash_{HA}$ B
- if $\vdash_{HA}$ $\exists$ x. A(x) then we can extract n and a proof of $\vdash_{HA}$ A(n)

Consider : $\forall$ x. $\exists$ y. x=y+y $\vee$ x = S(y+y)

Heyting's semantics

To make the point of *constructivity*

- a proof of $n=n$ is 0 (some trivial object)
- a proof of $A \wedge B$ is (can be reduced to) (a,b) with a:A and b:B
- a canonical proof of $A \vee B$ is (ε,c) with $\varepsilon=0$ and c:A or $\varepsilon=1$ and c:B
- a proof of $A \Rightarrow B$ is a computational function f, s.t. if a:A, then $f(a)$: B
- a canonical proof of $\exists$ x.A is a pair (t,a) s.t. a: A[x \ t]
- a proof of $\forall$ x.A is a computational function f, s.t. for all n, $f(n)$: A[x \ n]

Side Remark: Why is arithmetic undecidable ?

$t=u$ is decidable

In HA, we can *prove* $\forall x, \forall y, x=y \vee x \neq y$
(which is the good way to state decidability)
Let's do it

If A and B are decidable, so are $A \wedge B$, $A \vee B$, $A \Rightarrow B$

Undecidability comes "only" from the quantifiers

Even if for all x, we can determine $A(x)$ or $\neg A(x)$, we do not know

whether $\forall x. A(x)$ is true or not

Simple game semantics

Let us keep a first–order language (actually arithmetic)
We drop the implication ⇒

For every predicate P we add its negation *P (same arity)
We *define* the negation of any proposition as:

$$\neg P(t_1, \dots, t_n) \equiv {}^*P(t_1, \dots, t_n)$$
$$\neg (A \vee B) \equiv \neg A \wedge \neg B$$
$$\neg (A \wedge B) \equiv \neg A \vee \neg B$$
$$\neg \forall x.\, A \equiv \exists x.\, \neg A$$
$$\neg \exists x.\, A \equiv \forall x.\, \neg A$$

Now ! Every closed proposition can be viewed as a *game* !
a game between the mathematician and nature

The game

The mathematician plays when the proposition is:

- $\exists$ x . A provides an object t, game becomes A[x \ t]
- A $\vee$ B chose left or right, game becomes A or B

Nature plays when the proposition is:

- $\forall$ x. A provides an object t, game becomes A[x \ t]
- A $\wedge$ B chose left or right, game becomes A or B

The game stops when the proposition is atomic $P(t_1, \ldots t_n)$

- if $P(t_1, \ldots t_n)$ is true, mathematician wins
- if $P(t_1, \ldots t_n)$ is false, nature wins

Paul Lorenzen (1958)

A true intuitionistically: mathematician has a *winning strategy*

Going beyond intuitionistic logic

Remember we have classical logic in sequent calculus by authorizing sequents with several conclusions: $A_1, \dots, A_n \vdash B_1, \dots B_m$

We go to multigames: $A_1, \dots, A_n$

idea: mathematician has to "prove" only one A_i

- if nature has to play on at least one A_i, it plays
- if not, mathematician plays on one A_i
- if A_i is $B \vee C$, mathematician can break it without choosing

  $B \vee C \rightsquigarrow B, C$
- if A_i is $\exists x.A$, then mathematician can "keep" the existential for another later attempt $\exists x.A \rightsquigarrow \exists x.A, A[x \setminus t]$

Excluded Middle in multi-games

$A \vee \neg A \quad \rightsquigarrow \quad A, \neg A$

Now let us look at A:

if $B \wedge C$, then nature plays B or C — mathematician plays $\neg B$ or $\neg C$

if $B \vee C$, then nature plays $\neg B$ or $\neg C$ — mathematician plays B or C

if $\forall x.B$, then nature plays $B[x\backslash t]$ — mathematician plays $\neg B[x\backslash t]$

if $\exists x.B$, then nature plays $\neg B[x\backslash t]$ — mathematician plays $B[x\backslash t]$

Mathematician wins !

when $\vdash A$ (in classical logic), there is a winning strategy (essentially a termination argument)

see for instance the page of Thierry Coquand about game semantics

Links with Curry-Howard for classical logic