MPRI

PRFSYS

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Cuts in Heyting Arithmetic

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Axioms

$$\forall x. x=x$$

$$\forall x. \forall y. x=y \land P(x) \Rightarrow P(y)$$

$$P(0) \land (\forall x.P(x) \Rightarrow P(S(x)) \Rightarrow \forall y. P(y)$$

closed normal object:

closed normal atomic proposition n=m (⊤ and ⊥ are not atomic)

Rewrite rules

$$0 + x > x$$

$$S(x) + y > S(x+y)$$

$$0 \times x > 0$$

$$S(x) \times y > x \times y + y$$

$$EQZ(S(x)) > \bot$$



Cuts in deduction modulo



Previous presentation: new additional rule

(conv)
$$\frac{\Gamma \vdash A}{\Gamma \vdash B}$$
 if $A =_R B$

we do not want it to interfere with cuts.

We can rather reformulate the rules:

$$\wedge$$
-i $\frac{\Gamma \vdash A}{\Gamma \vdash C}$ if $C =_R A \wedge B$

is now a cut

(we do the same for all rules)

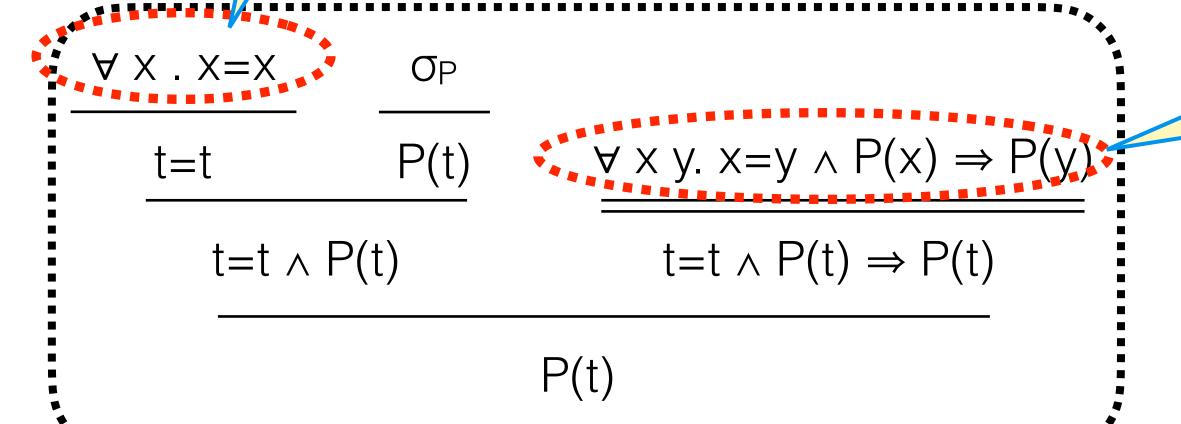
Axiomatic Cuts



Equality Cut





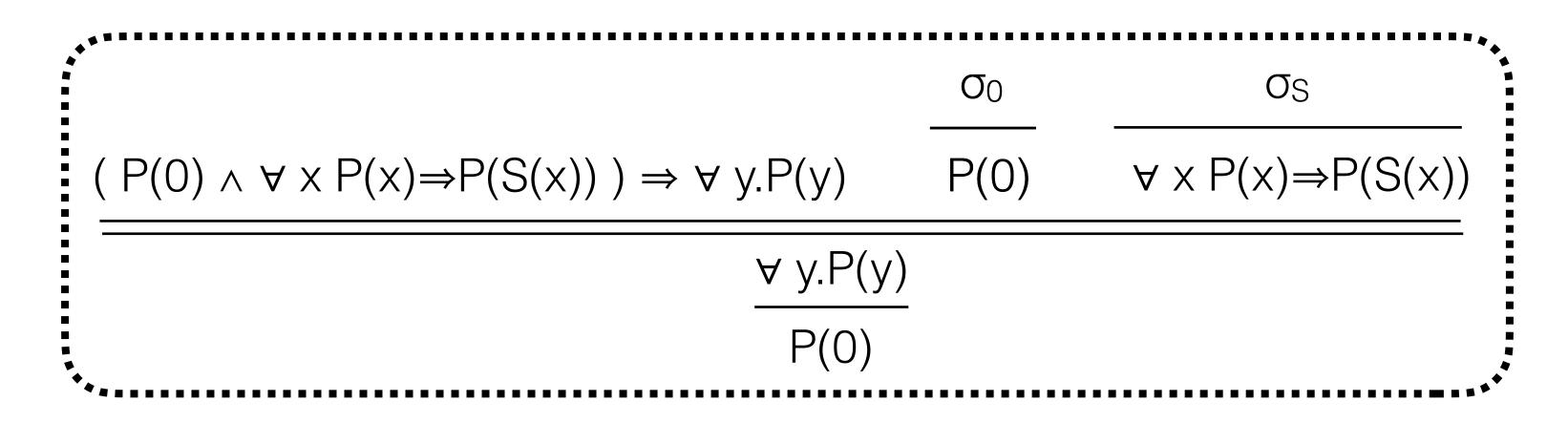


"elimination"

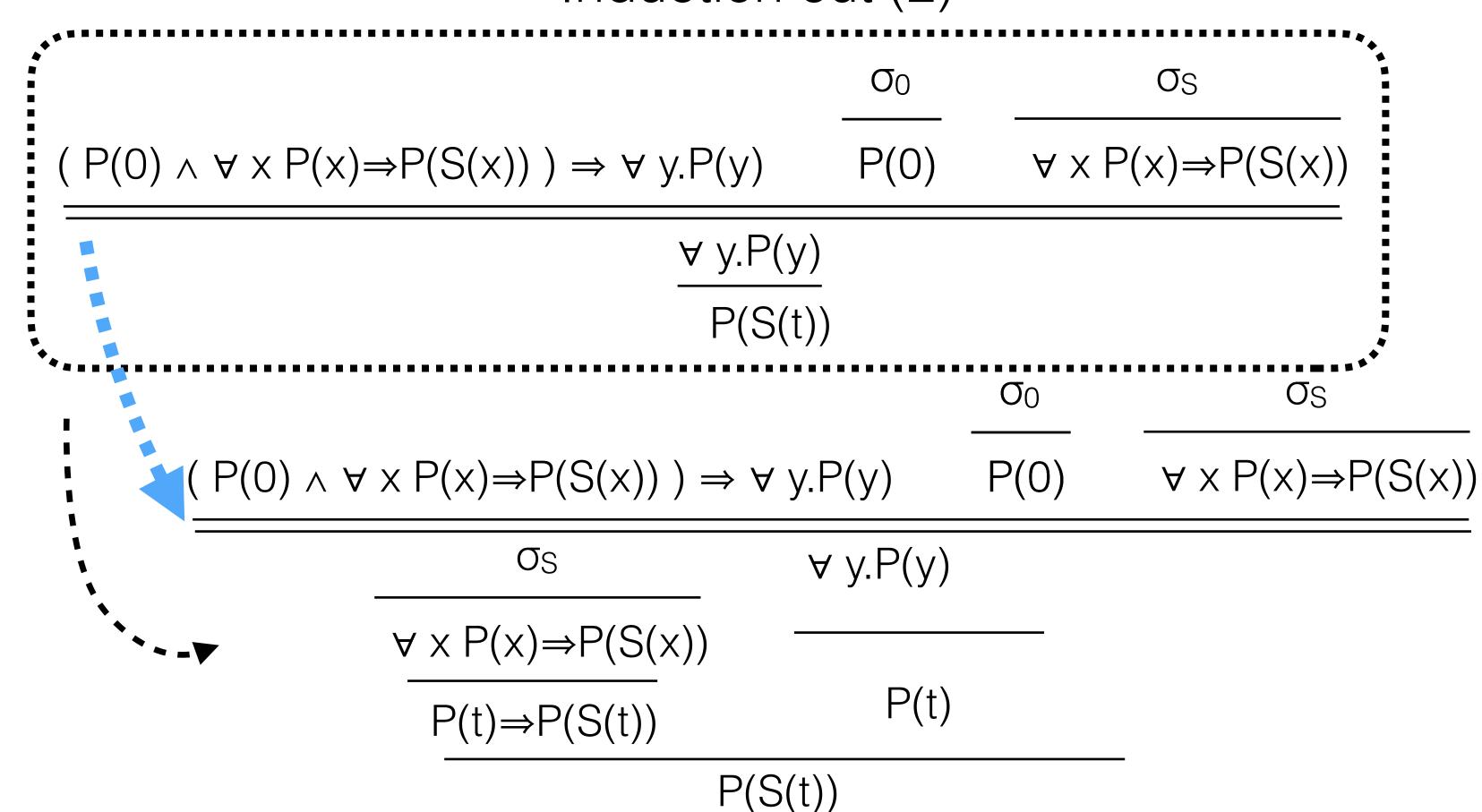


Induction Cut (1)





Induction cut (2)





Cut Free Proofs



Properties

easy:

If t is a term without free variables, then $t > {}^*S^n(0)$

Cut free proofs:

Take A without free variables. Any cut-free proof of A in HA either:

- ends with an introduction
- is refl or t=t (from refl)
- is Leibniz or partial application of L : \forall y. t=y∧ P(t) \Rightarrow P(y), u=t∧ P(t) \Rightarrow P(u)
- Is Induction or a partial application of it: $\forall y$. P(y)

by induction over the structure of the proof (somewhat tedious)

A without free variables. A cut-free proof of A in HA is either:

- ends with an introduction
- is refl or t=t (from refl)
- is Leibniz or partial application of L : \forall y. t=y∧ P(t) \Rightarrow P(y), u=y∧ P(t) \Rightarrow P(u)
- Is Induction of proof partial application: $\forall y$. P(y)

Constructivity:

- If $\vdash_{HA} A \lor B$, then either $\vdash_{HA} A$ or $\vdash_{HA} B$
- if $\vdash_{HA} \exists x. A(x)$ then we can extract n and a proof of $\vdash_{HA} A(n)$

Consider: $\forall x. \exists y. x=y+y \lor x = S(y+y)$



Heyting's semantics



To make the point of *constructivity*

- ▶ a proof of n=n is 0 (some trivial object)
- ▶ a proof of A \land B is (can be reduced to) (a,b) with a:A and b:B
- a canonical proof of A \vee B is (ε,c) with $\varepsilon=0$ and c:A or $\varepsilon=1$ and c:B
- ▶ a proof of $A \Rightarrow B$ is a computational function f, s.t. if a:A, then f(a):B
- ▶ a canonical proof of $\exists x.A$ is a pair (t,a) s.t. $a: A[x \setminus t]$
- ▶ a proof of \forall x.A is a computational function f, s.t. for all n, f(n): A[x\n]

Side Remark: Why is arithmetic undecidable?



t=u is decidable

In HA, we can *prove* \forall x, \forall y, x=y \vee x \neq y (which is the good way to state decidability) Let's do it

If A and B are decidable, so are $A \land B$, $A \lor B$, $A \Rightarrow B$

Undecidability comes "only" from the quantifiers

Even if for all x, we can determine A(x) or $\neg A(x)$, we do not know

whether $\forall x.A(x)$ is true or not



Simple game semantics



Let us keep a first-order language (actually arithmetic) We drop the implication ⇒

For every predicate P we add its negation *P (same arity) We *define* the negation of any proposition as:

$$\neg P(t_{1}, ..., t_{n}) \equiv ^{*}P(t_{1}, ..., t_{n})$$

$$\neg (A \lor B) \equiv \neg A \land \neg B$$

$$\neg (A \land B) \equiv \neg A \lor \neg B$$

$$\neg \forall X. A \equiv \exists X. \neg A$$

$$\neg \exists X. A \equiv \forall X. \neg A$$

Now! Every closed proposition can be viewed as a *game*! a game between the mathematician and nature



The game



The mathematician plays when the proposition is:

- ▶ ∃ x . A provides an object t, game becomes A[x \ t]
- A v B chose left or right, game becomes A or B

Nature plays when the proposition is:

- ▶ ∀ x. A provides an object t, game becomes A[x \ t]
- A ∧ B chose left or right, game becomes A or B

The game stops when the proposition is atomic $P(t_1, ..., t_n)$

- ▶ if P(t₁, ... t_n) is true, mathematician wins
- if $P(t_1, ..., t_n)$ is false, nature wins

Paul Lorenzen (1958)

A true intuitionistically: mathematician has a winning strategy



Going beyond intuitionistic logic



Remember we have classical logic in sequent calculus by authorizing sequents with several conclusions: $A_1, ..., A_n \vdash B_1, ... B_m$

We go to multigames: A_1, \ldots, A_n

idea: mathematician has to "prove" only one Ai

- if nature has to play on at least one A_i, it plays
- if not, mathematician plays on one Ai
- if A_i is B ∨ C, mathematician can break it without choosing
 B ∨ C → B, C
- if A_i is ∃ x.A, then mathematician can "keep" the existential for another later attempt ∃ x.A → ∃ x.A, A[x \ t]



Excluded Middle in multi-games



$$A \vee \neg A \rightarrow A, \neg A$$

Now let us look at A:

if $B \wedge C$, then nature plays B or C

if B ∨ C, then nature plays ¬B or ¬C

if $\forall x.B$, then nature plays $B[x\t]$

if $\exists x.B$, then nature plays $\neg B[x\t]$

mathematician plays \neg B or \neg C mathematician plays B or C mathematician plays \neg B[x\t] mathematician plays B[x\t]

Mathematician wins!

when ⊢ A (in classical logic), there is a winning strategy (essentially a termination argument)

see for instance the page of Thierry Coquand about game semantics

Links with Curry-Howard for classical logic