Foundations of Proof Systems

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How do we define mathematics ?

All humans are mortal, Socrates is human, thus Socrate is mortal.

correction : *syntaxic* criterion

$$\frac{\vdash A \Rightarrow B \qquad \vdash A}{\vdash B}$$

The stones to build mathematical proofs

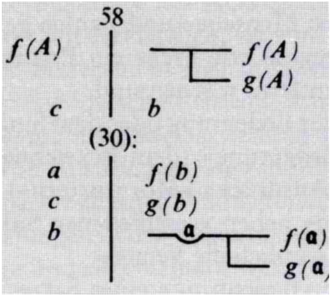
$$\frac{\dfrac{\vdash \forall x. H(x) \Rightarrow M(x)}{\vdash H(s) \Rightarrow M(S)} \qquad \vdash H(S)}{\vdash M(S)}$$

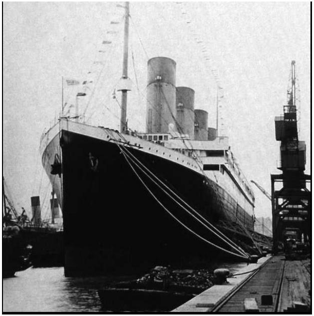
A mathematical proof is a *construction*

Birth of modern mathematical logic

Mathematical truth defined through totally objective rules

1872 : The *Begriffsschrift* of Frege





mechanical verification

proof = tree structure

A century later

Mechanical verification
becomes real

First proof system : Automath (1968)

N. G. de Bruijn

$\boxed{\text{Formal proofs are } \textit{actually} \text{ built.}}$

Today

A modern proof system : Coq

- Same principle
- More modern formalism

An example: equality

How do we formally define what x=y means ?

- We can say equality is the smallest equivalence relation
- We can say equality is the smallest reflexive relation
- $x=y \quad \equiv \quad \forall R, \ (\forall a, R(a,a)) \Rightarrow R(x,y)$ (as above written differently)
- Axioms:
  1. $\forall x, x=x$
  2. If x=y, then any property verified by x is verified by y

  or : $\forall x\ y, \quad x=y \wedge P(x) \Leftrightarrow P(y)$ (axiom scheme)

  or : $\forall x\ y, \quad x=y \wedge P(x) \Rightarrow P(y)$ (also scheme)

This is only about equality itself (no axiom like extentionality)
Obviously, these assertions are written in different "languages"
Equivalences between these variants: exercises in Coq

Some important points

- The language to express propositions ("for any relation…")
- The features for definitions ("the smallest relation…")

but also other points:

- the roles of computations

What do we ask from a formalism

Before (informal proofs) : we want the formalism to be expressive (many theorems)

Now (formal proofs) we want also :

- Concise proofs
- Close to our intuition (no spurious syntactical hacking)
- ...

This course : study formalisms with these aims in mind

First-order logic - language

A set of variables : $x, y, z, \ldots$
A set of function symbols : $f, g, h, \ldots$ each function symbol has an arity (number of arguments).
A set of predicate symbols : $A, B, C, P, R \ldots$ each with an arity.

Objects :

- a variable is a term,
- if f is of arity n and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

Propositions :

- if P is of arity n then $P(t_1, \ldots, t_n)$ is a (atomic) proposition
- is A and B are propositions,
  $A \wedge B,\ A \vee B,\ A \Rightarrow B,\ \bot,\ \forall x.A,\ \exists x.B$ are propositions.

Examples (languages of FOL)

Arithmetic (Peano, 1889)

Function symbols : $0, S, +, \times$

Predicate symbol : $=$

Set Theory (Cantor, Russell, Zermelo, Fraenkel...)

Predicate symbols : $\in, =$

A theory is :

- A language (functions + predicate symbols)
- A set of axioms (propositions of the language)

Axioms of arithmetic :

$$\forall x, 0 + x = x \qquad \forall x, 0 \times x = 0$$
$$\forall x\ y, S(x) + y = S(x + y) \qquad \forall x\ y, S(x) \times y = y + x \times y$$

$$\forall x, \neg(0 = S(x))$$
$$\forall x\ y, S(x) = S(y) \Rightarrow x = y$$

$$P(0) \wedge (\forall x, P(x) \Rightarrow P(S(x))) \ \Rightarrow\ \forall x, P(x).$$

$$\forall x, x = x$$
$$\forall x\ y, P(x) \wedge x = y \Rightarrow P(y).$$

Truth : natural deduction

Γ set of propositions
$\Gamma \vdash A$ A is provable under hypothesises+axioms Γ

$$\frac{A \in \Gamma}{\Gamma \vdash A}\ (\mathrm{Ax})$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge\text{-I}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge\text{-E}_1) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge\text{-E}_2)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\text{-I}_1) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee\text{-I}_2)$$

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee\text{-E})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ (\Rightarrow\text{-I}) \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ (\Rightarrow\text{-E})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x.A}\ (\forall\text{-I}) \quad \text{if } x \text{ not free in } \Gamma$$

$$\frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[x \setminus t]}\ (\forall\text{-E})$$

$$\frac{\Gamma \vdash A[x \setminus t]}{\Gamma \vdash \exists x.A}\ (\exists\text{-I})$$

$$\frac{\Gamma, A \vdash B \qquad \Gamma \vdash \exists x.A}{\Gamma \vdash B}\ (\exists\text{-E}) \quad \text{if } x \text{ not free in } \Gamma, B$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\bot\text{-E})$$

(this gives intuitionistic logic

$$\frac{}{\Gamma \vdash A \vee \neg A}\ (\mathrm{EM})$$

(this gives classical logic)

Relating correctness and truth : models and semantics

A set $\mathcal{U}$ (universe)
For every f of arity n, a function $|f| : \mathcal{U}^n \to \mathcal{U}$
For every P of arity n, a function $|P| : \mathcal{U}^n \to \{0,1\}$ (equivalently $|P| \subset \mathcal{P}(\mathcal{U}^n)$)
Given any $\mathcal{I}$ mapping variables x to $\mathcal{U}$ we define $|t|_{\mathcal{I}} \in \mathcal{U}$ by :

- $|x|_{\mathcal{I}} \equiv \mathcal{I}(x)$
- $|f(t_1,\ldots,t_n)|_{\mathcal{I}} \equiv |f|(|t_1|_{\mathcal{I}},\ldots|t_n|_{\mathcal{I}})$

Given any $\mathcal{I}$ we define $|A| \in \{0,1\}$ by :

- $P(t_1,\ldots,t_n)|_{\mathcal{I}} \equiv |P|(|t_1|_{\mathcal{I}},\ldots|t_n|_{\mathcal{I}})$
- $|A \wedge B|_{\mathcal{I}} \equiv |A|_{\mathcal{I}} \wedge |B|_{\mathcal{I}}$
- similar for $\vee$, $\Rightarrow$, $\bot\ldots$
- $|\forall x.A|_{\mathcal{I}} \equiv \min_{\alpha\in\mathcal{U}} |A|_{\mathcal{I};x\leftarrow\alpha}$
- $|\exists x.A|_{\mathcal{I}} \equiv \max_{\alpha\in\mathcal{U}} |A|_{\mathcal{I};x\leftarrow\alpha}$ (this is very much classical logic)

Model of a theory

A model is a triple : $\mathcal{U}$, interpretation of fs, interpretation of Ps.
It is a model of a theory $\mathcal{T}$ if for any $A \in \mathcal{T}$, $|A|_{\mathcal{I}} = 1$ (for any $\mathcal{I}$ since A is closed)

Correctness : If $\Gamma \vdash A$, and $\forall B \in \Gamma, |B|_{\mathcal{I}} = 1$, then $|A|_{\mathcal{I}} = 1$.
proof : quite straightforward (good exercise)

Coherence : There is no proof of $\mathcal{T} \vdash \perp$ (easy consequence of correctness)

Completeness : If for any model validating Γ, $|A|_{\mathcal{I}} = 1$, then $\Gamma \vdash A$ is provable.
proof : more difficult (Gödel's PhD)

- Relates correctness with truth
- incompleteness : limit of « truth » in math

An extension of first-order logic

Deduction modulo : we add rewrite rules to the language

$$
\begin{array}{rcl}
0 + x & \rhd & x \\
S(x) + y & \rhd & S(x+y) \\
O \times x & \rhd & 0 \\
S(x) \times y & \rhd & y + x \times y
\end{array}
$$

we allow reasoning modulo the rewrite rules :

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \psi} \text{ if } \phi =_R \psi$$

How to prove $2 + 2 = 4$?

Replacing more axioms by rewrite rules

How to ensure $0 \neq 1$?

$$\forall x. 0 \neq S(x)$$

Add a new predicate symbol EQZ

$$\begin{array}{rcl} \mathrm{EQZ}(0) & \rhd & \top \\ \mathrm{EQZ}(S(x)) & \rhd & \bot \end{array}$$

Exercise : finish the proof
Important : avoiding messy rewrite rules ($A \wedge B \rhd \bot \ldots$)

Replacing more axioms by rewrite rules(2)

How to ensure $\forall x. \forall y. S(x) = S(y) \Rightarrow x = y$?
(injectivity of S)
Add a new function symbol pred

$$
\begin{array}{rcll}
\mathrm{pred}(S(x)) & \triangleright & x & \\
\mathrm{pred}(0) & \triangleright & 0 & \text{(or whatever)}
\end{array}
$$

Exercise : finish the proof

A "simple" presentation of Arithmetic

Rules :

$$\begin{array}{rcl} 0 + x & \rhd & x \\ S(x) + y & \rhd & S(x+y) \\ O \times x & \rhd & 0 \\ S(x) \times y & \rhd & y + x \times y \end{array} \qquad \begin{array}{rcl} \mathrm{EQZ}(0) & \rhd & \top \\ \mathrm{EQZ}(S(x)) & \rhd & \bot \\ \mathrm{pred}(S(x)) & \rhd & x \\ \mathrm{pred}(0) & \rhd & 0 \end{array}$$

Axioms :

$$\forall x. x = x$$

$$\forall x. \forall y. x = y \wedge P(x) \Rightarrow P(y)$$

$$P(0) \wedge (\forall x. P(x) \Rightarrow P(S(x))) \Rightarrow \forall y. P(y)$$

Cuts in proofs

Another form of dynamics / computation / transformation in proofs

What is a cut ?

1. Prove $\forall a. \forall b. (a+b)^2 = a^2 + b^2 + 2ab$ (ends with $\forall$-intro)
2. Deduces $\forall b. (3+b)^2 = 9 + b^2 + 6b$ (use $\forall$-elim)

We could have proved (2) directly (following the same scheme as 1)

Logical Cut

An introduction rule followed by the corresponding elimination rule

$$\dfrac{\dfrac{\dfrac{\sigma_1}{\Gamma \vdash A} \qquad \dfrac{\sigma_2}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ (\wedge\text{-i})}{\Gamma \vdash A}\ (\wedge\text{-e1})$$

Simplifies to :

$$\dfrac{\sigma_1}{\Gamma \vdash A}$$

exercise : find the simplification for the other logical cuts

Cut Elimination

- Does this process terminate ?
- If we have a proof of $\Gamma \vdash A$, can we find a cut-free proof ?

Termination : a major point of this course

Cut-free proofs

Why does it matter to us ?

In a cut-free proof, there are only axiom rules above elimination rules (or the EM)

If a proof is cut-free, without axiom and constructive, it ends with an introduction rule.

A proof of $\vdash A \vee B$ that is constructive and cut-free ends with $\vee - i1$ of $\vee - i2$.

A proof of $\vdash \exists x.A(x)$ that is constructive and cut-free contains a *witness*.

Cut Free - axiom free proofs

Lemma : a cut free derivation (proof) of $[] \vdash A$ always ends with an introduction rule.

Proof : by induction over the derivation (could be the length of the derivation, but not necessary).

Let us do a few cases.

Why "natural" deduction ?

The ND rules aim at corresponding to actual (human) deduction steps.
Indeed :

Coq's formalism includes / extends first-order logic with some rewrite/computation rules.

Proofs are built top-down (goal-driven) and basic tactics correspond to ND rules

Next : : cuts and constructivity in Heyting Arithmetic