

Foundations of Proof Systems

Exercises

Oct. 2nd 2026

1 Realizability and Markov's Principle

We recall our definition of realizability for Heyting Arithmetic.

For every proposition A and valuation I , we define a relation $t \Vdash_I A$ between (untyped) terms t and A , by induction on the structure of A :

- no term realizes **false** : $t \Vdash_I \mathbf{false}$ never holds;
- $t \Vdash_I A \wedge B$ iff $t \triangleright^* (u, v)$ for some $u \Vdash_I A$ and $v \Vdash_I B$;
- $t \Vdash_I A \vee B$ iff either $t \triangleright^* i(u)$ for some $u \Vdash_I A$, or $t \triangleright^* j(v)$ for some $v \Vdash_I B$;
- $t \Vdash_I A \implies B$ iff for every $u \Vdash_I A$, $(t u) \Vdash_I B$;
- $t \Vdash_I \forall x. A$ iff for every $n \in \mathbb{N}$, $(t S^{(n)}(0)) \Vdash_{I[x \mapsto n]} A$;
- $t \Vdash_I \exists x. A$ iff $t \triangleright^* (S^{(n)}(0), u)$ for some $n \in \mathbb{N}$ and some $u \Vdash_{I[x \mapsto n]} A$;
- for an atomic proposition $P(a_1, \dots, a_k)$ (equality included), $t \Vdash_I P(a_1, \dots, a_k)$ iff $t \triangleright^* ()$ and $P(a_1[I], \dots, a_k[I])$ genuinely holds, as a fact about natural numbers in the standard model — e.g. $t \Vdash_I a = b$ iff $t \triangleright^* ()$ and $a[I] = b[I]$, and likewise for any base predicate introduced by rewriting.

Définition 1 Markov's principle (MP) is a scheme parametrized by proposition A and variable x .

$$(\forall x. A(x) \vee \neg A(x)) \Rightarrow \neg \neg (\exists x. A(x)) \Rightarrow \exists x. A(x).$$

Question 1 Explain how MP can be proved in classical arithmetic. ◇

Question 2 Is there a term *realizing* MP? If so, try to write it? ◇

2 Realizability and Harrop Formulas

Question 3 Show that if $A \Rightarrow \perp$ is provable in Heyting Arithmetic, then it is realized by the term $\lambda x. ()$. ◇

Définition 2 A proposition A is a *Harrop Formula* if :

- $A = \perp$,
- A is a formula $t = u$,

- $A = B \wedge C$ with B and C being Harrop formulas,
- $A = \forall x.B$ with B a Harrop Formula,
- A is $B \Rightarrow C$ with C a Harrop Formula. ◇

Question 4 Let A be a Harrop Formula. Construct a term t which realizes A , *provided* A is provable. ◇

Définition 3 The Independance of Premisses (IP) is the following scheme :

$$(A \Rightarrow \exists x.B) \Rightarrow \exists x.A \Rightarrow B$$

where A and B are propositions, and x does not occur in A . ◇

Question 5 Show that IP is provable in classical arithmetic (logic). ◇

Question 6 Explain (even informally) why IP is not realizable for some proposition A . ◇

Question 7 Show that IP is realizable when A is a Harrop formula. ◇