MPRI 2-7-1

week 4 - Oct. 8^{th}

Dependent Types

Adding sum and product types to $\lambda\rightarrow$

$$T \ ::= \ \alpha \mid T \rightarrow T \mid T \times T \mid T + T$$

$$t \ ::= \ x \mid \lambda x.t \mid t\ t \mid (t, t) \mid \pi_1(t) \mid \pi_2(t) \mid i(t) \mid j(t) \mid \delta(t, x.t, x.t)$$

$$\lambda x.t\ u \ \triangleright_\beta \ t[x \backslash u]$$

$$\pi_1(t, u) \ \triangleright_\beta \ t$$

$$\pi_2(t, u) \ \triangleright_\beta \ u$$

$$\frac{\vdash t : A \qquad \vdash u : B}{\vdash (t,u) : A \times B}$$

$$\frac{\vdash t : A \times B}{\vdash \pi_1(t) : A} \qquad \frac{\vdash t : A \times B}{\vdash \pi_2(t) : B}$$

Adding sum and product types to λ→ (2)

$$T \;::=\; \alpha \mid T \to T \mid T \times T \mid T + T$$

$$t \;::=\; x \mid \lambda x.t \mid t\ t \mid (t, t) \mid \pi_1(t) \mid \pi_2(t) \mid i(t) \mid j(t) \mid \delta(t, x.t, x.t)$$

$$\lambda x.t\ u \;\triangleright_\beta\; t[x \setminus u]$$

$$\pi_1(t, u) \;\triangleright_\beta\; t$$

$$\pi_2(t, u) \;\triangleright_\beta\; u$$

$$\frac{\vdash t : A}{\vdash i(t) : A + B} \qquad \frac{\vdash u : B}{\vdash j(u) : A + B}$$

$$\delta(i(t), x^A.u, y^B.v) \;\triangleright_\beta\; u[x^A \setminus t]$$

$$\delta(j(t), x^A.u, y^B.v) \;\triangleright_\beta\; v[y^B \setminus t]$$

$$\frac{\vdash t : A + B \quad \vdash u : C \quad \vdash v : C}{\vdash \delta(t, x^A.u, y^B.v) : C}$$

Normalization of $\lambda^{\to\times+}$

$$| A \times B | \equiv \{ t \in SN \mid t \rhd^* (u, v) \Rightarrow u \in |A| \wedge v \in |B| \}$$

$$| A + B | \equiv \{ t \in SN \mid (t \rhd^* i(u) \Rightarrow u \in |A|) \wedge (t \rhd^* j(v) \Rightarrow v \in |B|) \}$$

$\mathcal{N}$ = terms which are not of the form: $\lambda x.t$, (u,v), $i(u)$ or $j(v)$

In general, the metatheory follows the same schemes: inversion lemmas, type inference, subject reduction, normalization…

Time to switch to contexts for typing

$$\frac{}{\Gamma \vdash x : A} \text{if } (x{:}A) \in \Gamma \qquad \frac{\Gamma\,(x : A) \vdash t : B}{\Gamma \vdash \lambda\, x{:}A.\ t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_1(t) : A} \qquad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_2(t) : B} \qquad \frac{\Gamma \vdash t : A + B \quad \Gamma(x{:}A) \vdash u{:}C \quad \Gamma(y{:}B) \vdash v{:}C}{\Gamma \vdash \delta(t, x.u, y.v) : C}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash (t,u) : A \times B} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash i_B(t) : A + B} \qquad \frac{\Gamma \vdash u : B}{\Gamma \vdash j_A(t) : A + B}$$

Annotations to keep type uniqueness / type inference

Heyting's semantics

A (canonical) proof of…	… is:
$A \wedge B$	(a, b) where a is a proof of A, b a proof of B
$A \vee B$	(ε, c) with ε=0 and c proof of A or ε=1 and c proof of B
$A \Rightarrow B$	f, with f(a) proof of B for any proof a of A
$\forall x . A$	f, with f(t) : A[x\t] for any object t
$\exists x . A$	(t, a) with a a proof of A[x\t]

Heyting's semantics

A (canonical) proof of…	… is:
$A \wedge B$	(a, b) where a : A, b : B
$A \vee B$	(ε, c) with ε=0 and c : A or ε=1 and c : B
$A \Rightarrow B$	f, with f(a) : B for any a : A
$\forall x . A$	f, with f(t) : A[x\t] for any object t
$\exists x . A$	(t, a) with a : A[x\t]

Implicational Fragment

$$\frac{}{\Gamma \vdash x : A} \text{if } (x{:}A) \in \Gamma$$

Axiom $$\frac{}{\Gamma \vdash x : A} \text{if } (x{:}\ A) \in \Gamma$$

$$\frac{\Gamma\ (x : A) \vdash t : B}{\Gamma \vdash \lambda\ x{:}A.\ t : A \rightarrow B}$$

$\Rightarrow$–i $$\frac{\Gamma\ (x : A) \vdash t : B}{\Gamma \vdash \lambda\ x{:}A.\ t : A \Rightarrow B}$$

$$\frac{\Gamma \vdash t : A \rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B}$$

$\Rightarrow$–e $$\frac{\Gamma \vdash t : A \Rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B}$$

LF

$$[]\ \text{wf} \qquad \frac{\Gamma \vdash T : s}{\Gamma\ (x{:}T)\ \text{wf}} \qquad \frac{\Gamma\ \text{wf}}{\Gamma \vdash x : T} \qquad \frac{\Gamma\ \text{wf}}{\Gamma \vdash \text{Type} : \text{Kind}}$$

$$\frac{\Gamma \vdash A : \text{Type} \qquad \Gamma\ (x{:}A) \vdash B : s}{\Gamma \vdash \Pi\, x : A\, .\, B : s} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash A : s}{\Gamma \vdash t : B}\ A =_\beta B$$

$$\frac{\Gamma \vdash \Pi\, x : A\, .\, B : \text{Type} \qquad \Gamma(x{:}A) \vdash t : B}{\Gamma \vdash \lambda\, x : A.t : \Pi\, x : A\, .\, B} \qquad \frac{\Gamma \vdash t : \Pi\, x : A\, .\, B \qquad \Gamma \vdash u : A}{\Gamma \vdash (t\ u) : B\, [x \setminus u]}$$

Metatheory

No unicity of typing ! (modulo only beta-conversion)

1. $\Gamma\Delta$ wf $\Rightarrow \Gamma$ wf and $\Gamma \vdash t:T \Rightarrow \Gamma\Delta \vdash t:T$
2. $\Gamma(x:T)\Delta \vdash u:U \;\wedge\; \Gamma \vdash t:T \;\Rightarrow\; \Gamma\Delta[x\backslash t] \vdash u[x\backslash t] : U[x\backslash t]$

   and $\Gamma(x:T)\Delta$ wf $\wedge\; \Gamma \vdash t:T \;\Rightarrow\; \Gamma\Delta[x\backslash t]$ wf
3. Inversion lemma :
   - If $\Gamma \vdash x : T$ then $(x:T') \in \Gamma,\; T =_\beta T'$ and $\Gamma \vdash T' : s$
   - If $\Gamma \vdash \lambda x{:}A.t : B$ then $B =_\beta \Pi x{:}A.C$ and $\Gamma(x:A) \vdash t : C$
   - If $\Gamma \vdash (t\ u) : C$ then $\Gamma \vdash t : \Pi x{:}A.B$ $\Gamma \vdash u : A$ and $C =_\beta B[x\backslash\ u]$
   - If $\Gamma \vdash \Pi x{:}A.B : T$ then $\Gamma \vdash A : \mathrm{Type},\; \Gamma(x:A) \vdash B : s$ and $T = s$

If $\Gamma \vdash t : T$ and $\Gamma \vdash t : U$ then $T =_\beta U$

Adding disjoint sum types

$$t ::= x \mid \lambda x : t\,.\,t \mid (t\ t) \mid \Pi\, x : t\,.\,t \mid t + t \mid i_{t}(t) \mid j_{t}(t) \mid \delta(t, t, t)$$

$$\frac{\Gamma \vdash A : \mathrm{Type} \quad \Gamma \vdash B : \mathrm{Type}}{\Gamma \vdash A+B : \mathrm{Type}}$$

$$\frac{\Gamma \vdash A+B : \mathrm{Type} \quad \Gamma \vdash t : A}{\Gamma \vdash i_{B}(t) : A+B}$$

$$\frac{\Gamma \vdash A+B : \mathrm{Type} \quad \Gamma \vdash t : B}{\Gamma \vdash j_{A}(t) : A+B}$$

$$\frac{\Gamma \vdash t : A+B \quad \Gamma \vdash C : \mathrm{Type} \quad \Gamma \vdash u : A \rightarrow C \quad \Gamma \vdash v : B \rightarrow C}{\Gamma \vdash \delta(t,u,v) : C}$$

for decidability of type checking / inference

Adding Sigma types

$$t ::= x \mid \lambda x : t \,.\, t \mid (t\ t) \mid \Pi x : t \,.\, t \mid t + t \mid i(t) \mid j(t) \mid \delta(t, t, t)$$

$$\mid \Sigma x : t \,.\, t \mid \underset{\Sigma x : t \,.\, t}{(t, t)} \mid \pi_1(t) \mid \pi_2(t)$$

$\Sigma x : A \,.\, B$ written $A \times B$ when $x \notin FV(B)$

$$\frac{\Gamma \vdash A : Type \quad \Gamma(x{:}A) \vdash B : Type}{\Gamma \vdash \Sigma x : A \,.\, B : Type}$$

$$\frac{\Gamma \vdash \Sigma x : A \,.\, B : Type \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B[x \setminus t]}{\Gamma \vdash \underset{\Sigma x : A \,.\, B}{(t, u)} : \Sigma x : A \,.\, B}$$

$$\frac{\Gamma \vdash t : \Sigma x : A \,.\, B}{\Gamma \vdash \pi_1(t) : A} \qquad \frac{\Gamma \vdash t : \Sigma x : A \,.\, B}{\Gamma \vdash \pi_2(t) : B[x \setminus \pi_1(t)]}$$

for decidability of type checking / inference

Metatheory (2)

1. $\Gamma\Delta$ wf $\Rightarrow \Gamma$ wf and $\Gamma\vdash t{:}T \Rightarrow \Gamma\Delta\vdash t{:}T$

2. $\Gamma(x{:}T)\Delta \vdash u{:}U \;\wedge\; \Gamma\vdash t{:}T \;\Rightarrow\; \Gamma\Delta[x\backslash t] \vdash u[x\backslash t] : U[x\backslash t]$

   and $\Gamma(x{:}T)\Delta$ wf $\wedge\; \Gamma\vdash t{:}T \;\Rightarrow\; \Gamma\Delta[x\backslash t]$ wf

3. Inversion lemma :
   - If $\Gamma \vdash x : T$ then $(x{:}T') \in\Gamma$, $T =_\beta T'$ and $\Gamma \vdash T' : s$
   - If $\Gamma \vdash \lambda x{:}A.t : B$ then $B =_\beta \Pi x{:}A.C$ and $\Gamma(x{:}A) \vdash t : C$
   - If $\Gamma \vdash (t\ u) : C$ then $\Gamma\vdash t : \Pi x{:}A.B$ $\Gamma\vdash u : A$ and $C =_\beta B[x\backslash u]$
   - If $\Gamma \vdash \Pi x{:}A.B : T$ then $\Gamma\vdash A : Type$, $\Gamma(x{:}A)\vdash B : s$ and $T{=}s$
   - If $\Gamma\vdash (t,u)_{\Sigma x:A.B} : C$ then $\Gamma\vdash t{:}A$, $\Gamma\vdash u{:}B[x\backslash u]$, $C =_\beta \Sigma x{:}A.B$
   - If $\Gamma \vdash \Sigma x{:}A.B : T$ then $\Gamma\vdash A : Type$, $\Gamma(x{:}A)\vdash B : s$ and $T{=}Type$

If $\Gamma \vdash t : T$ and $\Gamma \vdash t : U$ then $T =_\beta U$ (same for $A+B$)