

MPRI

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Higher-Order Logic

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Two simple properties of simply typed λ -calculus

A very simple model

For each atomic type α , take a set $U(\alpha)$

Define : $|a| \equiv U(a)$

$|A \rightarrow B| \equiv |B|^{|A|}$ (total functions)

Given \mathcal{J} s.t. $\mathcal{J}(x^A) \in |A|$ we define for every $t : B$, $|t|_{\mathcal{J}} \in |B|$

- ▶ $|x^B|_{\mathcal{J}} \equiv \mathcal{J}(x^B)$
- ▶ $|(t \ u)|_{\mathcal{J}} \equiv |t|_{\mathcal{J}}(|u|_{\mathcal{J}})$
- ▶ $|\lambda x^A . t|_{\mathcal{J}} \equiv \gamma \in |A| \mapsto |t|_{\mathcal{J}}; x^A \leftarrow \gamma$

Is this linked to normalization ?

A very simple model (2)

If we add the possibility for non-terminating functions :

$$Y_A : ((A \rightarrow A) \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A) \quad \text{s.t.} \quad Y_A F \triangleright F (Y_A F)$$

$$F : (A \rightarrow A) \rightarrow (A \rightarrow A)$$

$$Y F : (A \rightarrow A)$$

$$F (Y F) : (A \rightarrow A)$$

corresponds to $\text{let rec } f (x:A) = \dots (f \ t)\dots (f \ u)\dots$

What happens with the model ?

$|Y_A|$ is not (totally) defined

Normal λ -terms: $x, \lambda x.n, \lambda x_1.\lambda x_2.n \dots$

$(x n_1 n_2 \dots n_m)$

in the end : $\lambda x_1.\lambda x_2. \dots \lambda x_k. (x n_1 n_2 \dots n_m)$

Consider the following signature :

$0 : \iota, S : \iota \rightarrow \iota, + \times : \iota \rightarrow \iota \rightarrow \iota$

what terms $f : \iota \rightarrow \iota$ can we construct ?

$S (+ n) (\times n)$

$\lambda x^\iota. n$

$\lambda x^\iota. 0$

$(S n)$

$(+ n_1 n_2)$

$(\times n_1 n_2)$

x^ι

only polynomials (with constant exponents)

Aka : Simple Theory of Types Church 1940

Two atomic types : ι (natural numbers) and o (propositions)

Constants : $0 : \iota$, $S : \iota \rightarrow \iota$, $+ \times : \iota \rightarrow \iota \rightarrow \iota$

$\Rightarrow : o \rightarrow o \rightarrow o$

$\forall_T : (T \rightarrow o) \rightarrow o$

Propositions are objects

$$[] \text{ wf} \quad \frac{\Gamma \text{ wf} \quad \vdash A : o}{\Gamma ; A \text{ wf}}$$

$$Ax \quad \frac{\Gamma \text{ wf}}{\Gamma \vdash A} \quad \text{if } A \in \Gamma$$

Higher-Order Logic : rules

$$Ax \quad \frac{\Gamma \text{ wf}}{\Gamma \vdash A} \quad \text{if } A \in \Gamma$$

$$\frac{\Gamma ; A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$(\Rightarrow A B)$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall_T x^T. A} \quad \text{if } x \notin FV(\Gamma)$$

$(\forall_T \lambda x^T. A)$

$$\frac{\Gamma \vdash \forall_T A \quad \vdash t : T}{\Gamma \vdash (A t)}$$

$$\frac{\Gamma \vdash A \quad \vdash A' : o}{\Gamma \vdash A'} \quad \text{if } A =_\beta A'$$

If $\Gamma \vdash A$, then $\Gamma \text{ wf}$ and $\vdash A : o$

$$\perp \equiv \forall X^o. X^o$$

$$\wedge \equiv \lambda A^o. \lambda B^o. \forall X^o. (A^o \Rightarrow B^o \Rightarrow X^o) \Rightarrow X^o$$

$$\vee \equiv \lambda A^o. \lambda B^o. \forall X^o. (A^o \Rightarrow X^o) \Rightarrow (B^o \Rightarrow X^o) \Rightarrow X^o$$

These are inductive definitions
elim rules "for free". We need to prove the intro rules

$$= \equiv \lambda x^T. \lambda y^T. \forall P^{T \rightarrow o}. (P x^T) \Rightarrow (P y^T)$$

Existential

$$\exists_T \equiv \lambda P^{T \rightarrow o} . \forall X^o . (\forall x^T . (P x^T) \Rightarrow X) \Rightarrow X$$

How do we talk about x^y ?

Think of a function as a relation

$(\text{Exp } n \ 0 \ 1)$

$(\text{Exp } n \ m \ r) \Rightarrow (\text{Exp } n \ (S \ m) \ (n \times r))$

$$\text{Exp} \equiv \lambda n. \lambda m. \lambda r. (\forall R. (R \ 0 \ 1) \Rightarrow (\forall b \ c. (R \ b \ c) \Rightarrow (R \ (S \ b) \ (n \times c))))$$

$$\Rightarrow R \ m \ r$$

Naming functions: Hilbert operator

$\varepsilon(P)$ "The" object verifying P ("choice operator")

$$\frac{\vdash (P \ t)}{\vdash (P \ \varepsilon(P))}$$

$$\frac{\vdash (P \ t)}{\vdash (P \ \varepsilon(P))}$$