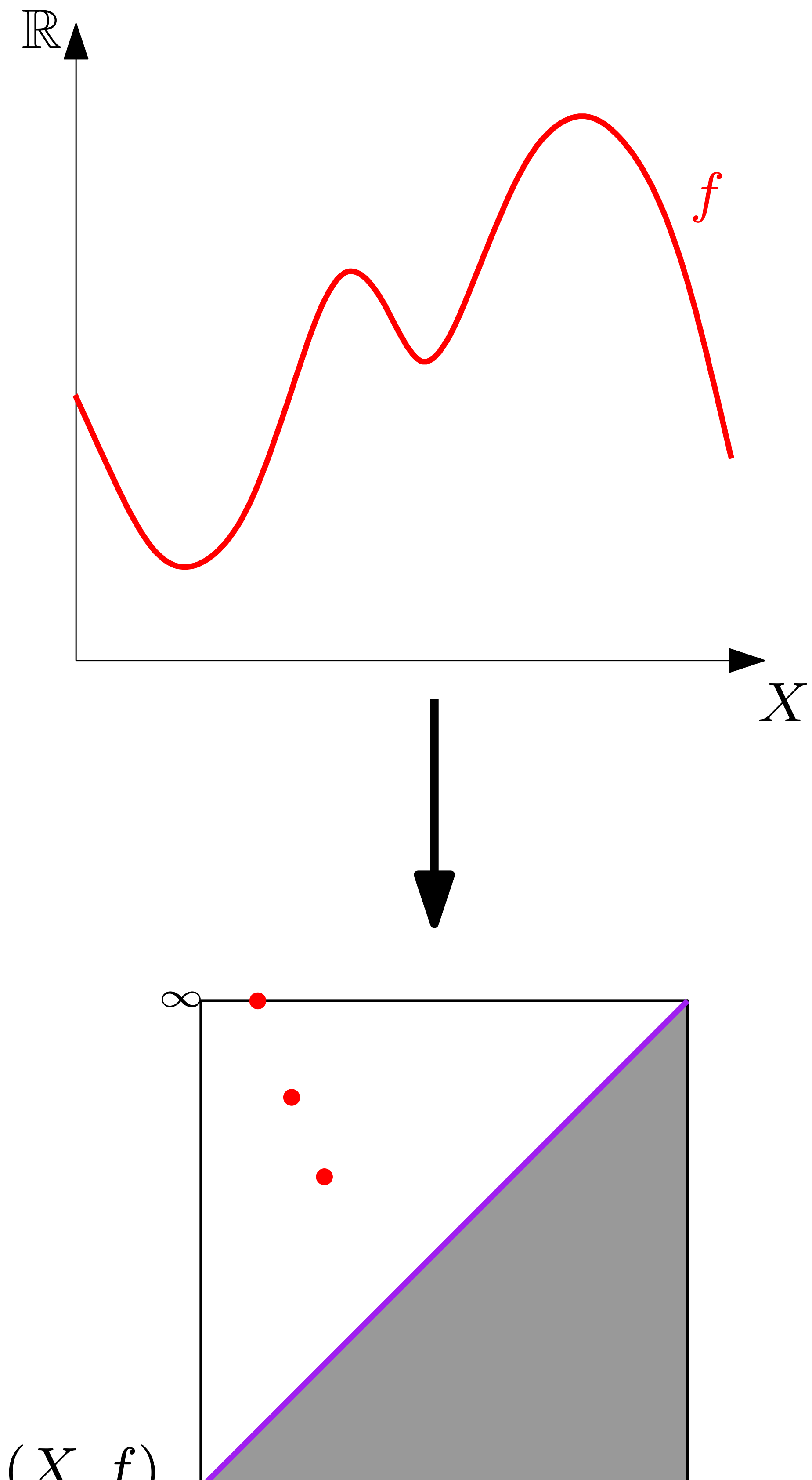
Topological Persistence

Topological Persistence (in a nutshell)

X topological space

$f : X \to \mathbb{R}$

$\mathrm{Dg}\, f$



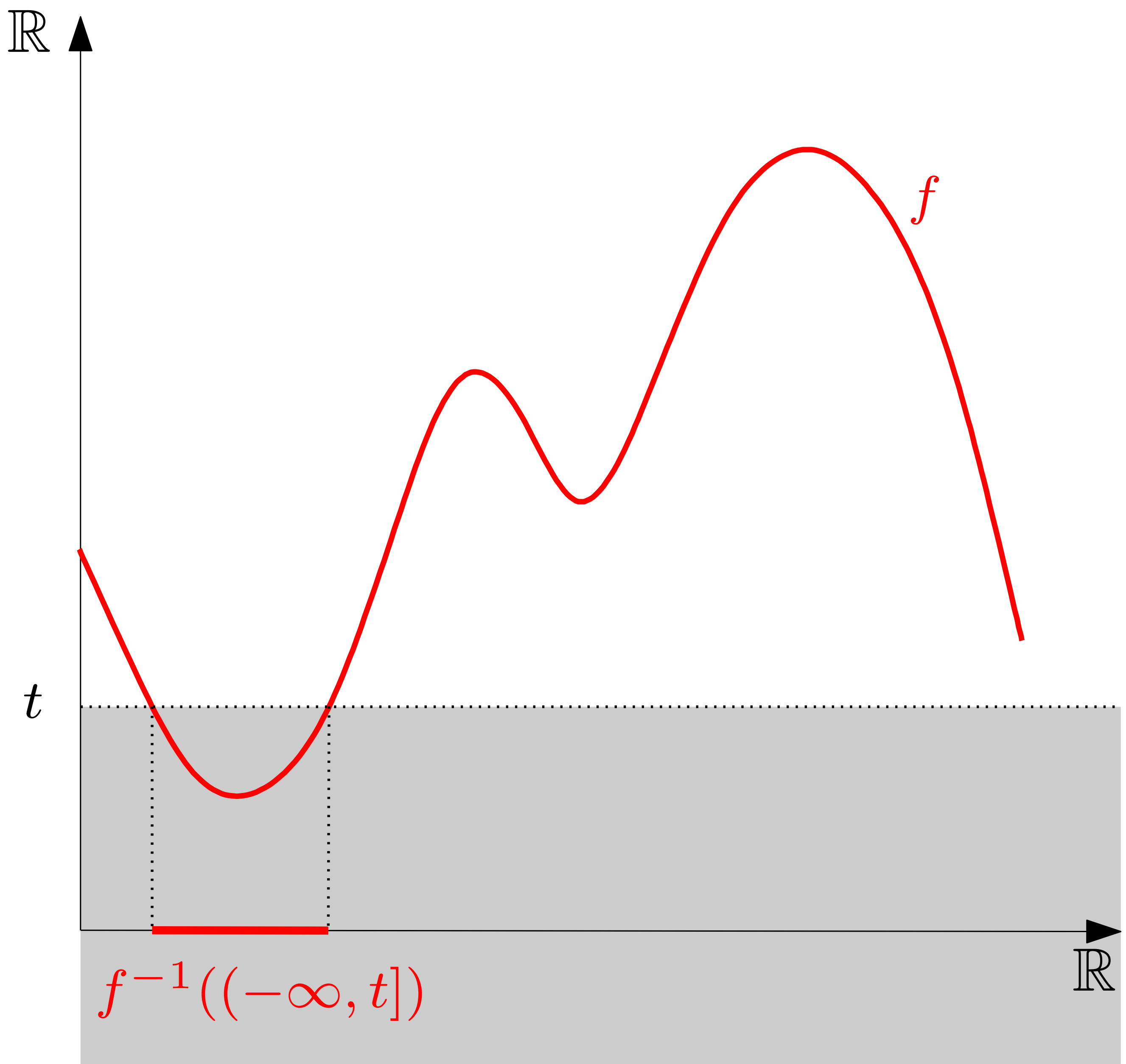
signature: *persistence diagram*

encodes the topological structure of the pair (X, f)

Topological Persistence (in a nutshell)

Inside the black box:

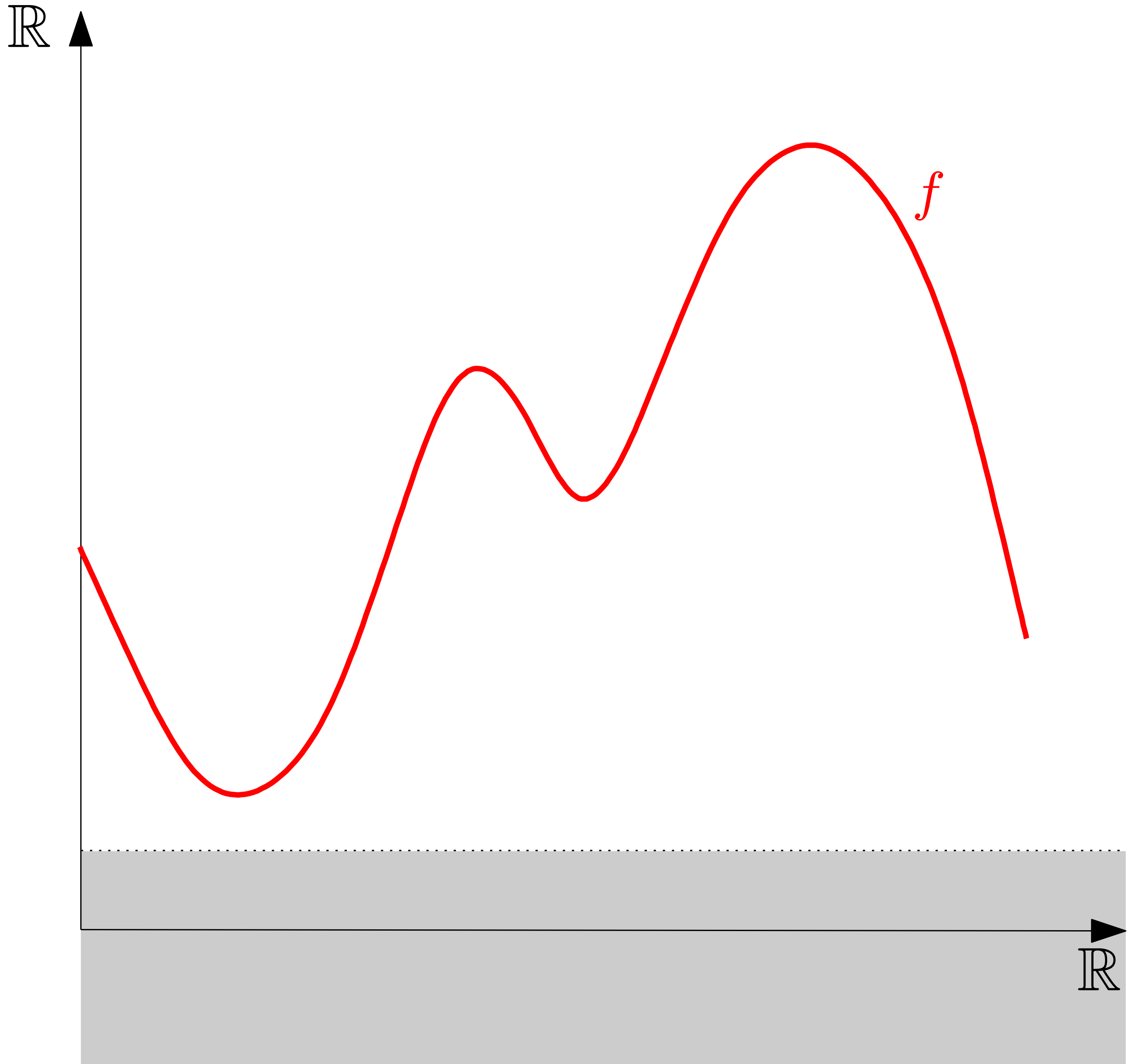
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



Topological Persistence (in a nutshell)

Inside the black box:

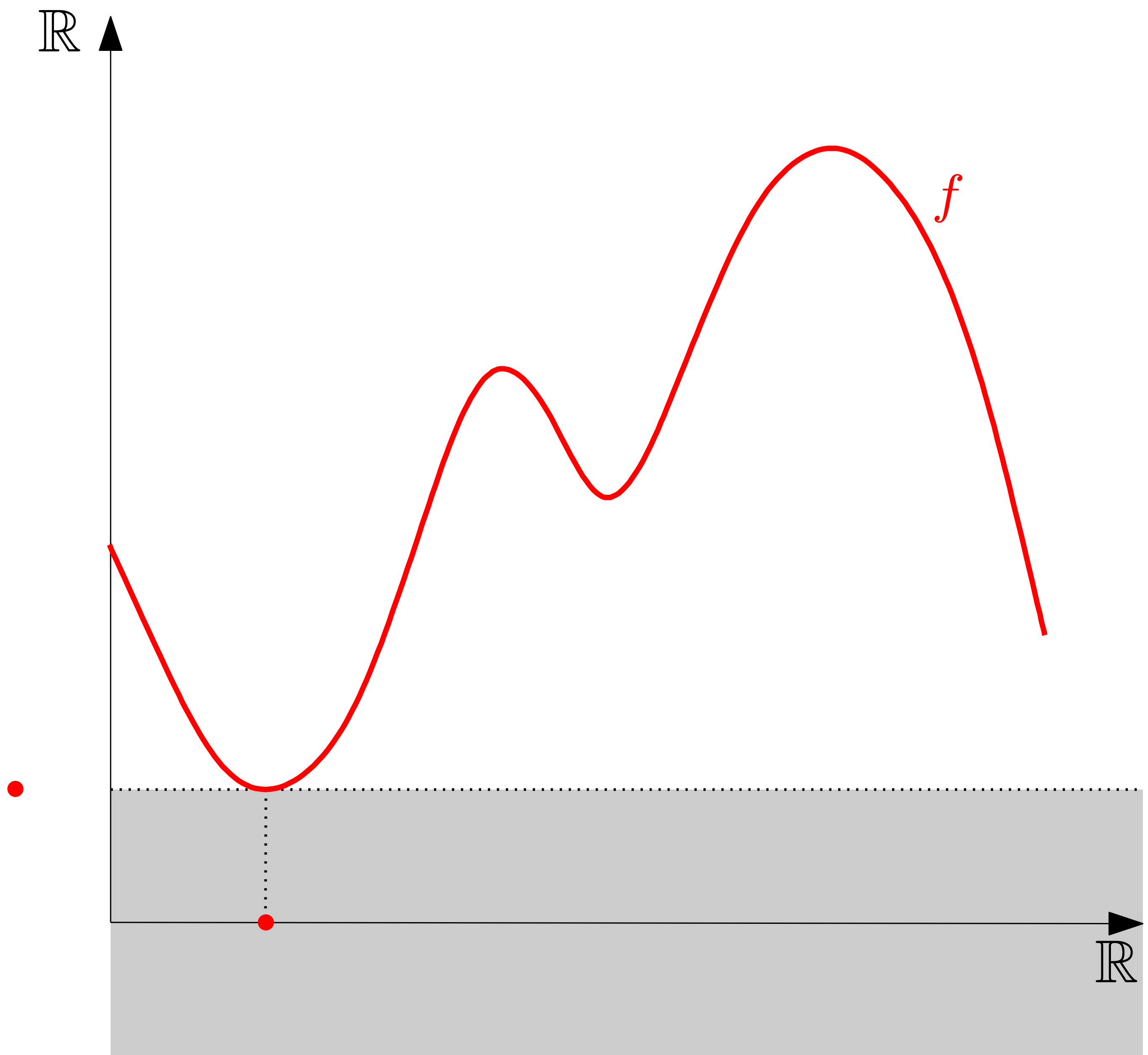
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



Topological Persistence (in a nutshell)

Inside the black box:

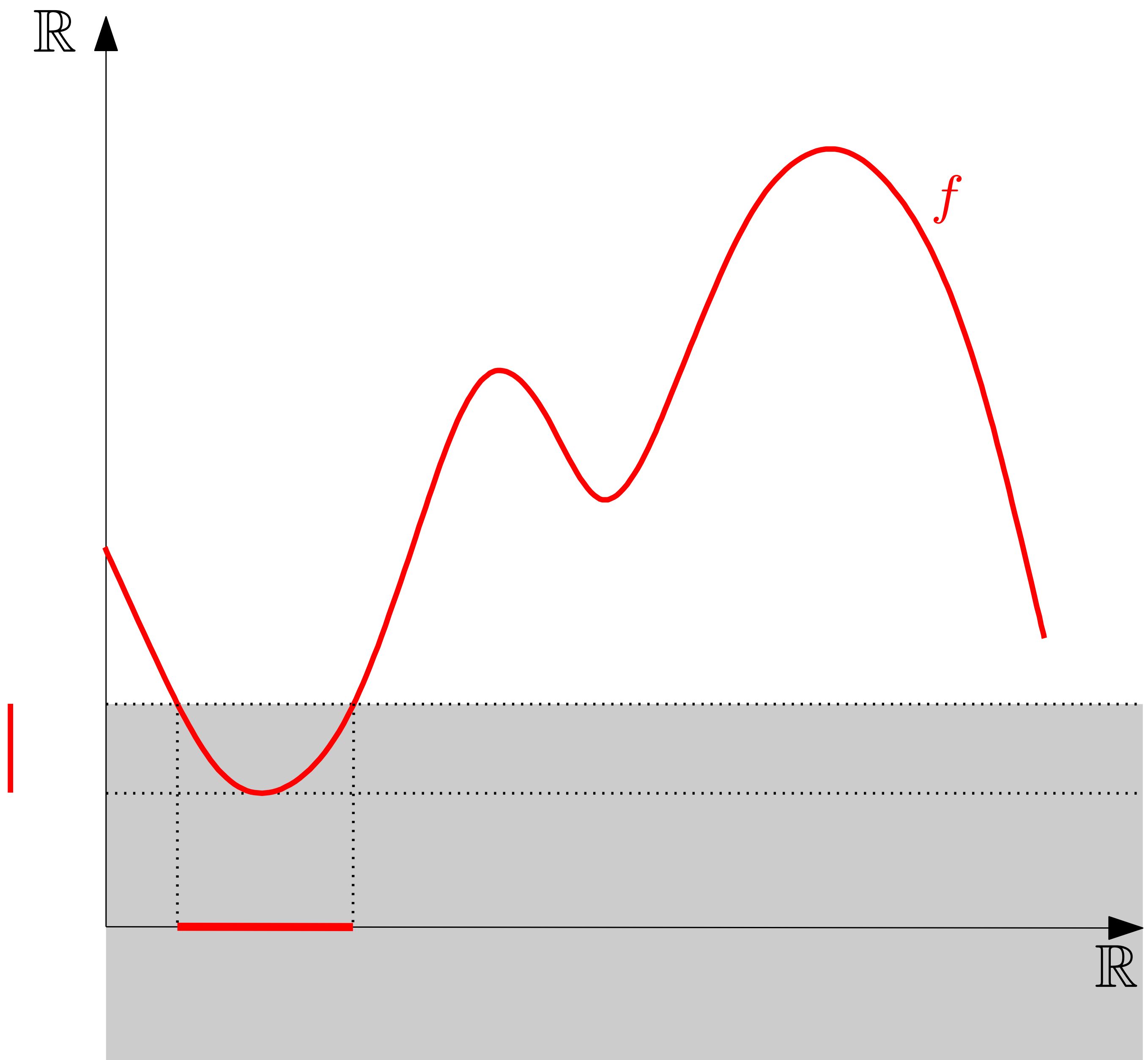
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



Topological Persistence (in a nutshell)

Inside the black box:

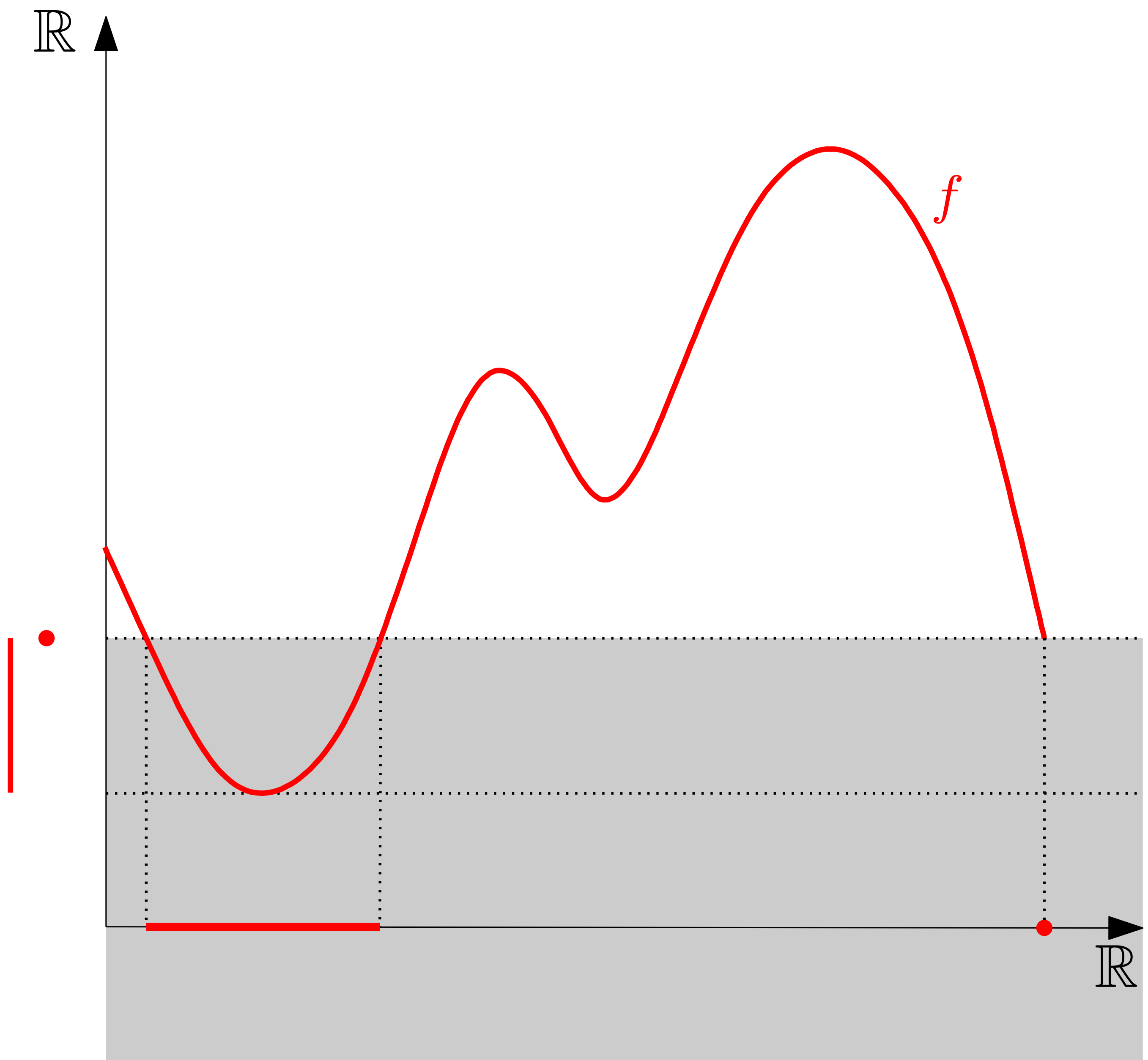
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



Topological Persistence (in a nutshell)

Inside the black box:

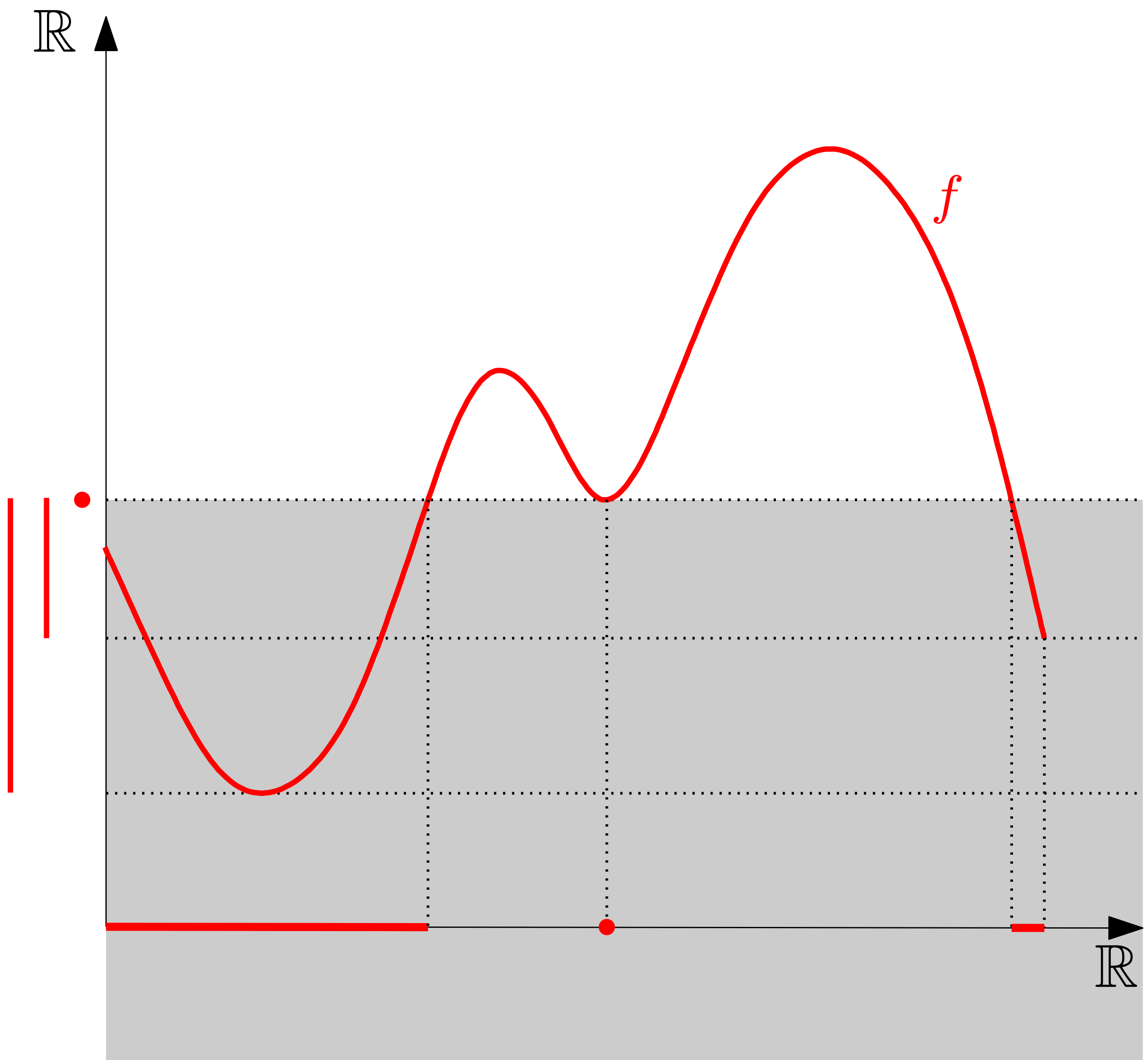
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty,t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



Topological Persistence (in a nutshell)

Inside the black box:

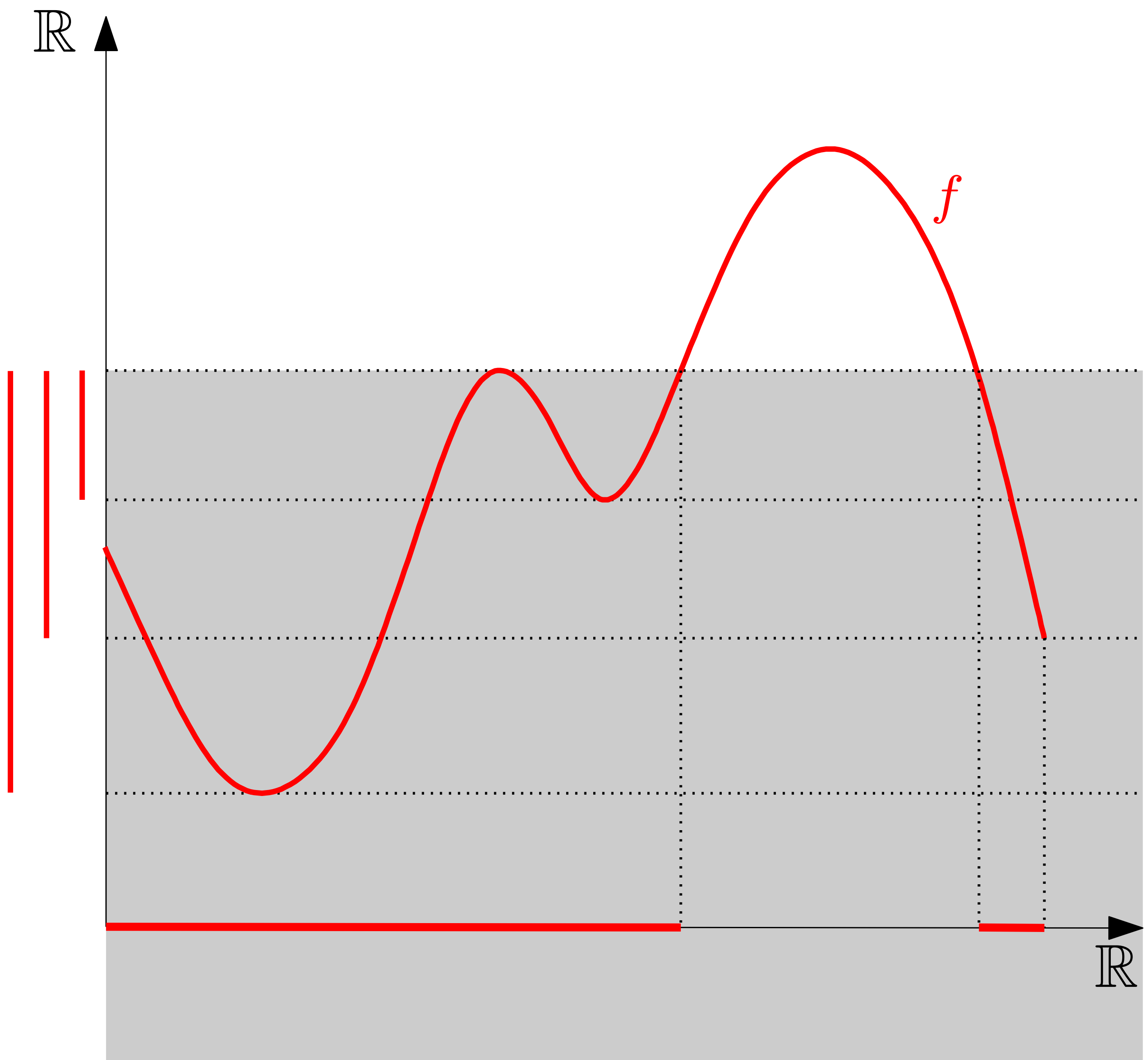
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



Topological Persistence (in a nutshell)

Inside the black box:

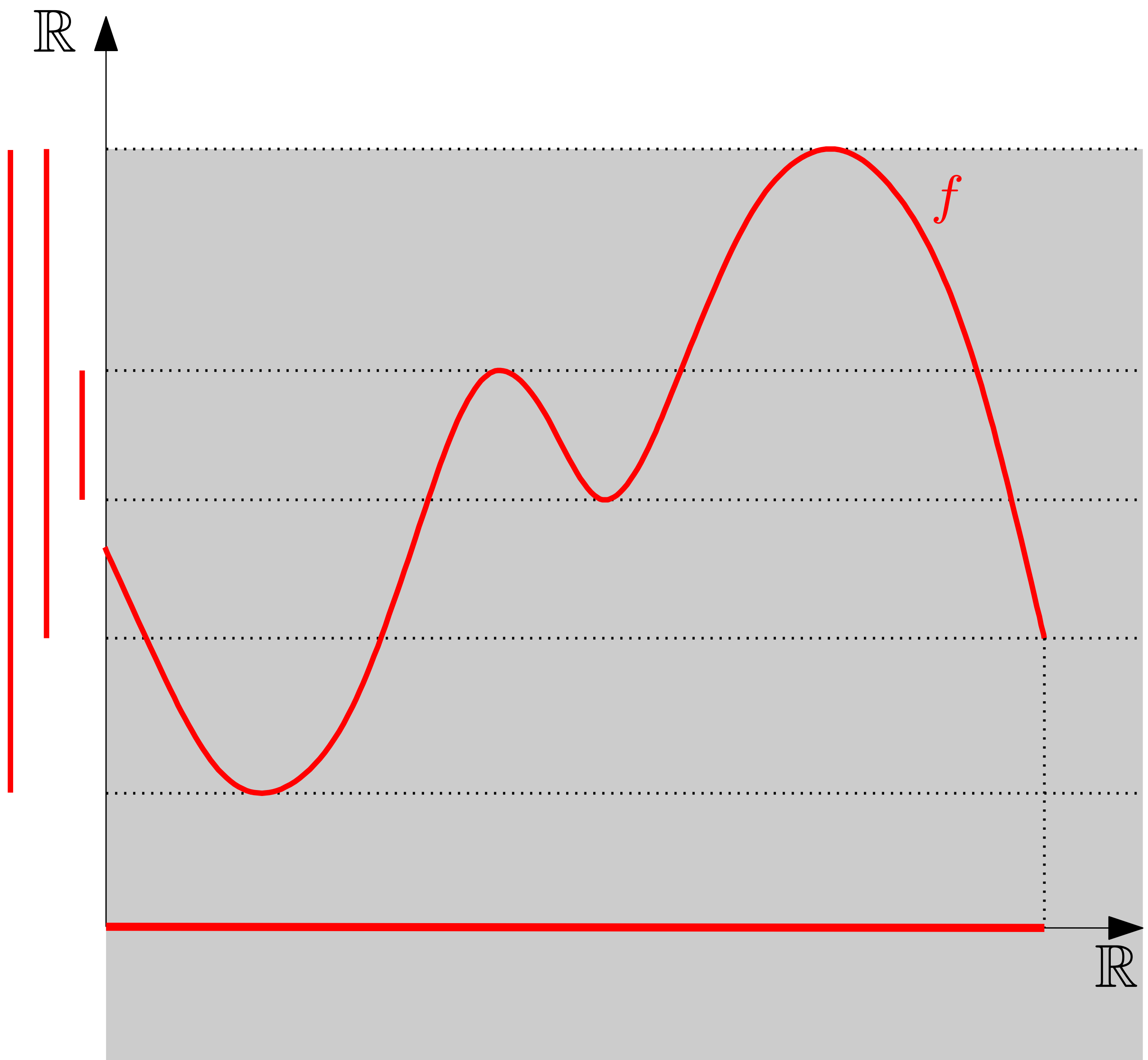
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



Topological Persistence (in a nutshell)

Inside the black box:

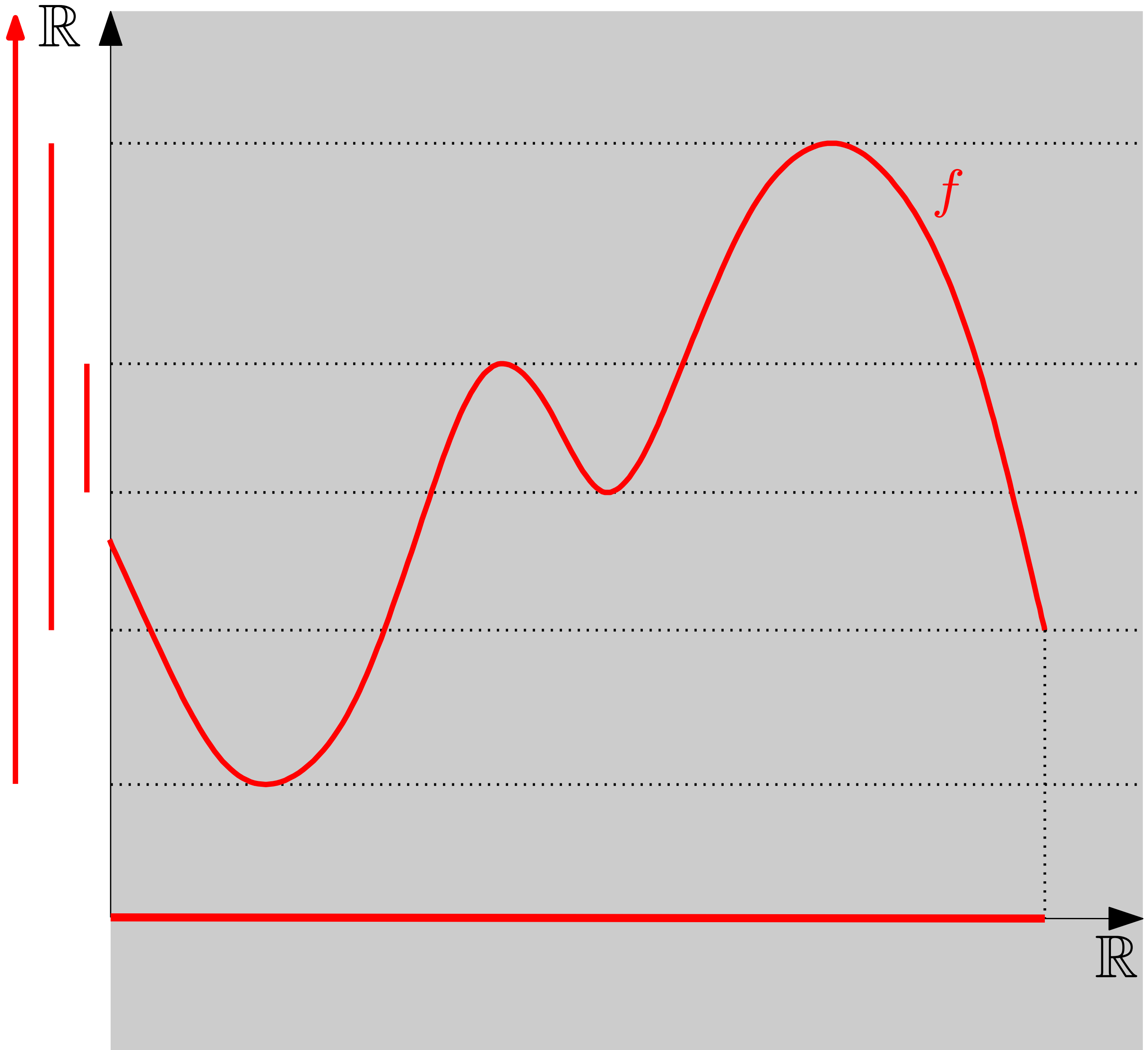
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



Topological Persistence (in a nutshell)

Inside the black box:

- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family
- Finite set of intervals (barcode) encodes births/deaths of topological features

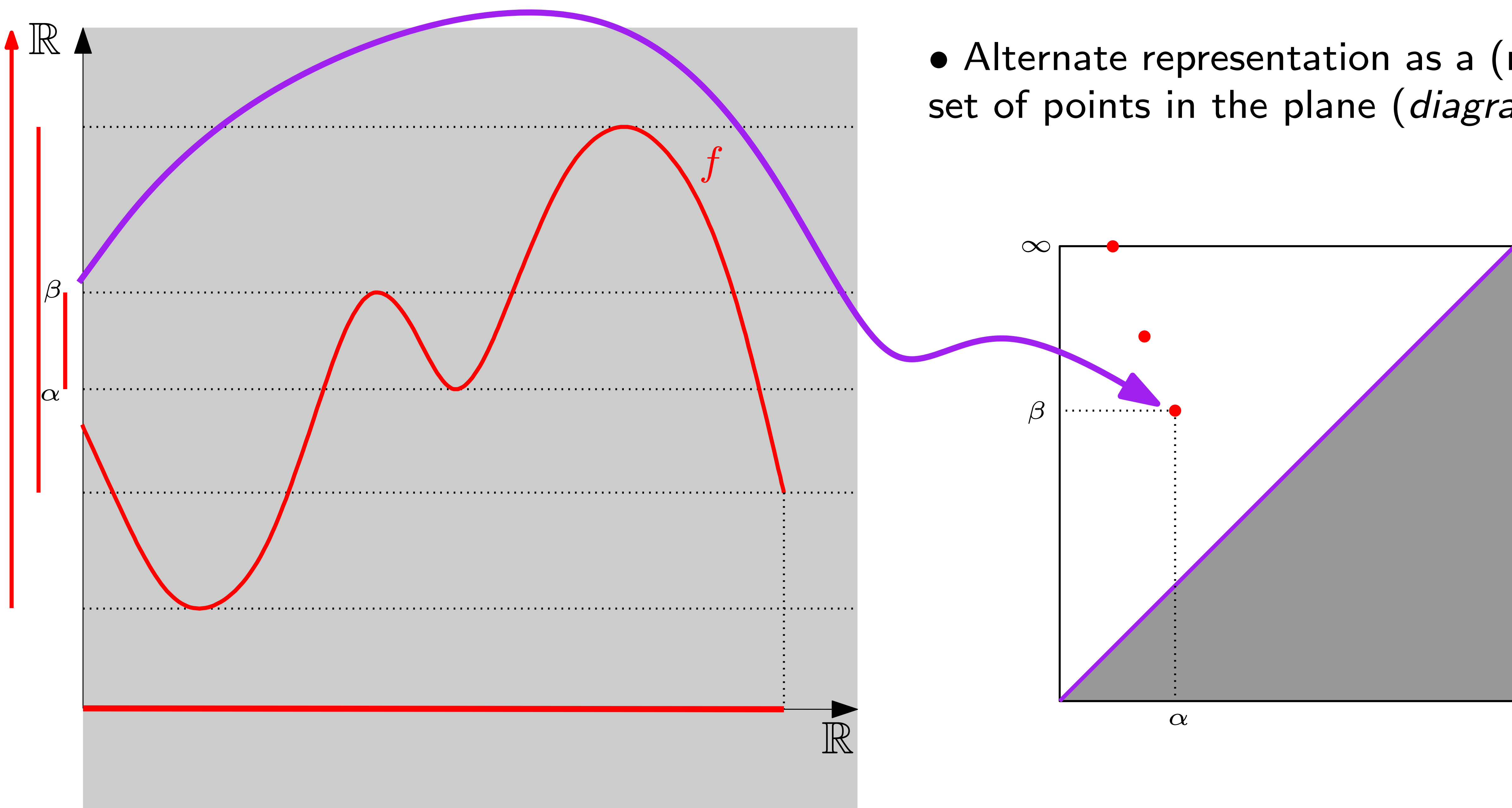


Topological Persistence (in a nutshell)

Inside the black box:

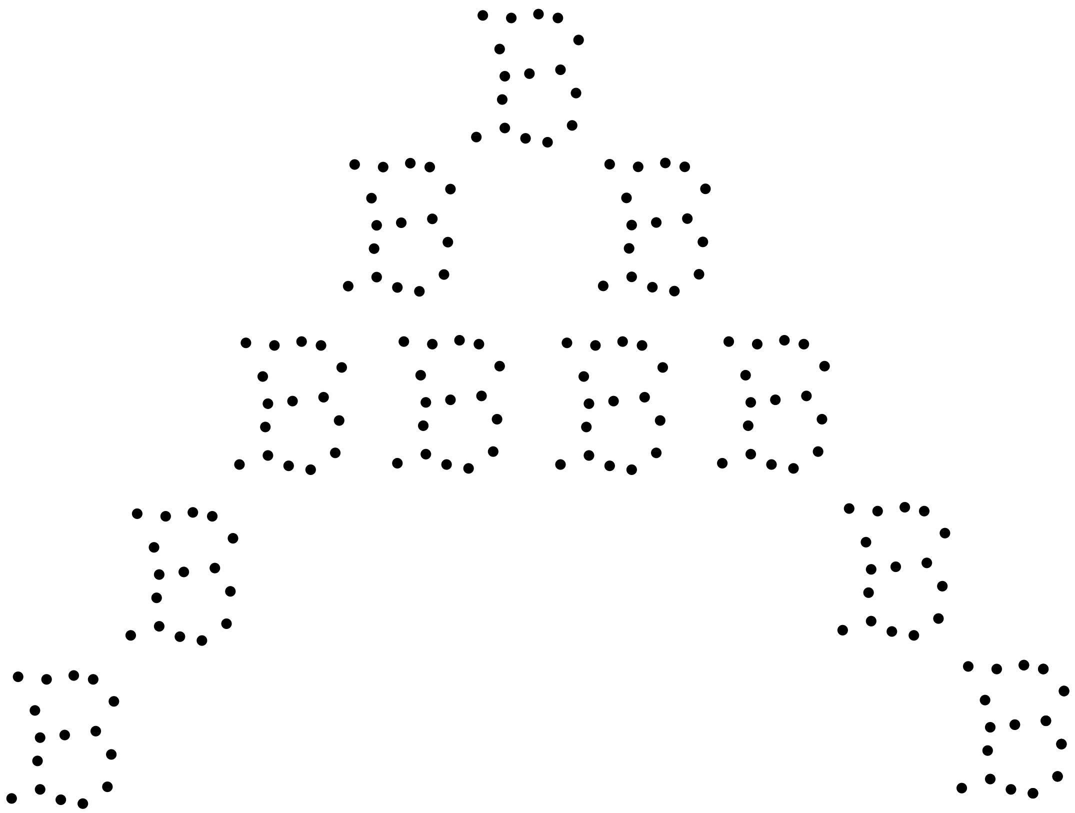
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family
- Finite set of intervals (barcode) encodes births/deaths of topological features

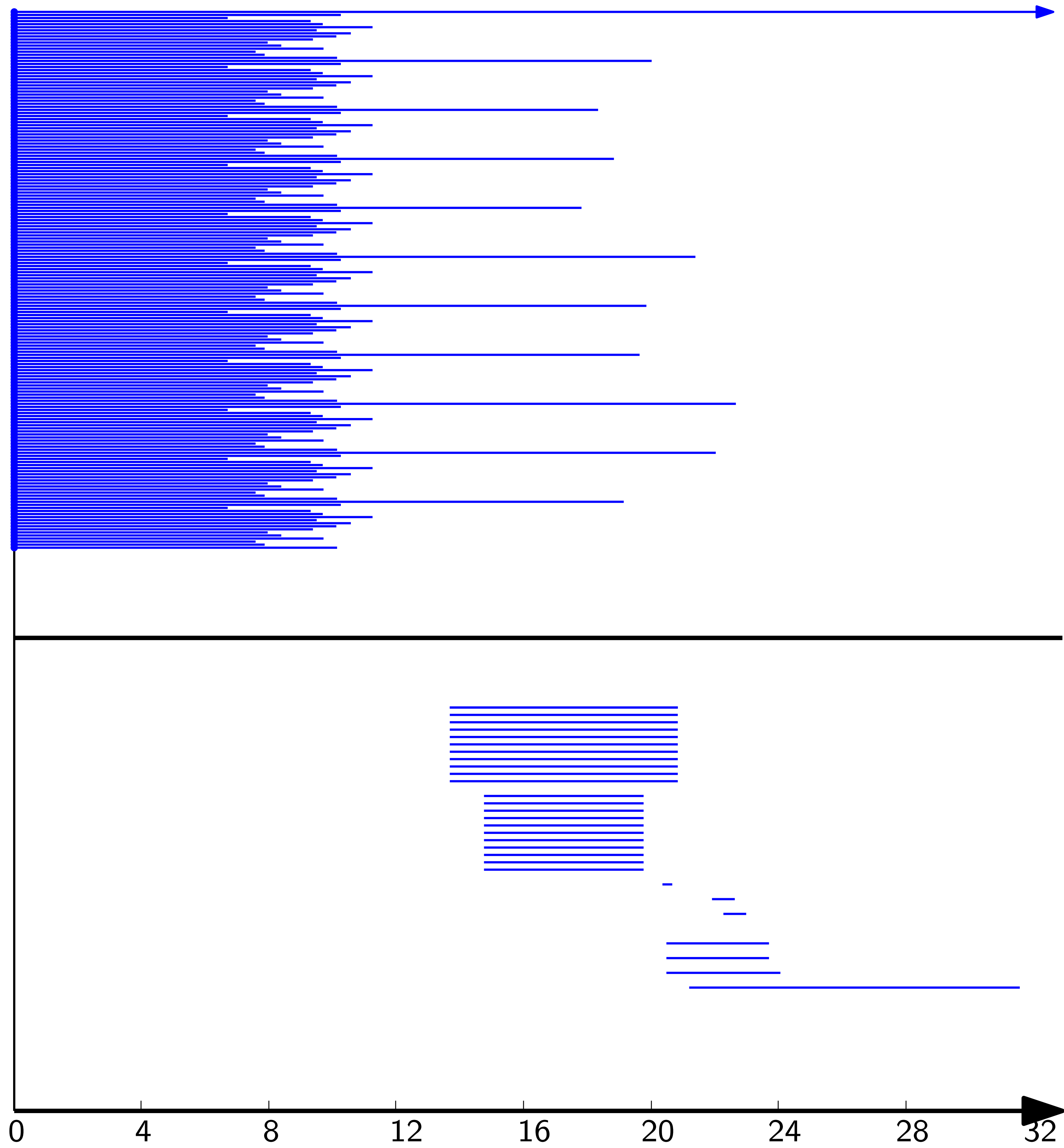
- Alternate representation as a (multi-) set of points in the plane (*diagram*).



Example: Distance Function

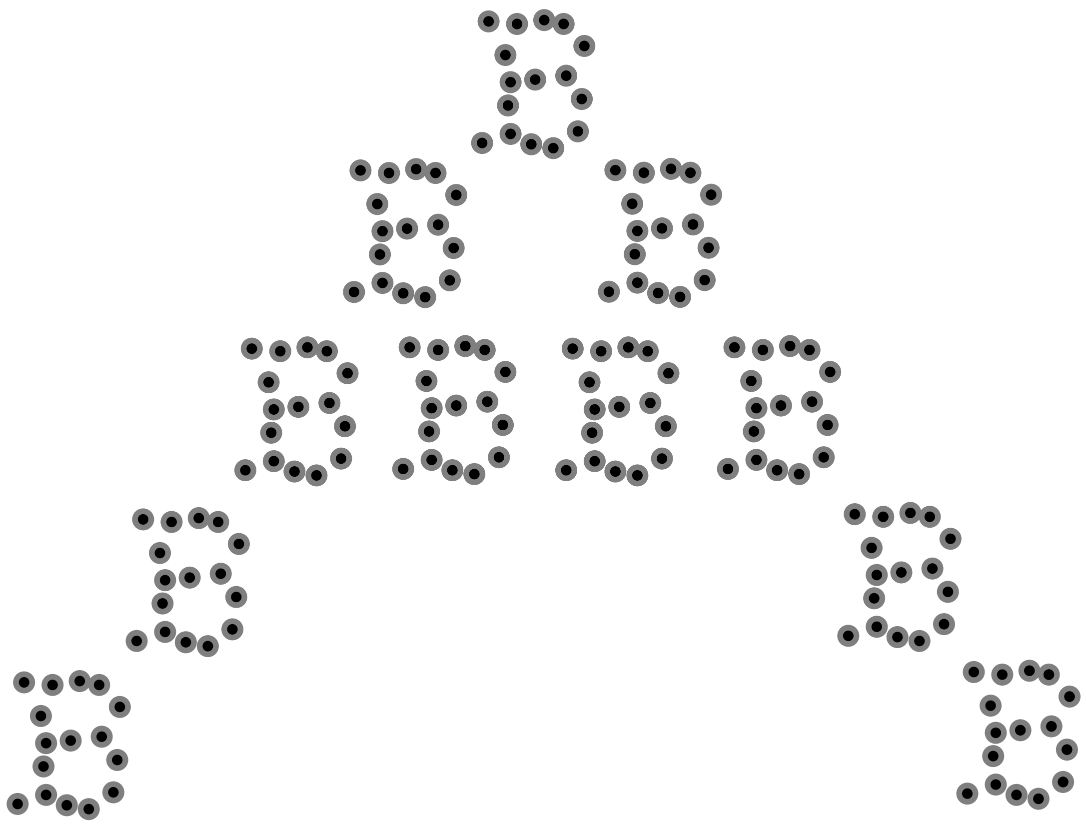
$$f_P: \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$

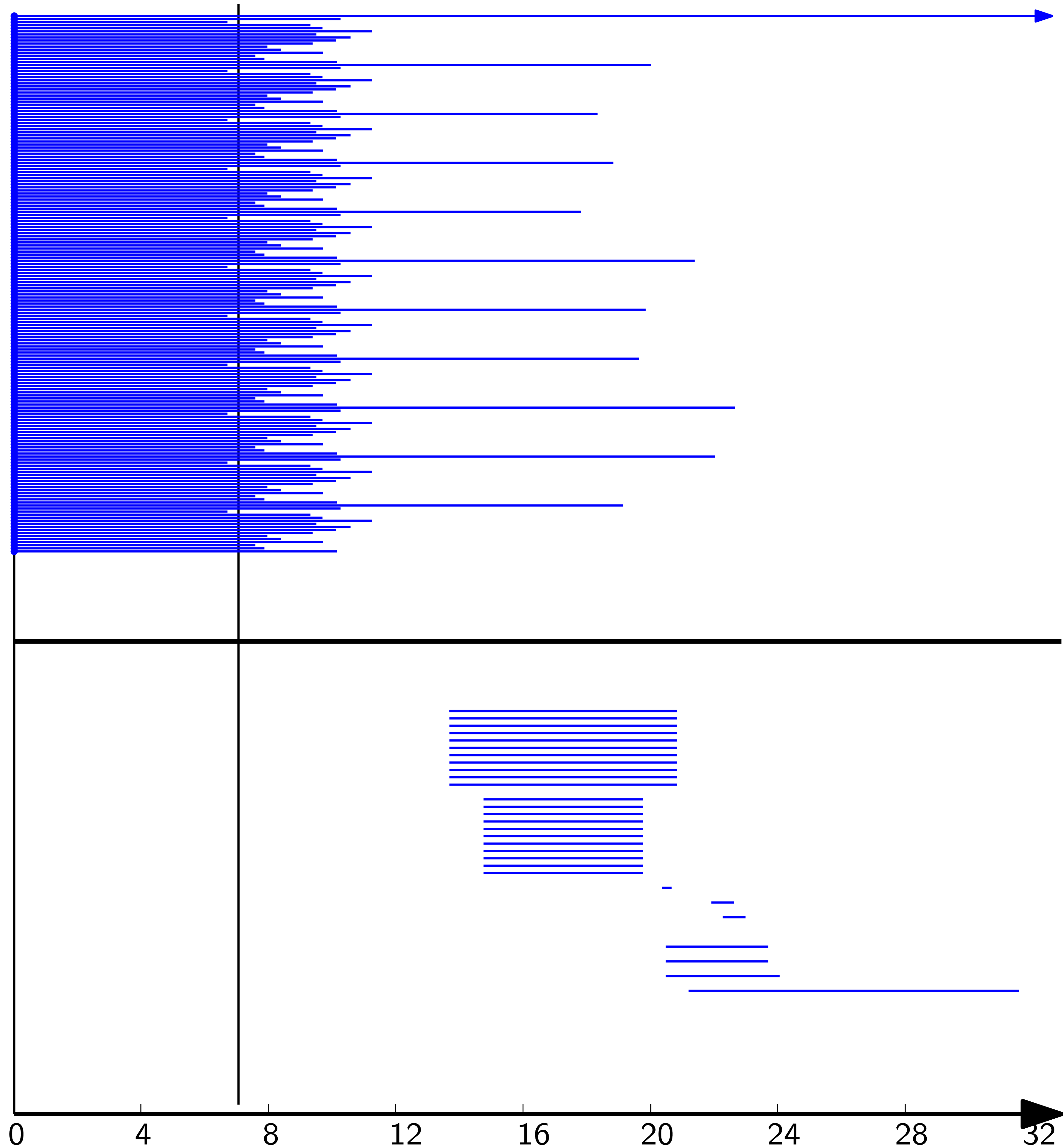




Example: Distance Function

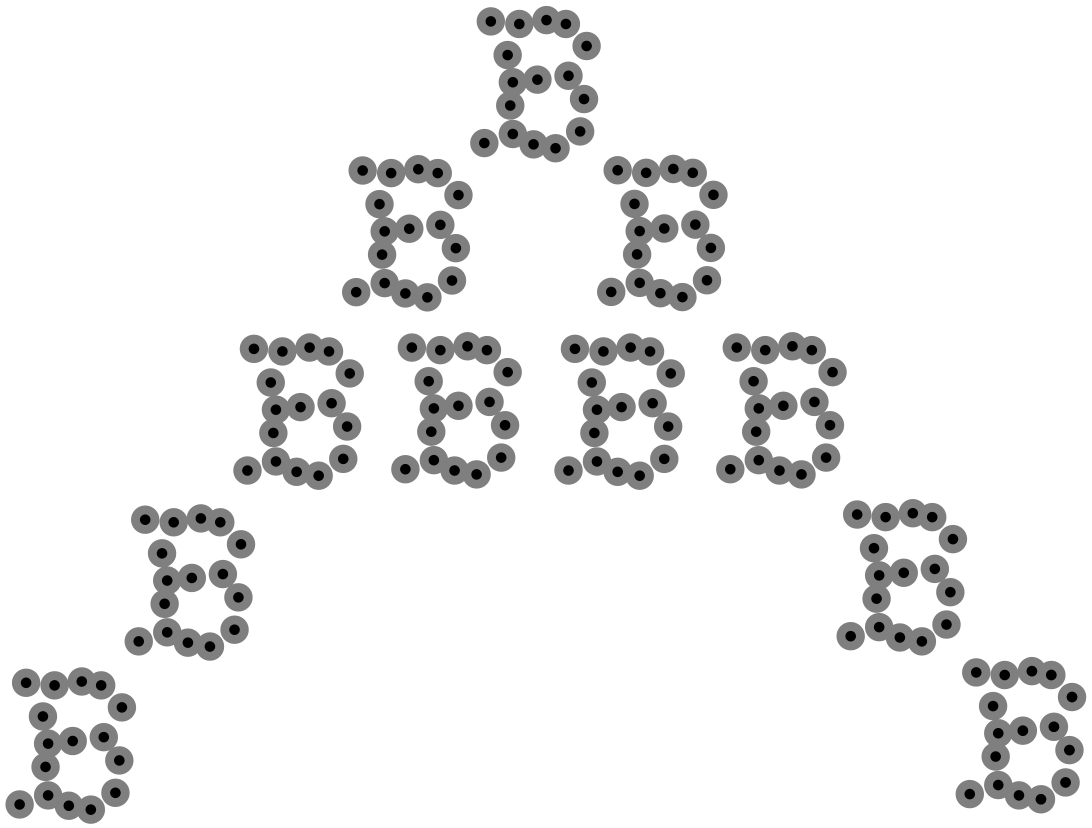
$$f_P : \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$

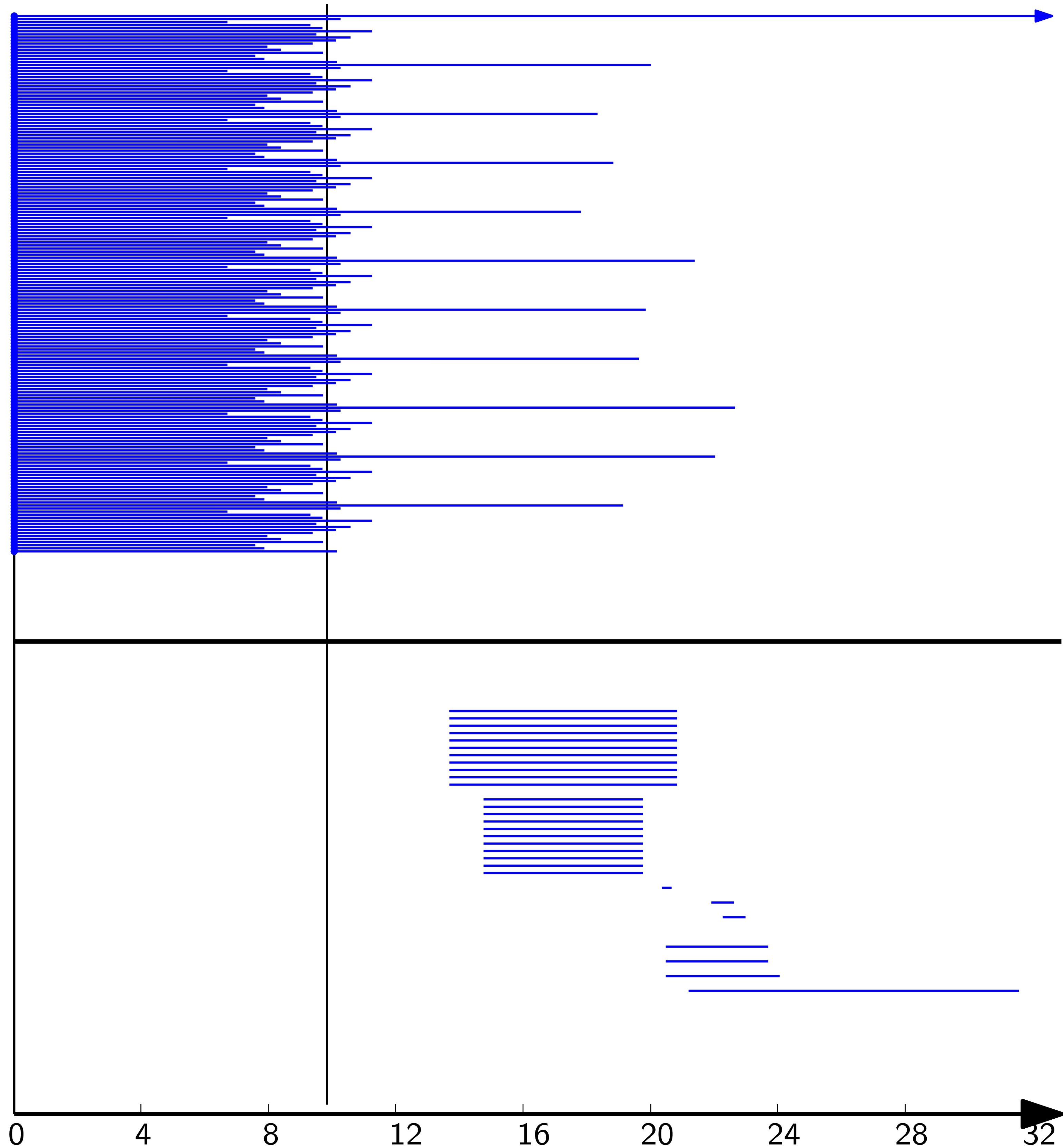




Example: Distance Function

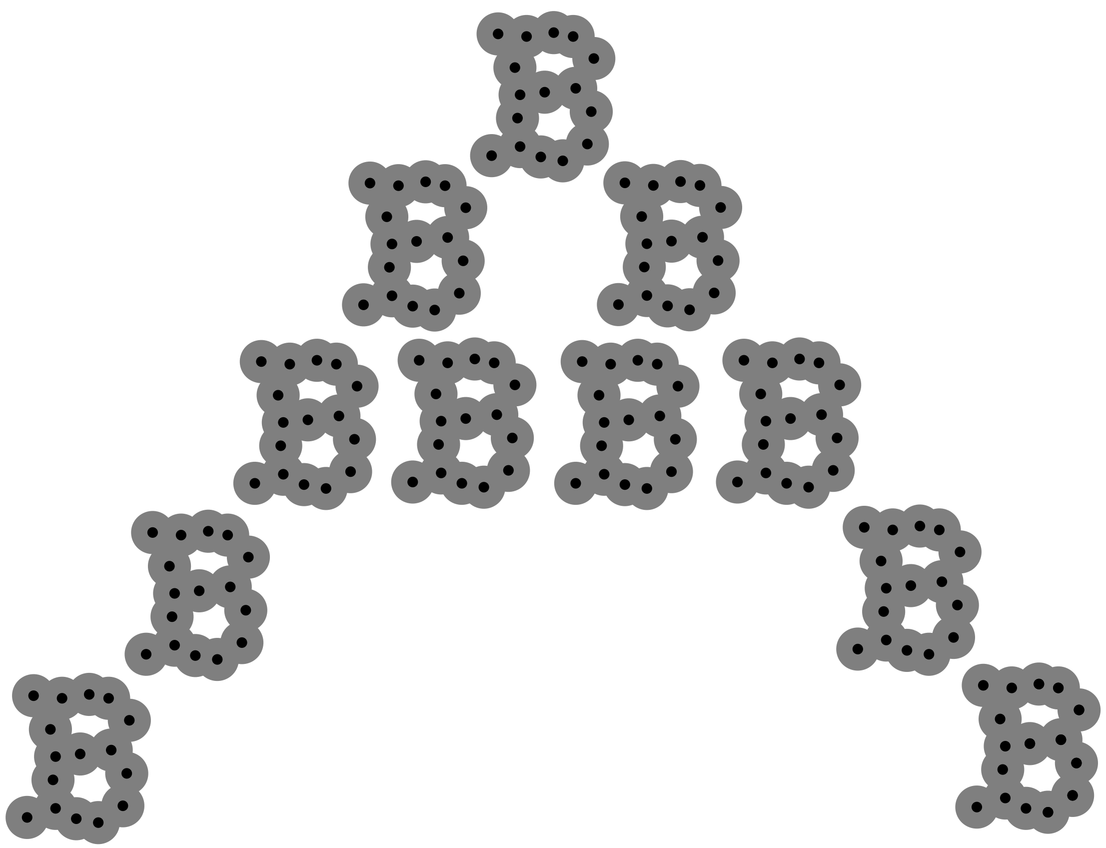
$$f_P : \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$

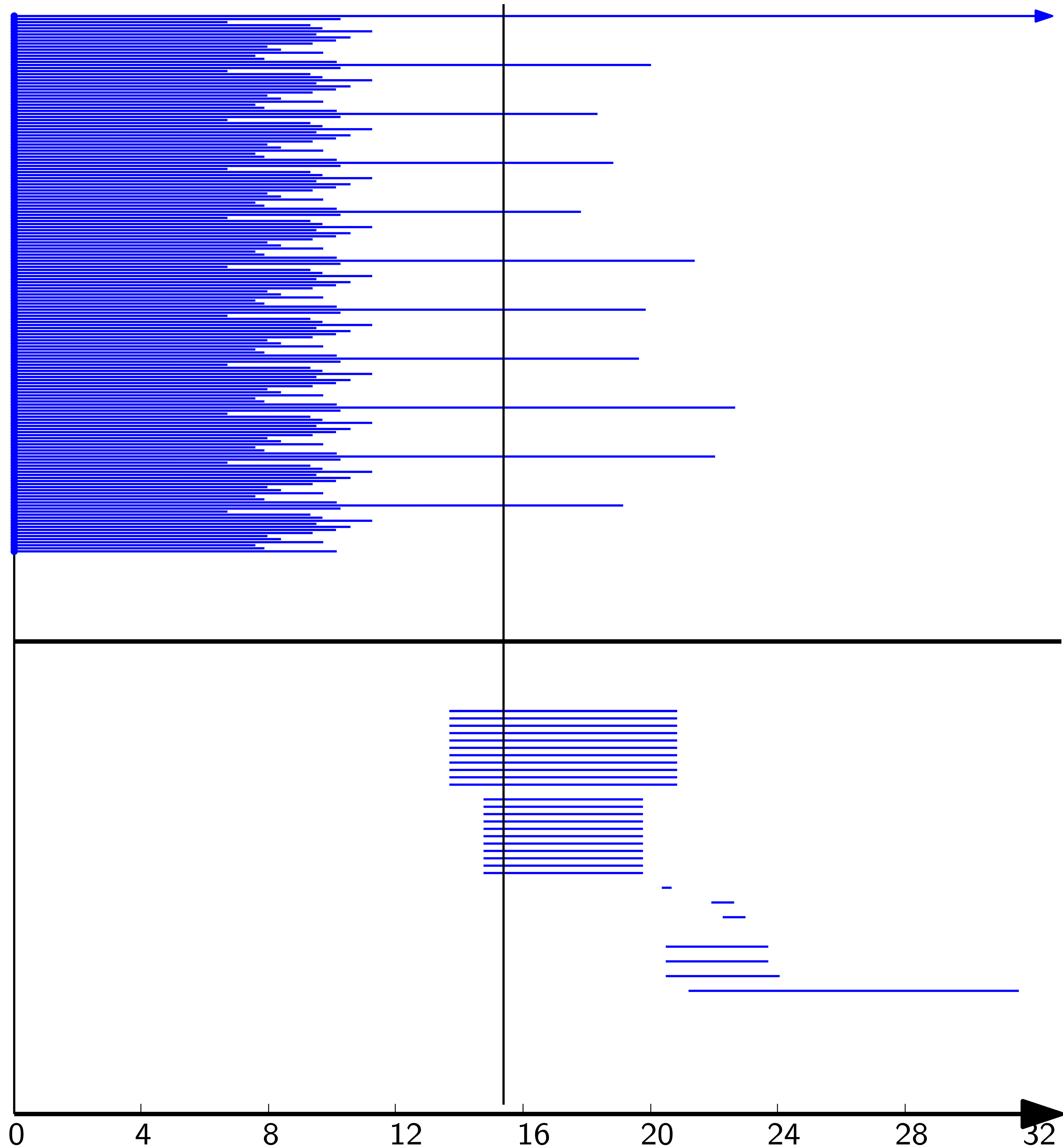




Example: Distance Function

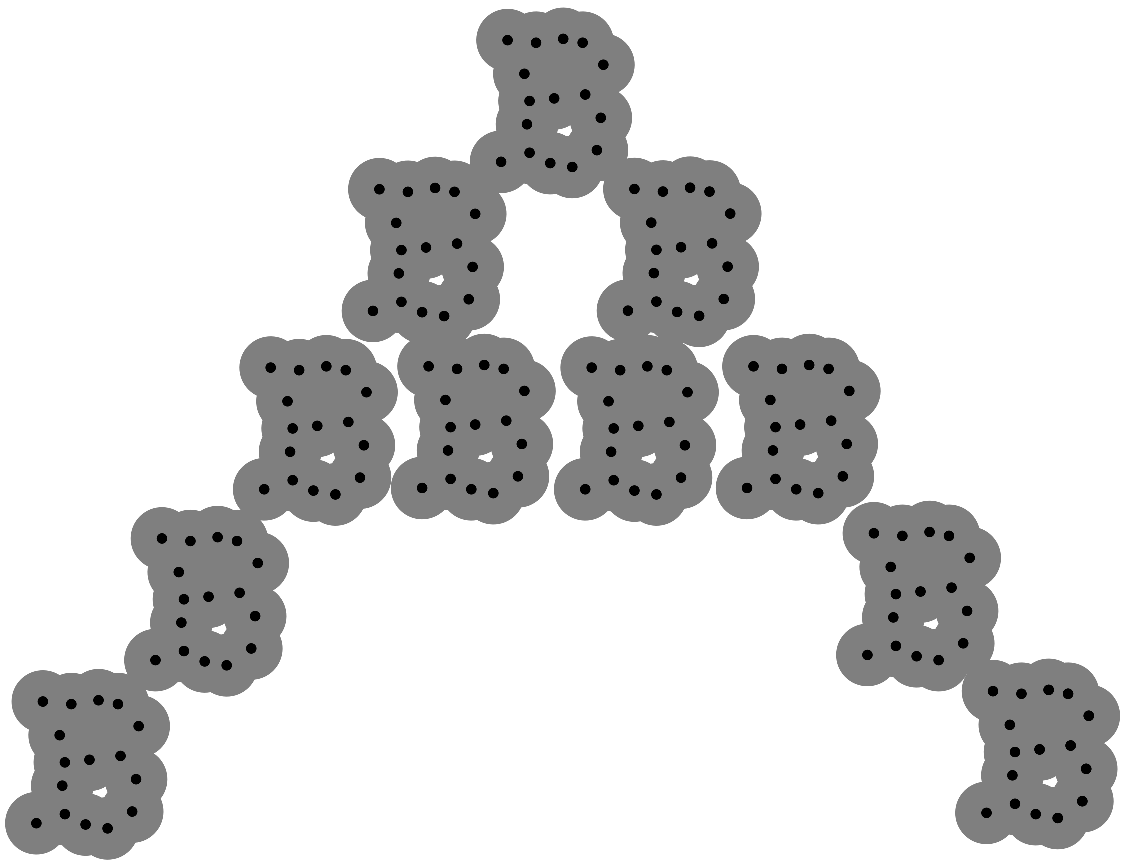
$$f_P : \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$

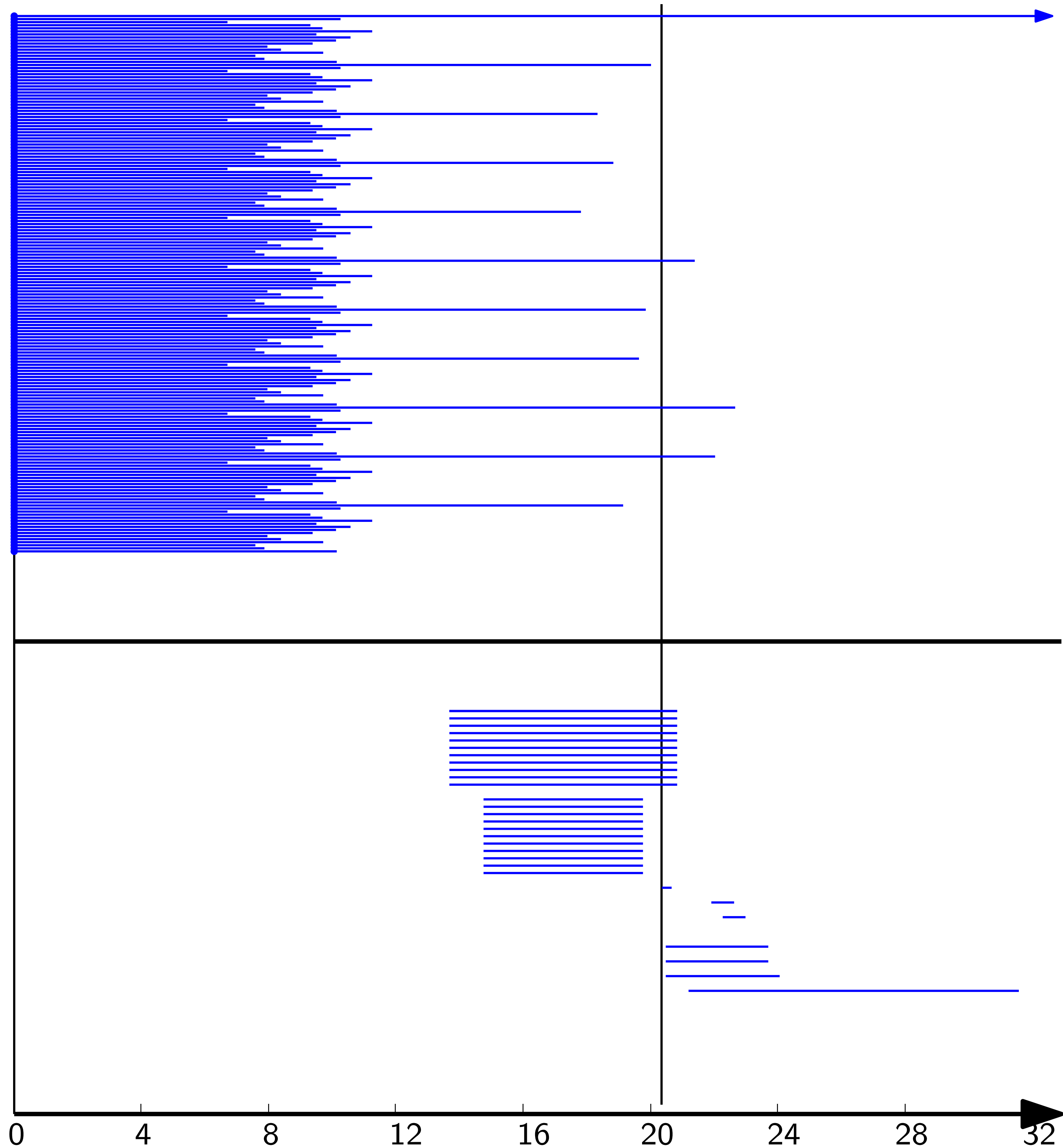




Example: Distance Function

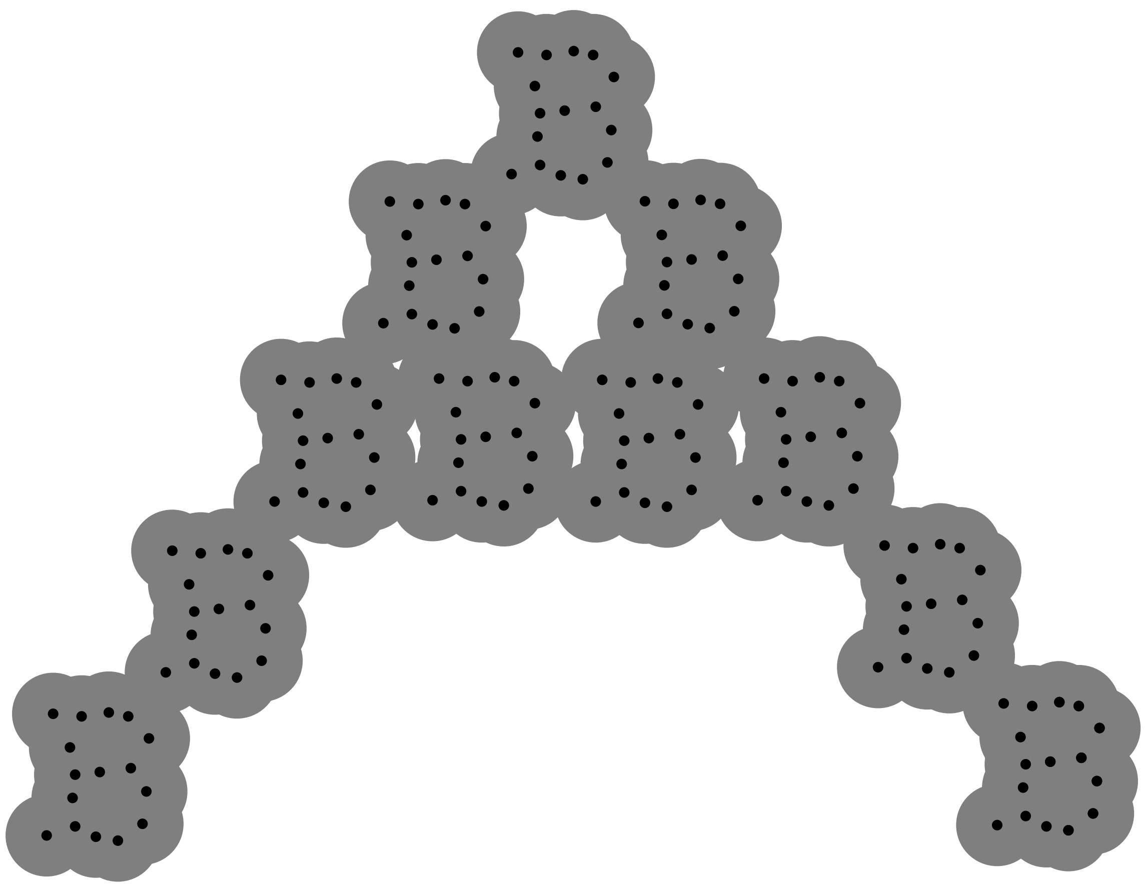
$$f_P: \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$

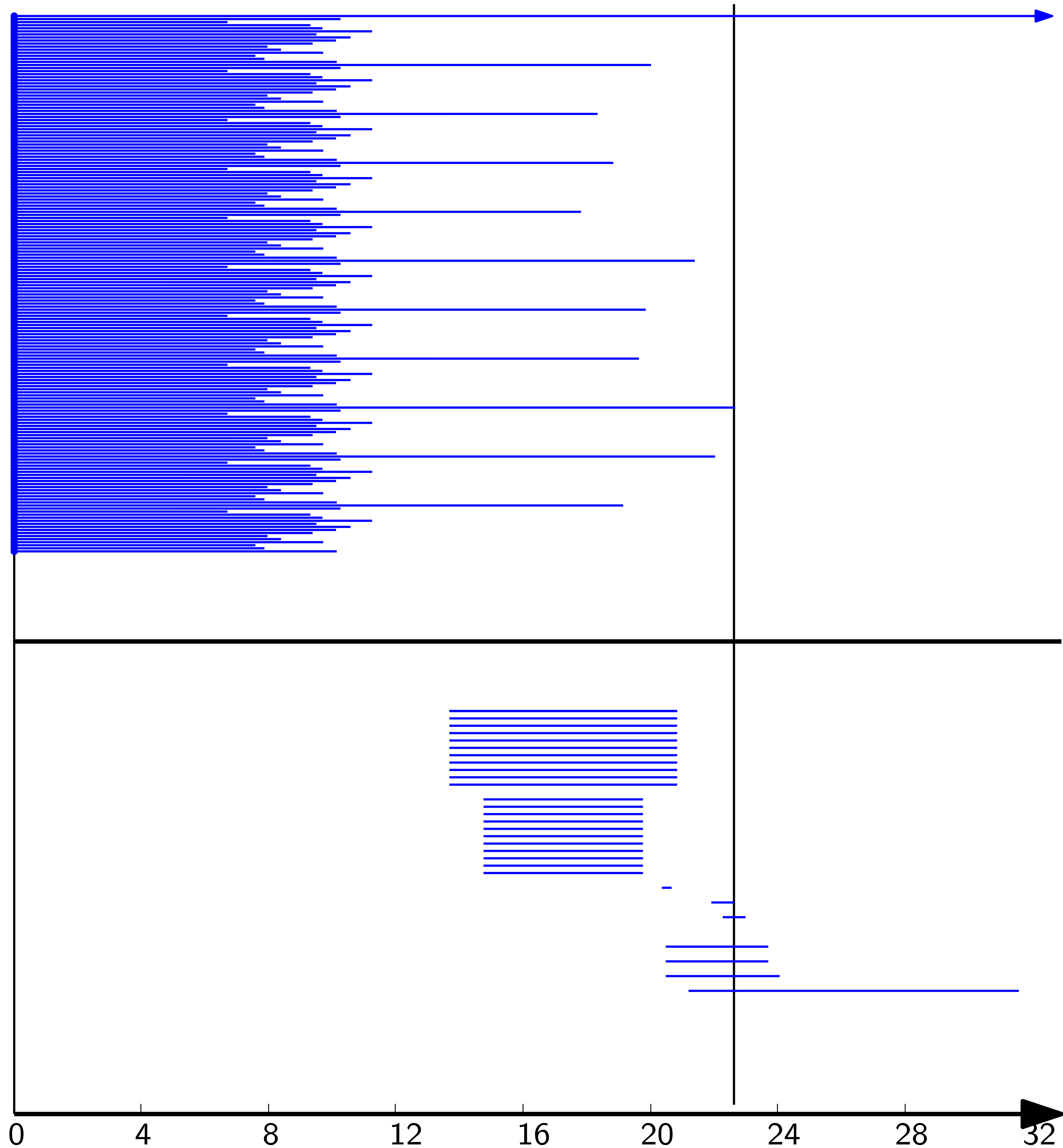




Example: Distance Function

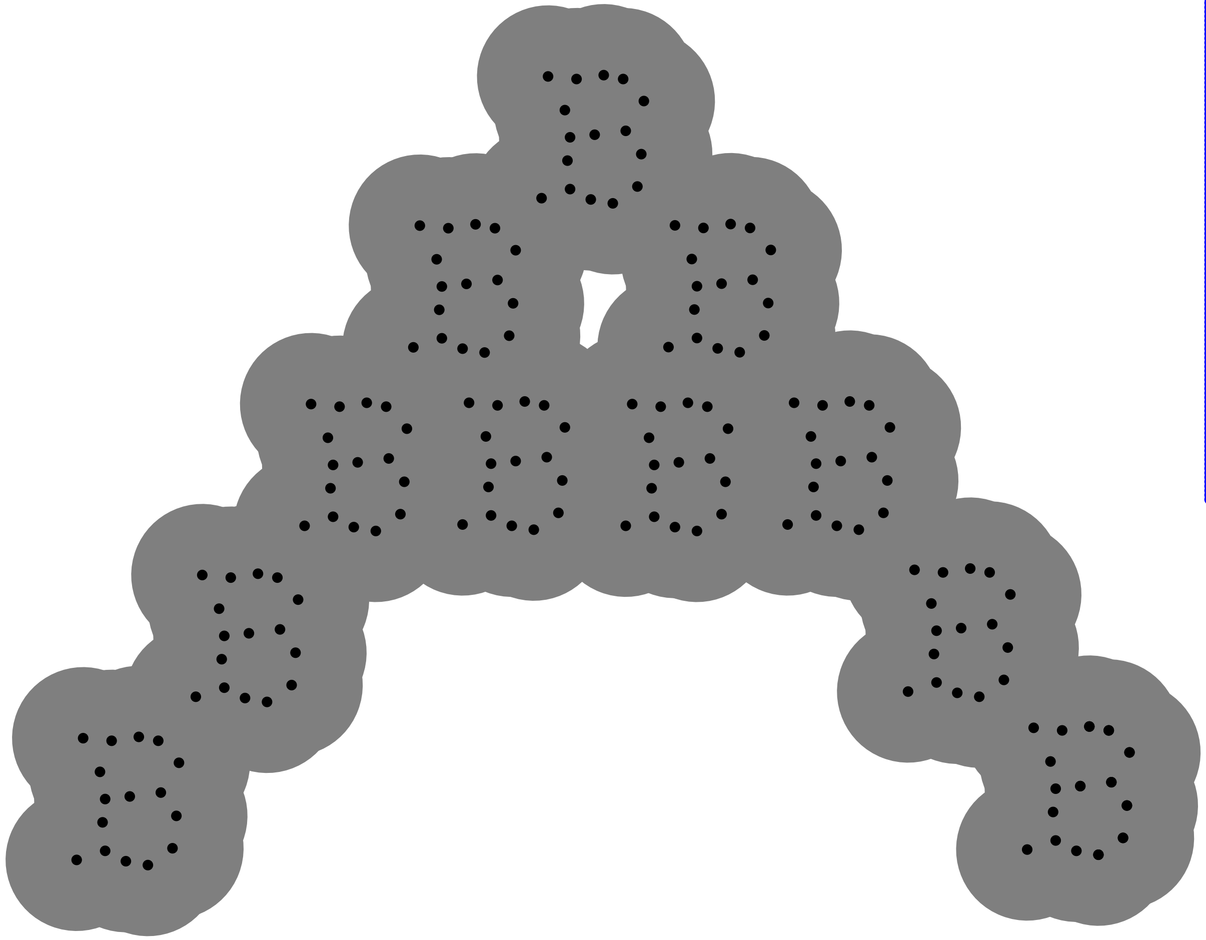
$$f_P: \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$

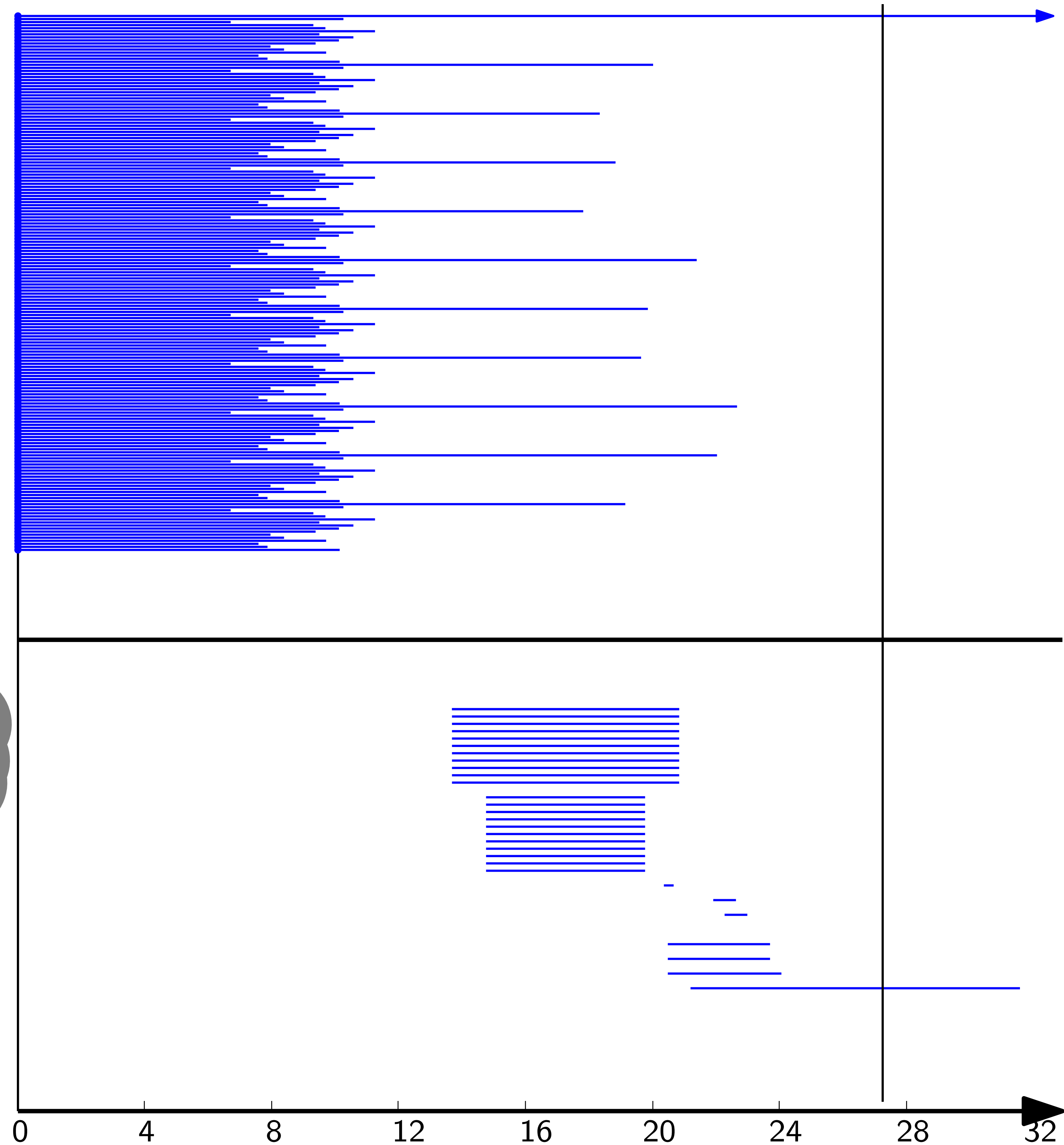




Example: Distance Function

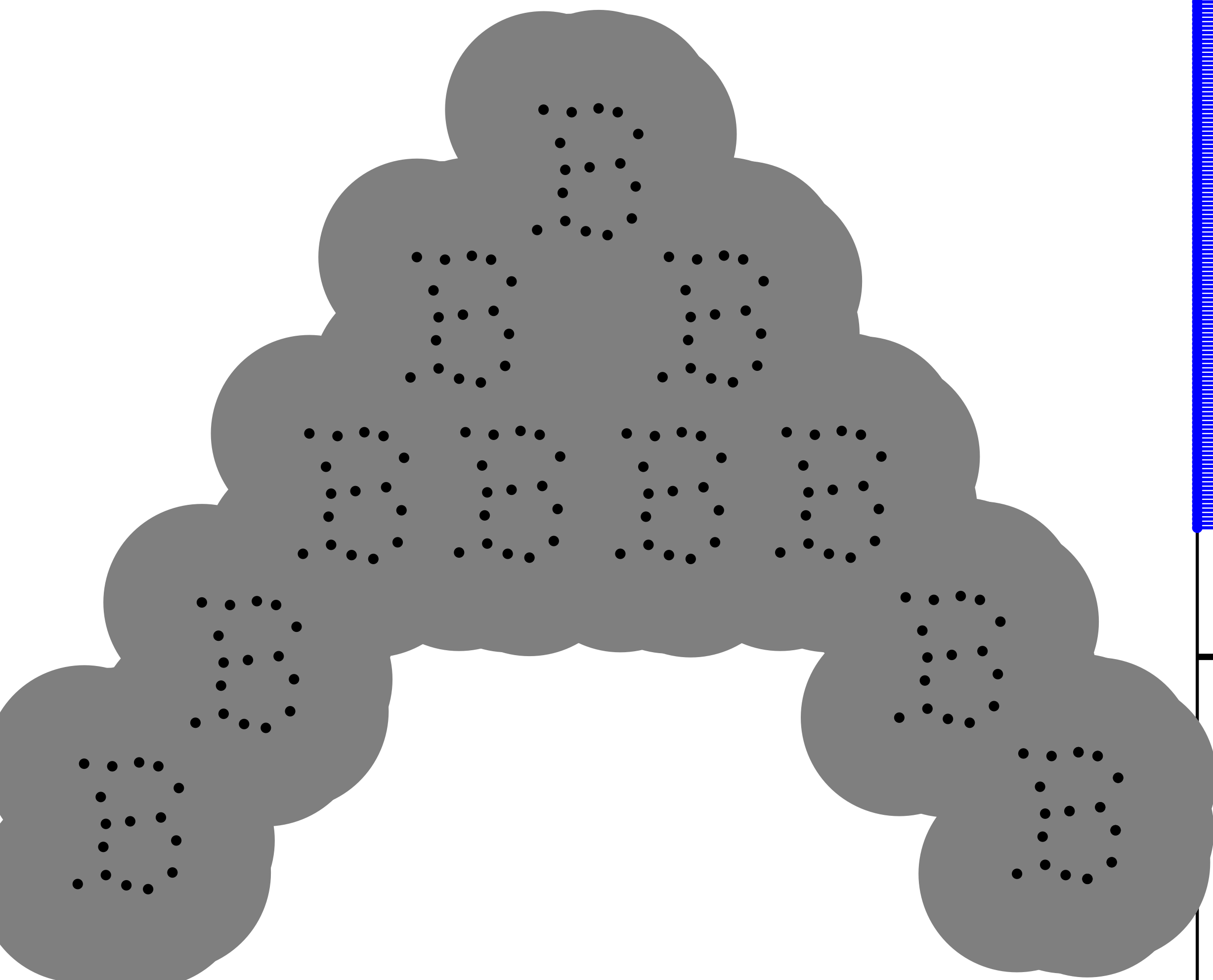
$$f_P : \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$

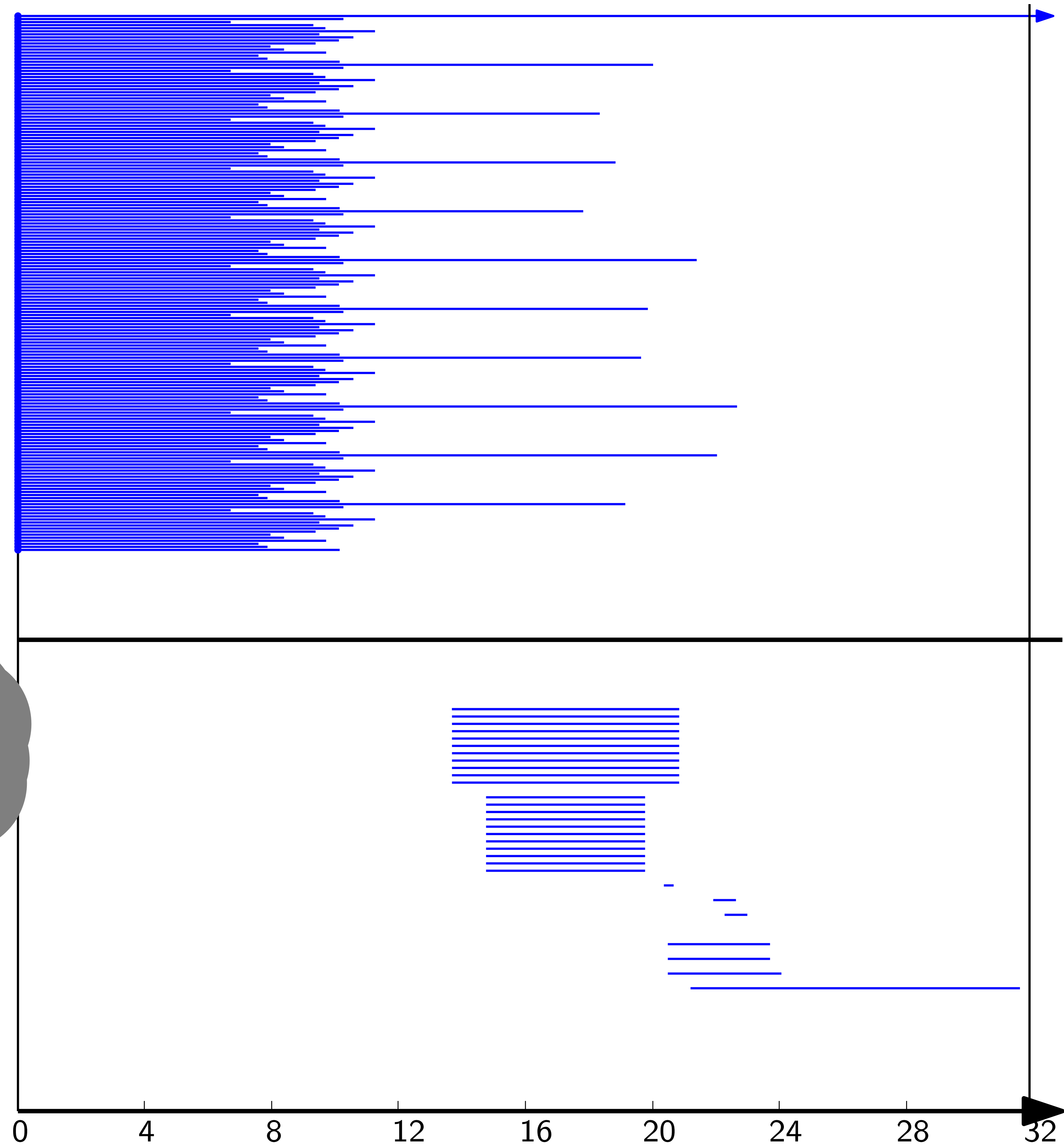




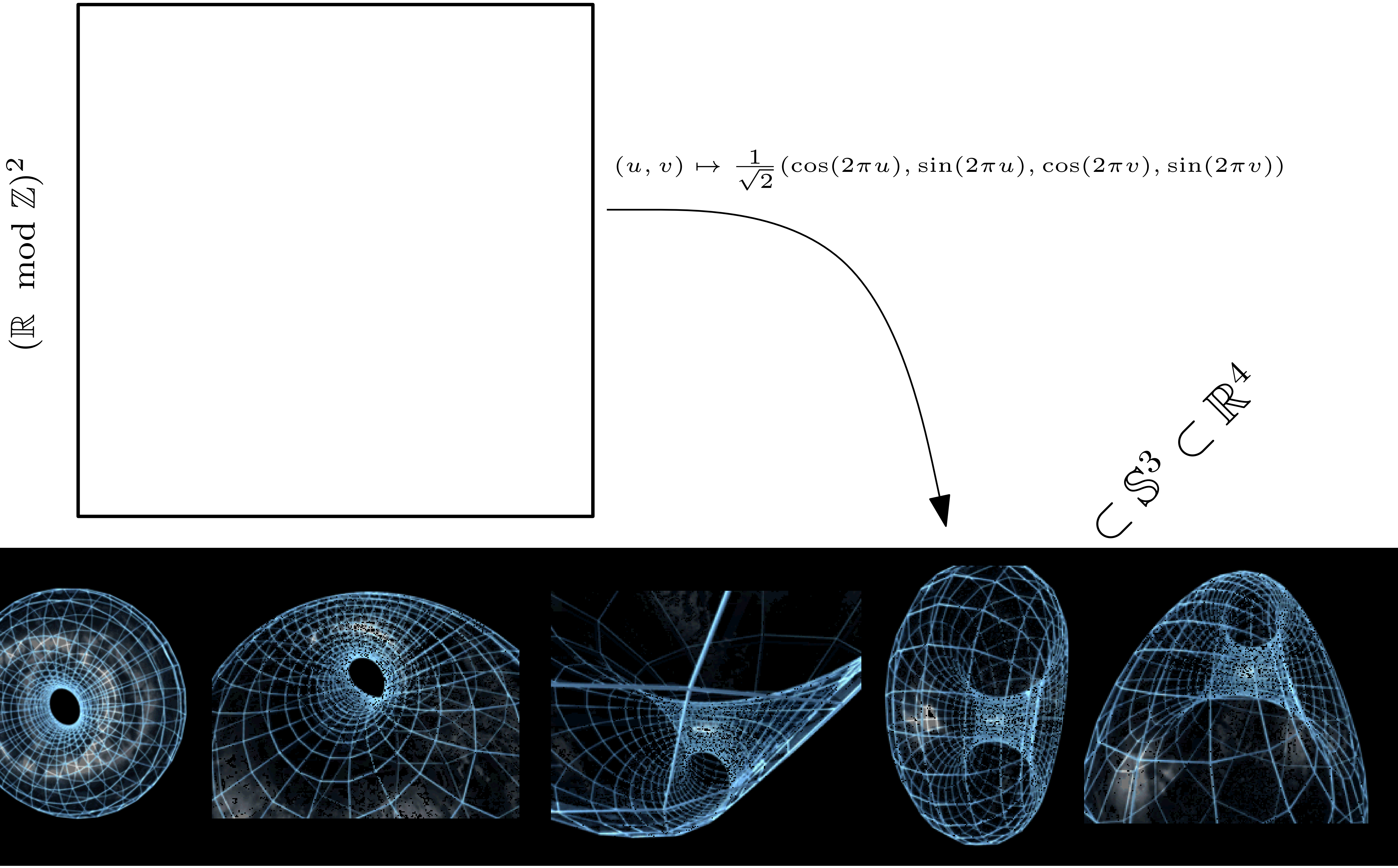
Example: Distance Function

$$f_P: \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$





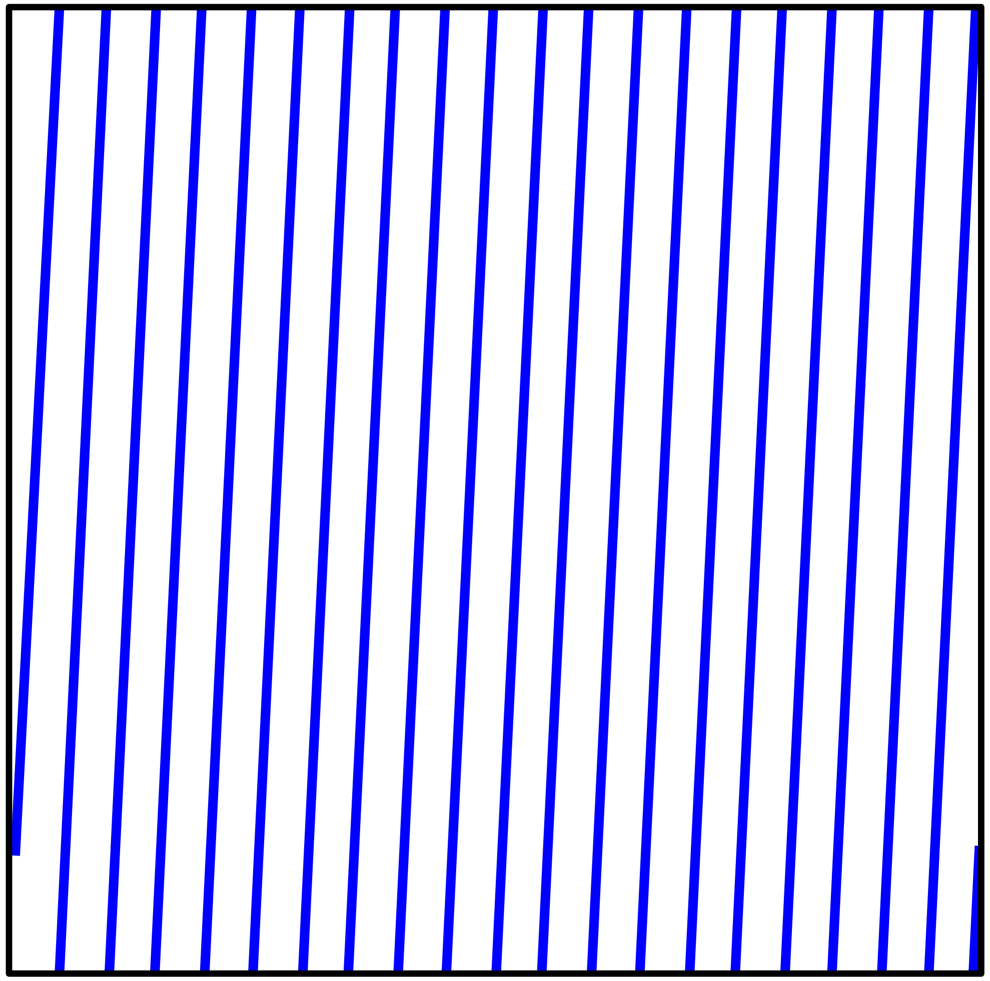
Example: Distance Function



source: `http://http://en.wikipedia.org/wiki/Clifford_torus`

Example: Distance Function

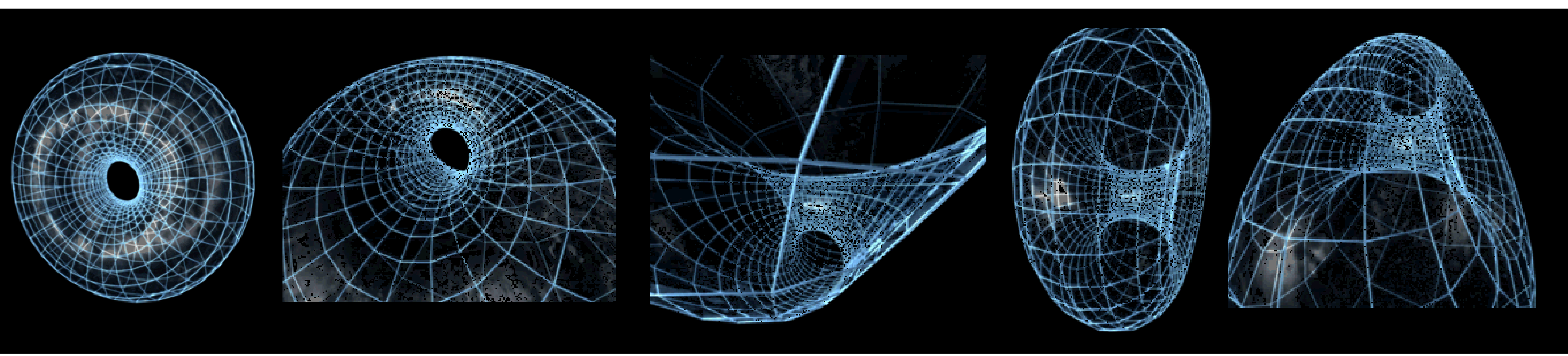
$(\mathbb{R} \mod \mathbb{Z})^2$



(spiral winding around the flat torus)

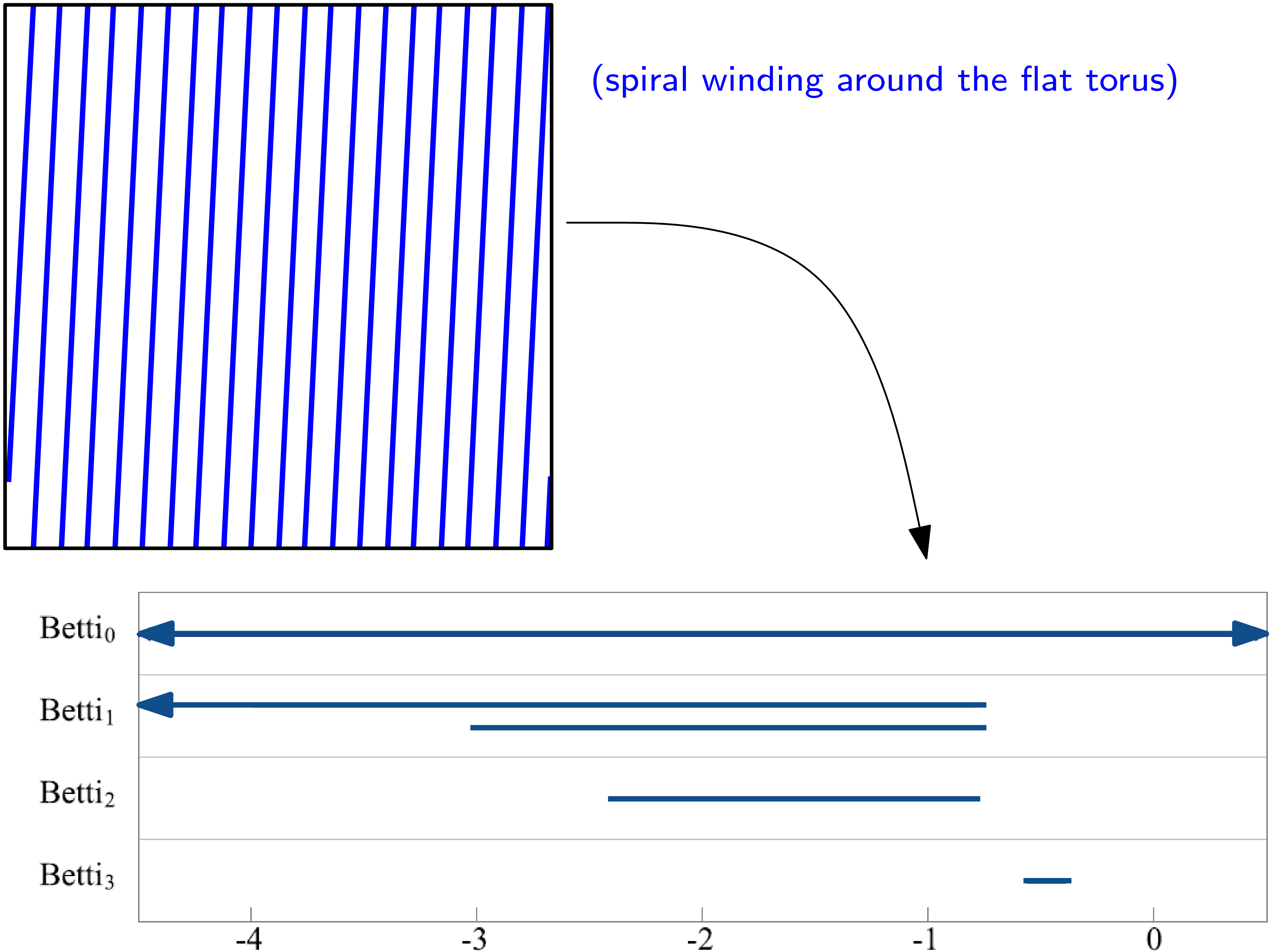
$(u, v) \mapsto \frac{1}{\sqrt{2}}(\cos(2\pi u), \sin(2\pi u), \cos(2\pi v), \sin(2\pi v))$

$\subset \mathbb{S}^3 \subset \mathbb{R}^4$



source: `http://http://en.wikipedia.org/wiki/Clifford_torus`

Example: Distance Function



Mathematical viewpoint: homology + quivers

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

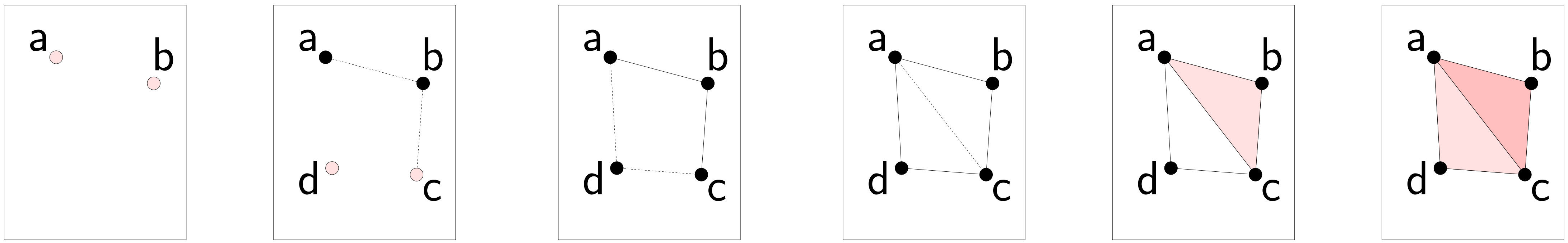
Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Mathematical viewpoint: homology + quivers

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Example 2: *simplicial filtration* (nested family of simplicial complexes)



F_1 F_2 F_3 F_4 F_5 F_6

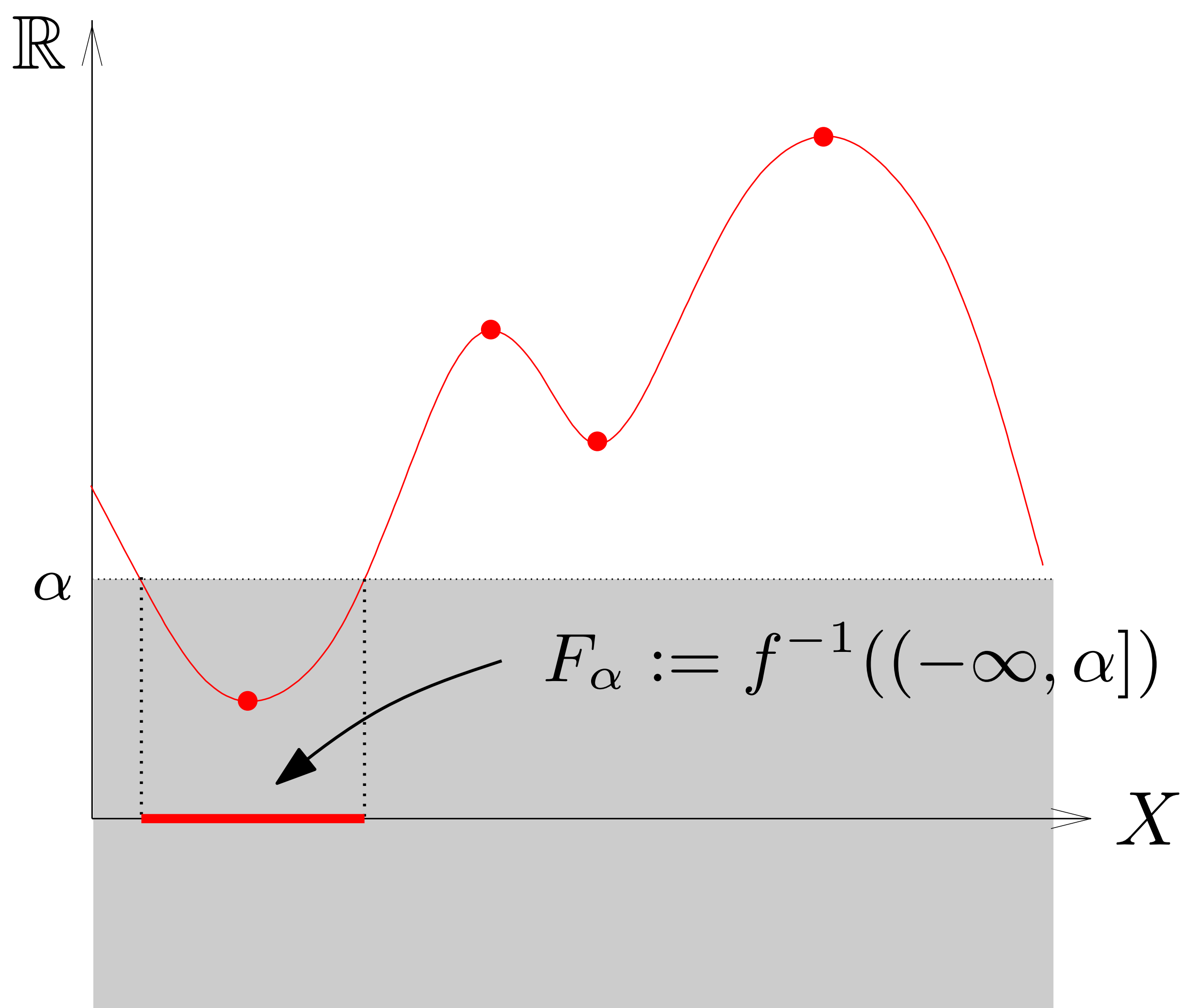
Mathematical viewpoint: homology + quivers

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

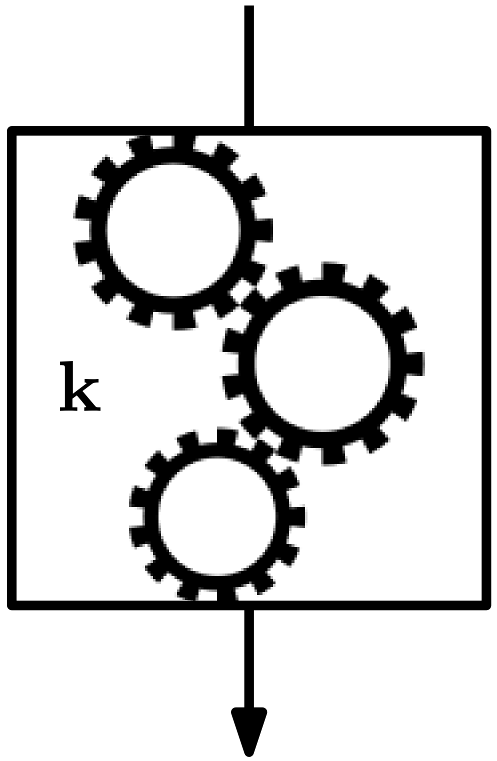
Example 2: *simplicial filtration* (nested family of simplicial complexes)

Example 3: *sublevel-sets filtration* (family of sublevel sets of a function $f : X \to \mathbb{R}$)



Mathematical viewpoint: homology + quivers

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$



(homology functor)

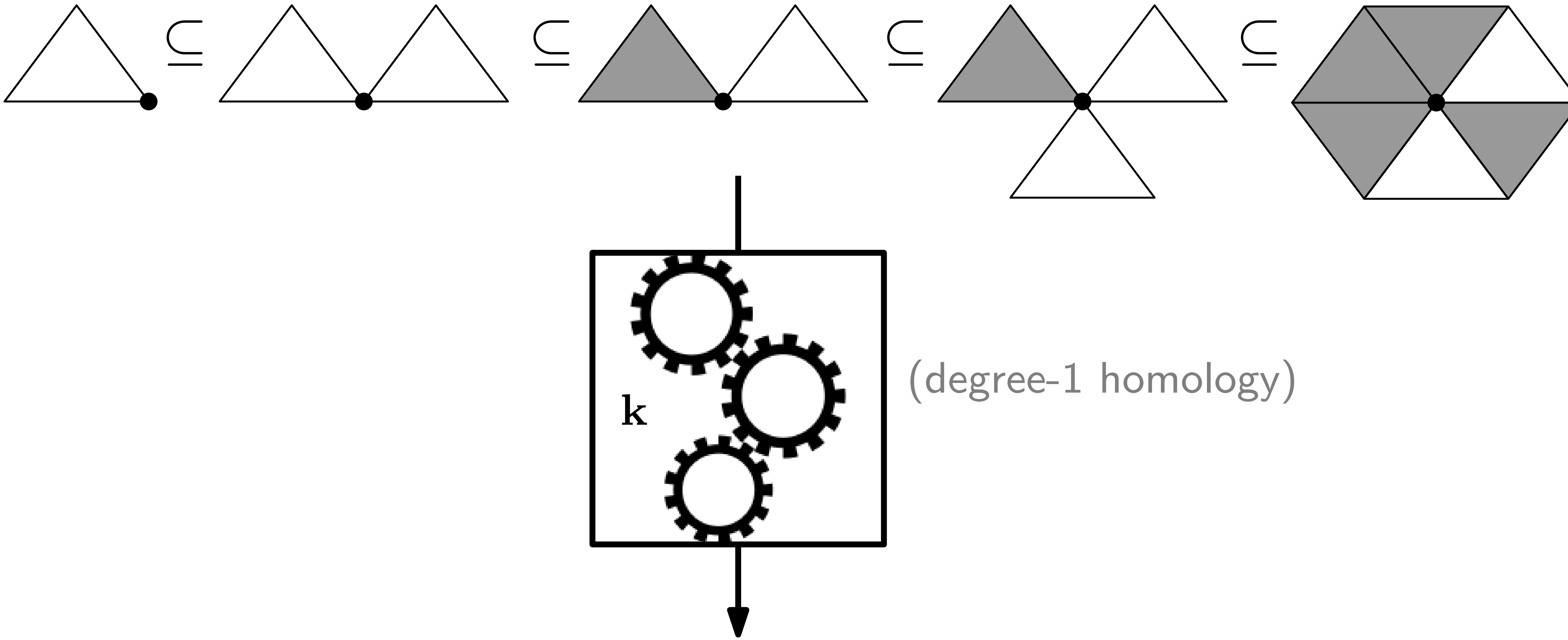
topological level

algebraic level

Persistence module: $H_*(F_1) \to H_*(F_2) \to H_*(F_3) \to H_*(F_4) \to H_*(F_5) \cdots$

Mathematical viewpoint: homology + quivers

Example:

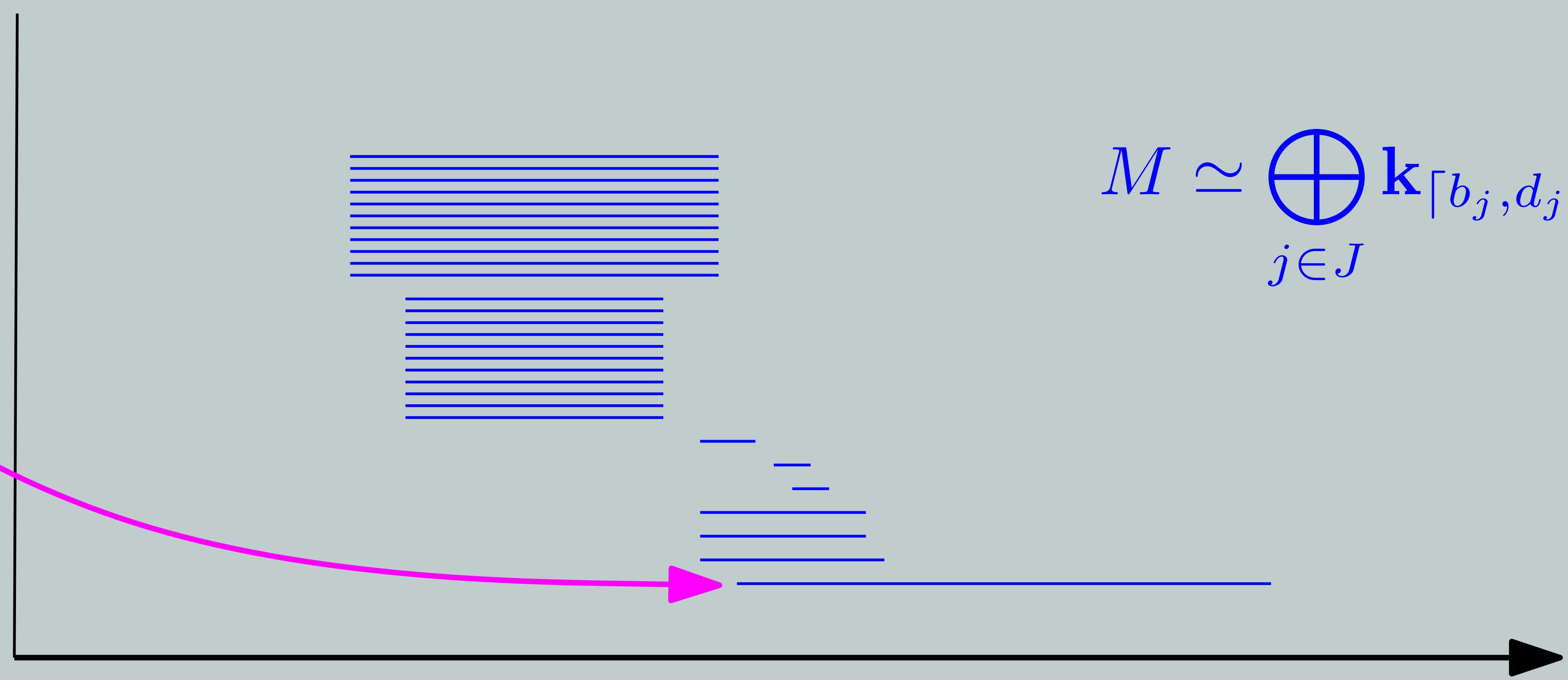


$$\mathbf{k} \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \mathbf{k}^2 \xrightarrow{\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)} \mathbf{k} \xrightarrow{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} \mathbf{k}^2 \xrightarrow{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \mathbf{k}^2 \;\cdots$$

Mathematical viewpoint: homology + quivers

Theorem. Let M be a persistence module over an index set $T \subseteq \mathbb{R}$. Then, M decomposes as a direct sum of *interval modules* $\mathbf{k}_{\lceil b,d \rfloor}$:

$$\underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{t < \lceil b,d \rfloor} \xrightarrow{0} \underbrace{\mathbf{k} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \mathbf{k}}_{\lceil b,\, d \rfloor} \xrightarrow{0} \underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0}}_{t > \lceil b,d \rfloor}$$



$$M \simeq \bigoplus_{j \in J} \mathbf{k}_{\lceil b_j, d_j \rfloor}$$

(the barcode is a complete descriptor of the algebraic structure of M)

Mathematical viewpoint: homology + quivers

Theorem. Let M be a persistence module over an index set $T \subseteq \mathbb{R}$. Then, M decomposes as a direct sum of *interval modules* $\mathbf{k}_{\lceil b,d \rfloor}$:

$$\underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{t<\lceil b,d\rfloor} \xrightarrow{0} \underbrace{\mathbf{k} \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \mathbf{k}}_{\lceil b,\, d\rfloor} \xrightarrow{0} \underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0}}_{t>\lceil b,d\rfloor}$$

in the following cases:

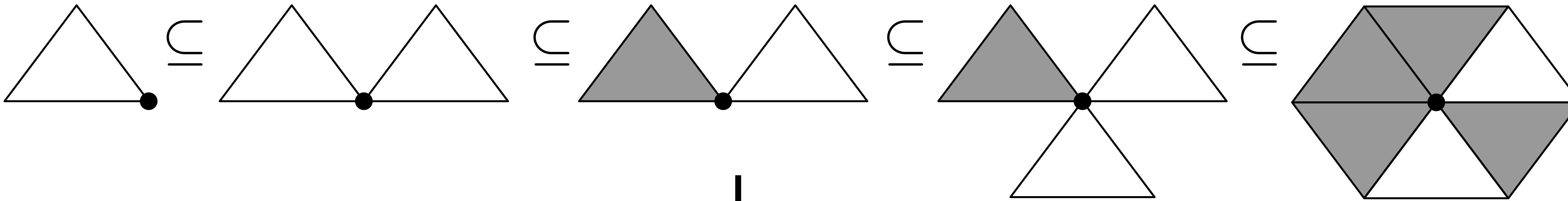
- T is finite [Gabriel 1972] [Auslander 1974],
- M is *pointwise finite-dimensional* (every space M_t has finite dimension) [Webb 1985] [Crawley-Boevey 2012].

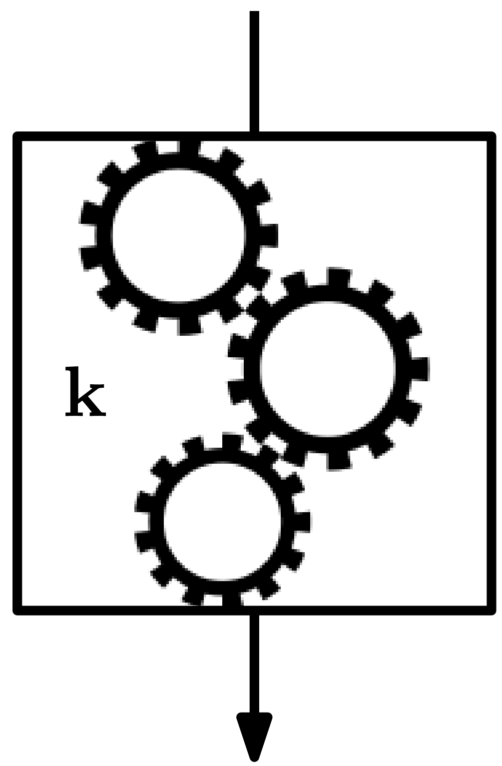
Moreover, when it exists, the decomposition is **unique** up to isomorphism and permutation of the terms [Azumaya 1950].

(Note: this is independent of the choice of field $\mathbf{k}$.)

Mathematical viewpoint: homology + quivers

Example:





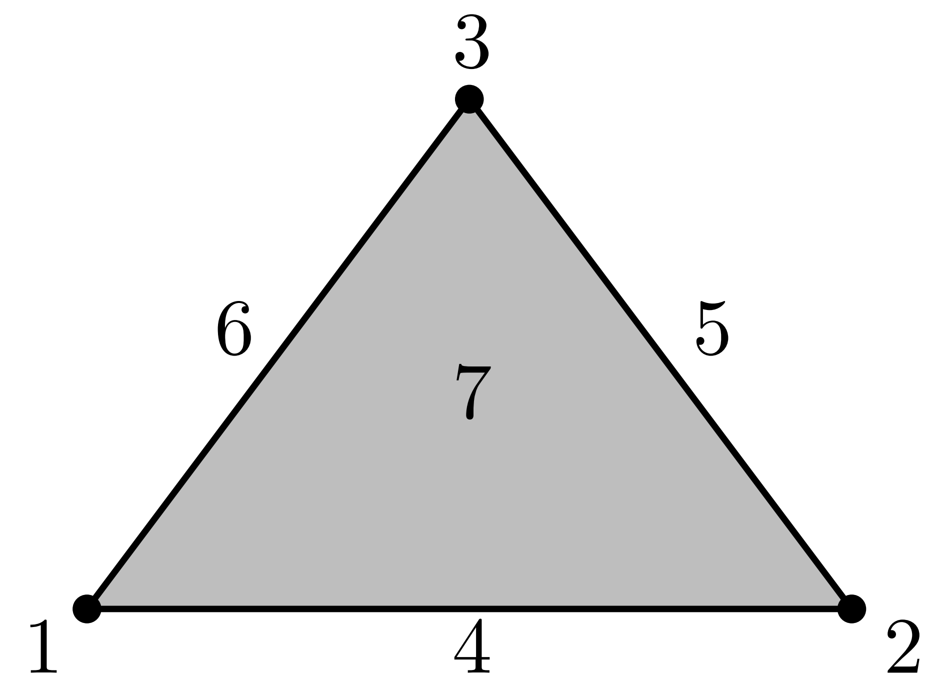
(degree-1 homology)

$$\mathbf{k} \xrightarrow{\binom{1}{0}} \mathbf{k}^2 \xrightarrow{(\,0\ \ 1\,)} \mathbf{k} \xrightarrow{\binom{0}{1}} \mathbf{k}^2 \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}} \mathbf{k}^2 \ \cdots$$

Computation of barcodes: matrix reduction

[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .

Input: simplicial filtration

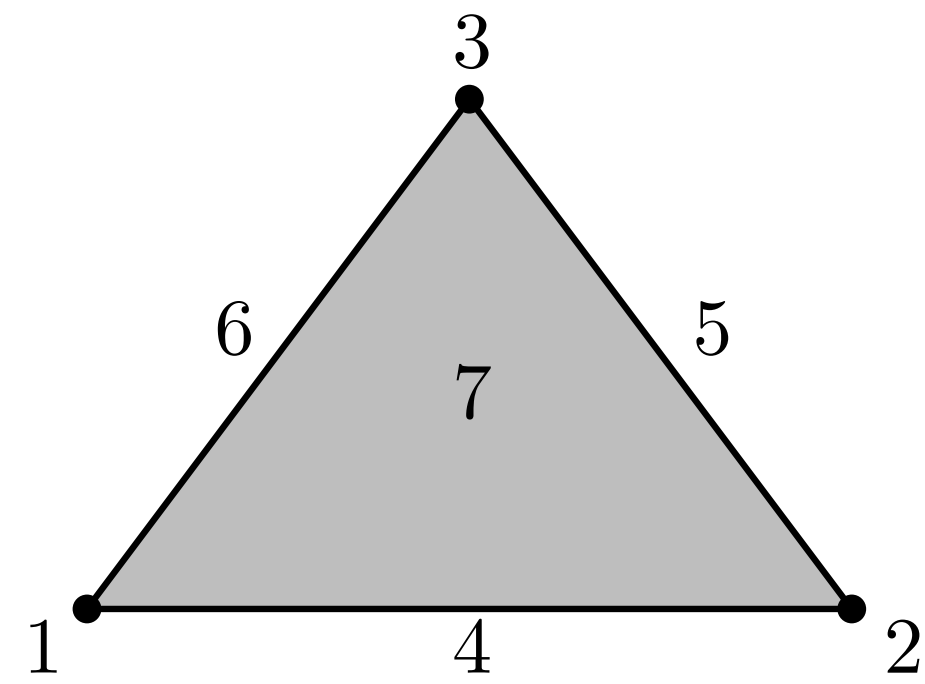


Computation of barcodes: matrix reduction

[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .

Input: simplicial filtration

Output: boundary matrix



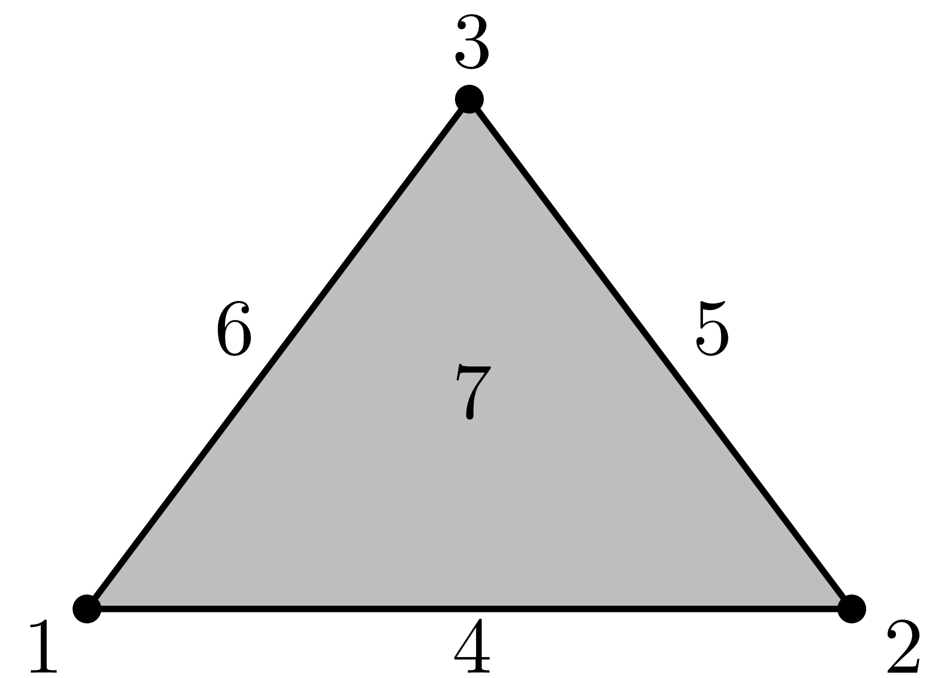
	1	2	3	4	5	6	7
1				*		*	
2				*	*		
3					*	*	
4							*
5							*
6							*
7							

Computation of barcodes: matrix reduction

[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .

Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form

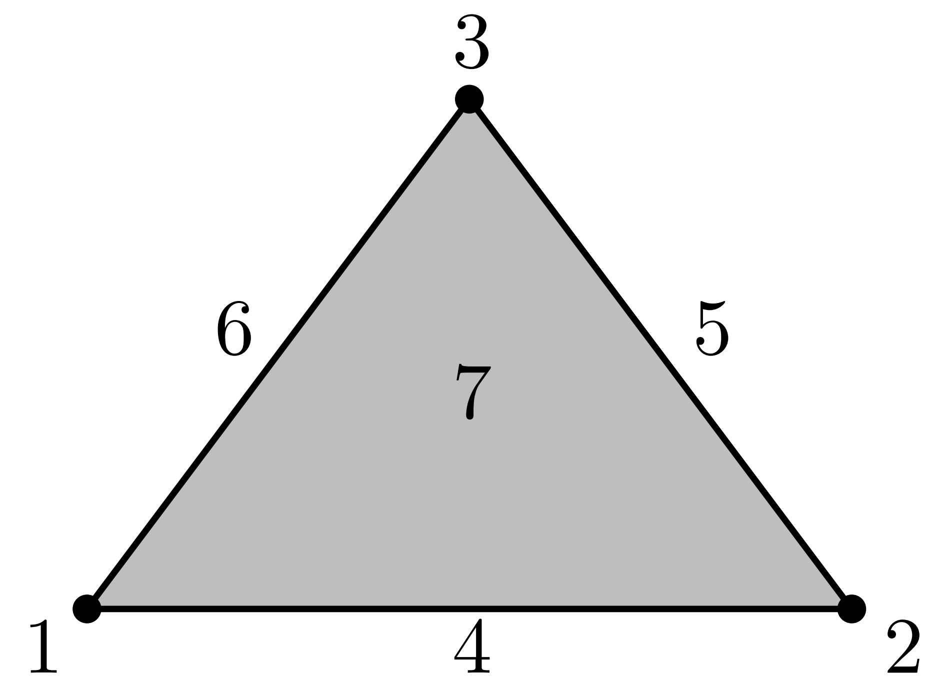


	1	2	3	4	5	6	7
1				$*$		$*$	
2				$*$	$*$		
3					$*$	$*$	
4							$*$
5							$*$
6							$*$
7							

	1	2	3	4	5	6	7
1				$*$			
2				1	$*$		
3					1		
4							$*$
5							$*$
6							1
7							

Computation of barcodes: matrix reduction

[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .



Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form

simplex pairs give finite intervals:

$[2, 4)$, $[3, 5)$, $[6, 7)$

unpaired simplices give infinite intervals: $[1, +\infty)$

	1	2	3	4	5	6	7
1				*		*	
2				*	*		
3					*	*	
4							*
5							*
6							*
7							

	1	2	3	4	5	6	7
1				*			
2				1	*		
3					1		
4							*
5							*
6							1
7							

Computation of barcodes: matrix reduction

[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .

Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form

PLU factorization:

- Gaussian elimination
- fast matrix multiplication (divide-and-conquer) [Bunch, Hopcroft 1974]
- random projections?

Computation of barcodes: matrix reduction

[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .

Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form

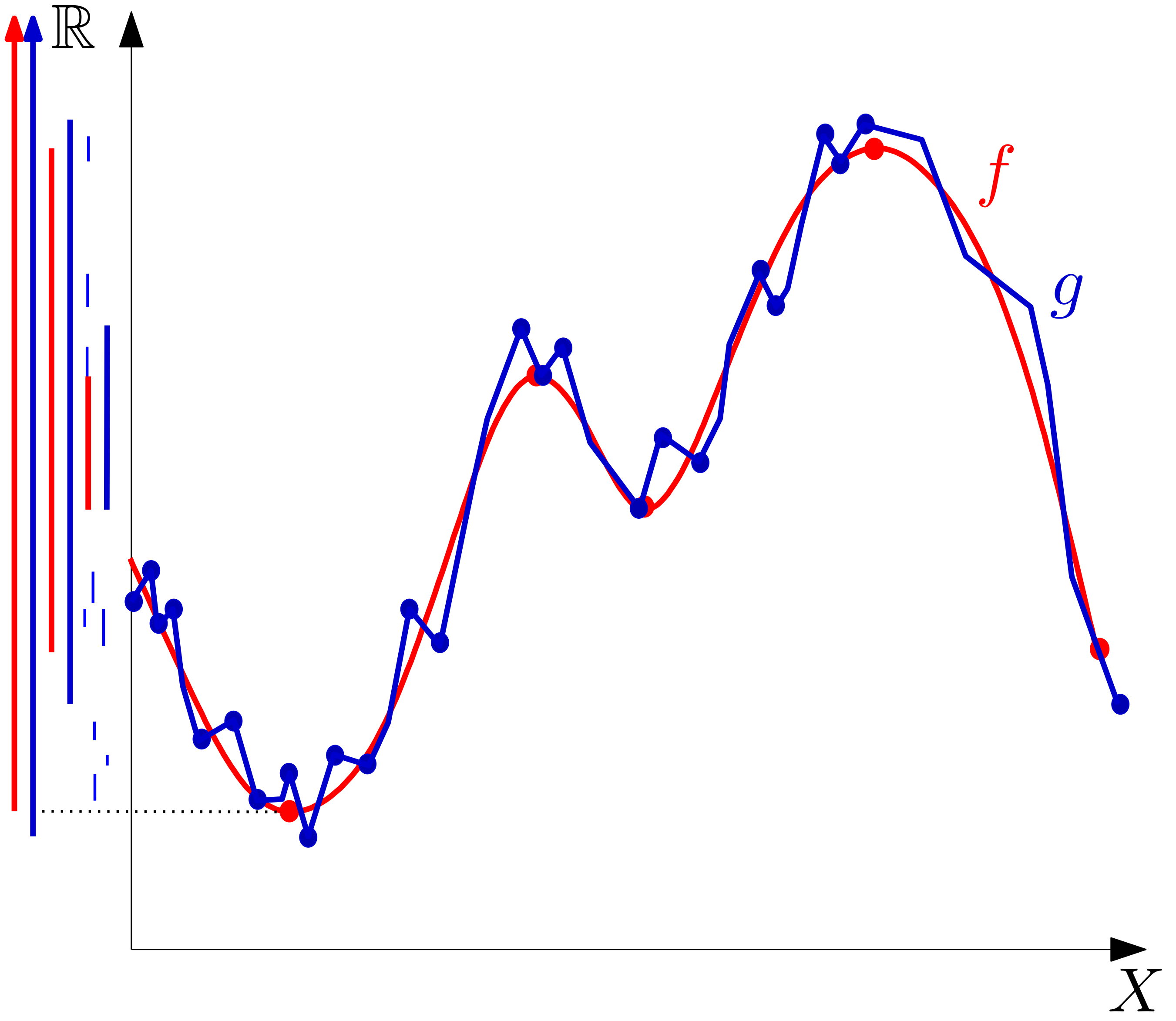
PLU factorization:

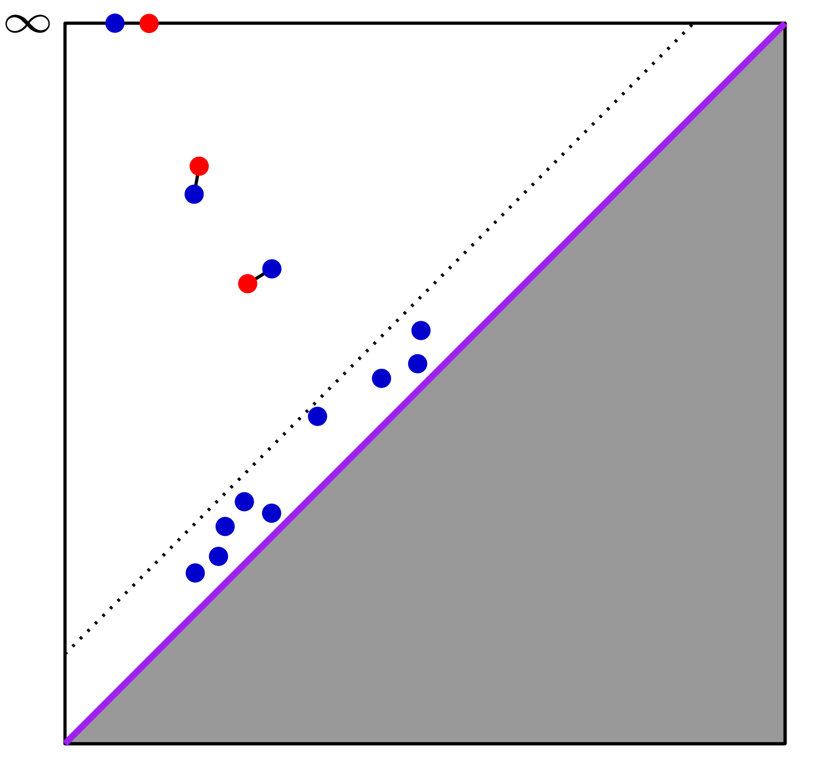
- Gaussian elimination
  - PLEX / JavaPLEX (http://appliedtopology.github.io/javaplex/)
  - Dionysus (http://www.mrzv.org/software/dionysus/)
  - Perseus (http://www.sas.upenn.edu/~vnanda/perseus/)
  - Gudhi (http://gudhi.gforge.inria.fr/)
  - PHAT (https://bitbucket.org/phat-code/phat)
  - DIPHA (https://github.com/DIPHA/dipha/)
  - CTL (https://github.com/appliedtopology/ctl)

Stability of persistence barcodes

Theorem: For any pfd functions $f, g : X \to \mathbb{R}$,

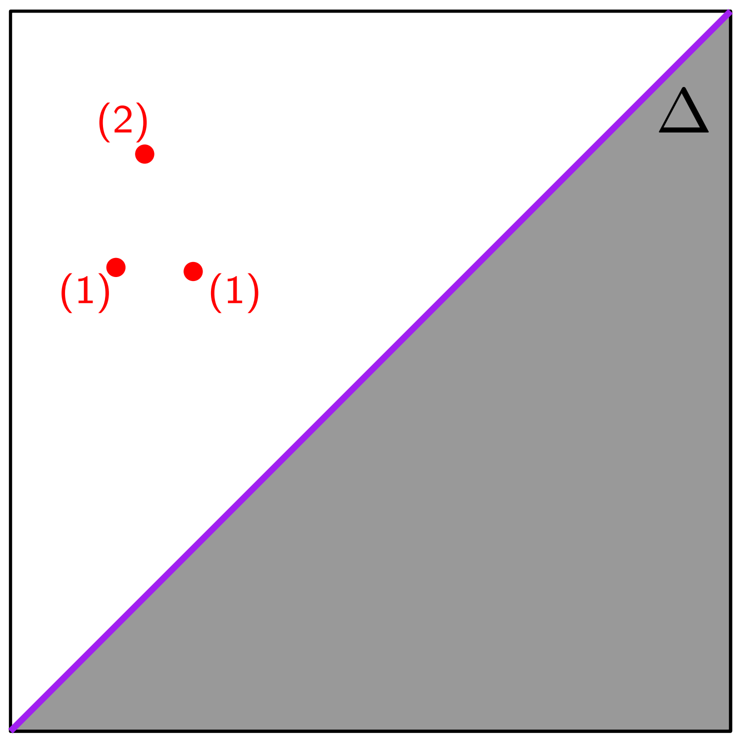
$$d_\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq \|f - g\|_\infty$$





Space of persistence diagrams

Persistence diagram $\equiv$ **finite** multiset in the open half-plane $\Delta \times \mathbb{R}_{>0}$



Space of persistence diagrams

Persistence diagram $\equiv$ **finite** multiset in the open half-plane $\Delta \times \mathbb{R}_{>0}$

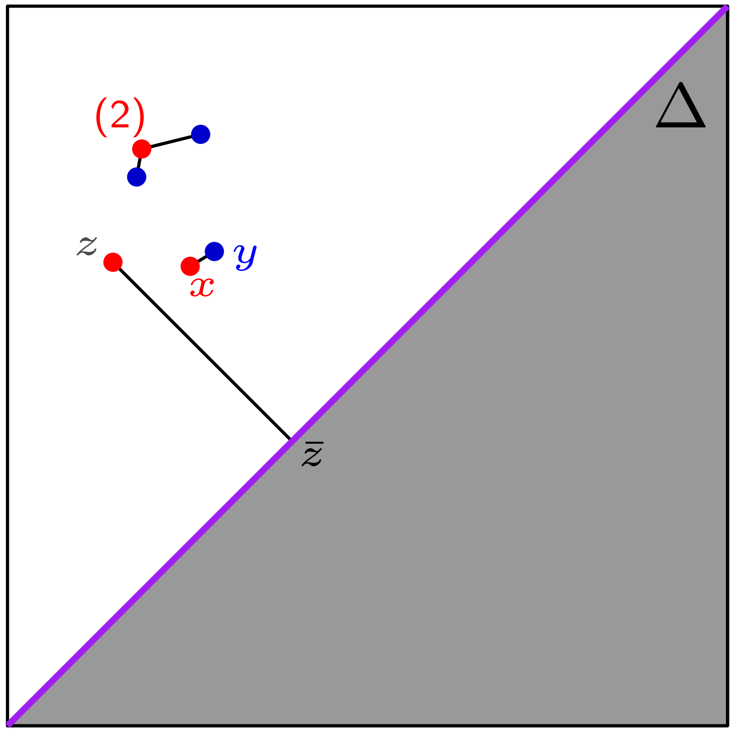
Given a **partial matching** $M : X \leftrightarrow Y$:

cost of a matched pair $(x, y) \in M$: $c_p(x, y) := \|x - y\|_\infty^p$

cost of an unmatched point $z \in X \sqcup Y$: $c_p(z) := \|z - \bar{z}\|_\infty^p$

cost of M:

$$c_p(M) := \left(\sum_{(x,\,y) \text{ matched}} c_p(x, y) + \sum_{z \text{ unmatched}} c_p(z) \right)^{1/p}$$



Space of persistence diagrams

Persistence diagram $\equiv$ **finite** multiset in the open half-plane $\Delta \times \mathbb{R}_{>0}$

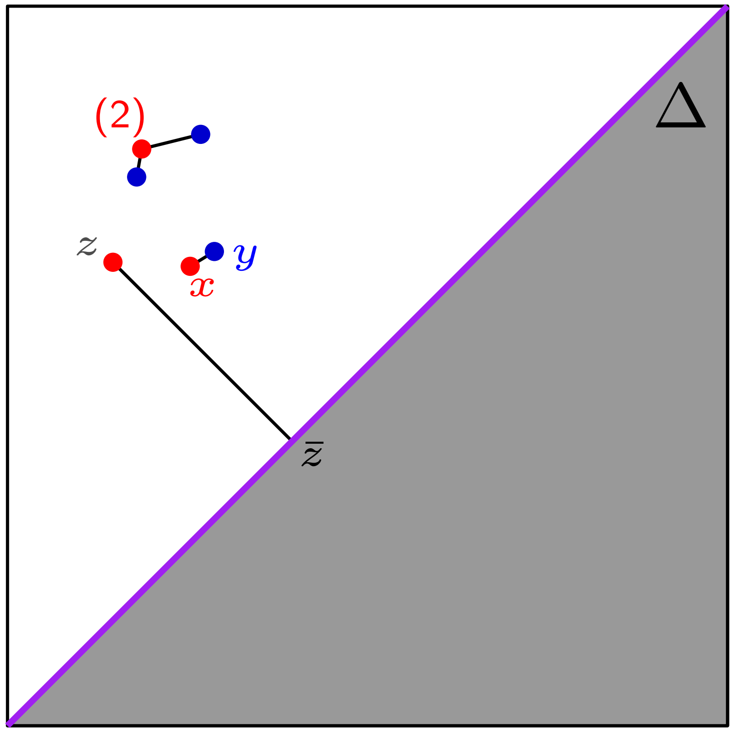
Given a **partial matching** $M : X \leftrightarrow Y$:

cost of a matched pair $(x, y) \in M$: $c_p(x, y) := \|x - y\|_\infty^p$

cost of an unmatched point $z \in X \sqcup Y$: $c_p(z) := \|z - \bar{z}\|_\infty^p$

cost of M:

$$c_p(M) := \left(\sum_{(x,\,y)\text{ matched}} c_p(x, y) \; + \sum_{z\text{ unmatched}} c_p(z) \right)^{1/p}$$



Def: p-th diagram distance (extended metric):

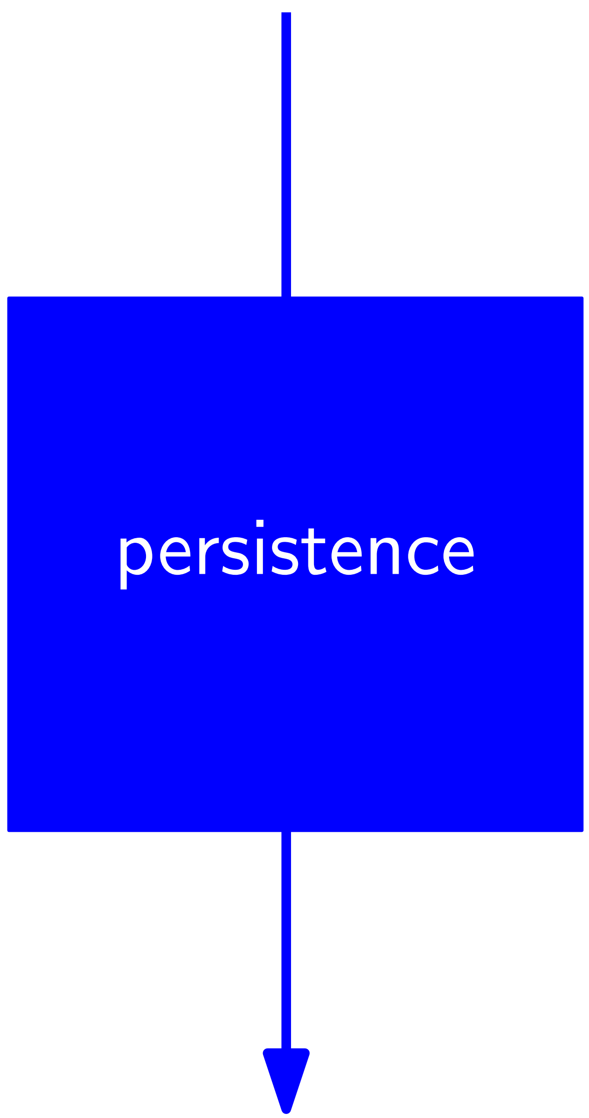
$$\mathrm{d}_p(X, Y) := \inf_{M : X \leftrightarrow Y} c_p(M)$$

Def: bottleneck distance:

$$\mathrm{d}_\infty(X, Y) := \lim_{p \to \infty} \mathrm{d}_p(X, Y)$$

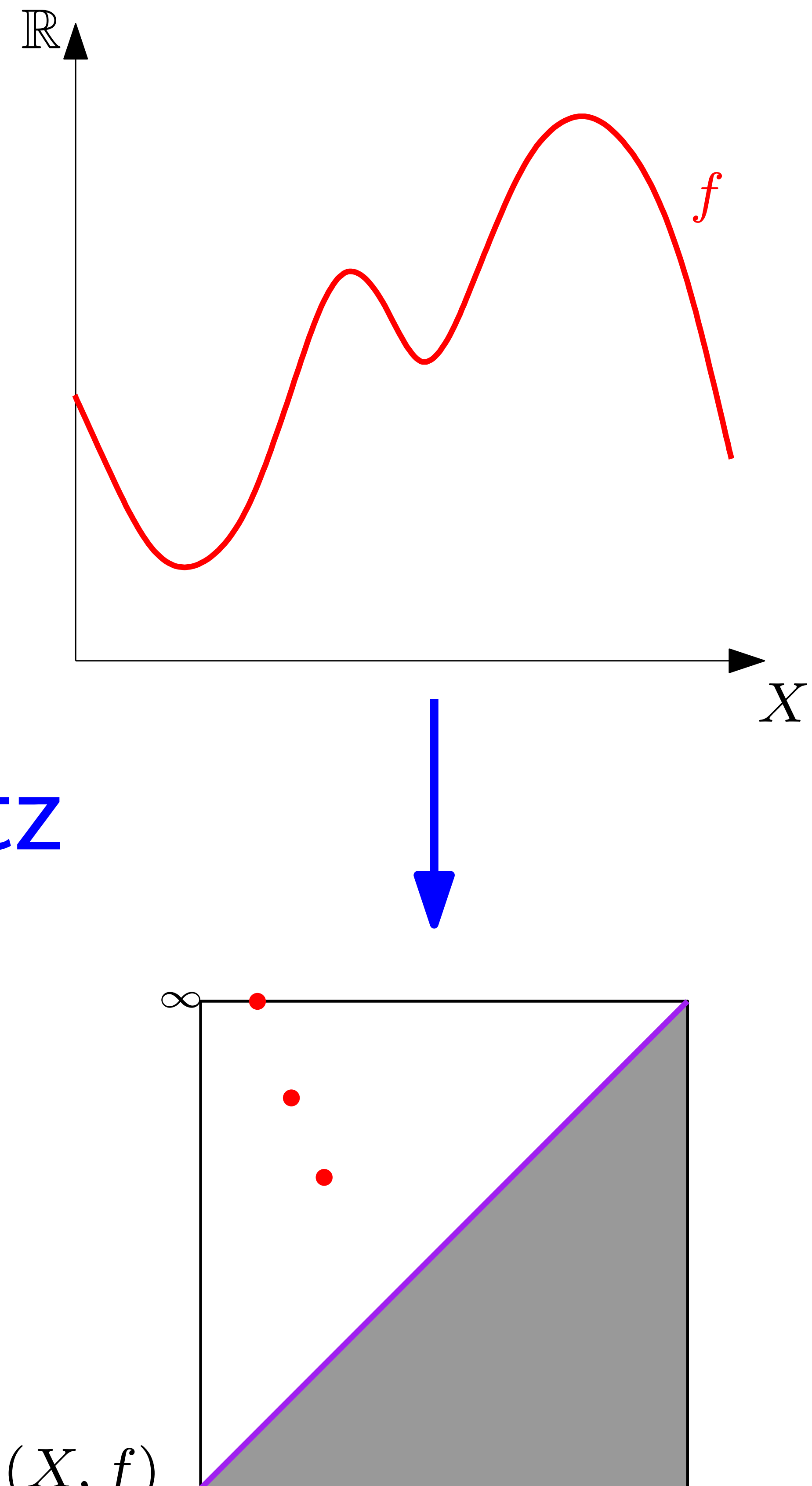
X topological space

$f : X \to \mathbb{R}$



$\mathrm{Dg}\, f$

Lipschitz



signature: *persistence diagram*

encodes the topological structure of the pair (X, f)