INF631 – Data Analysis: Geometry and Topology in Arbitrary Dimensions

Delaunay-Based Reconstruction

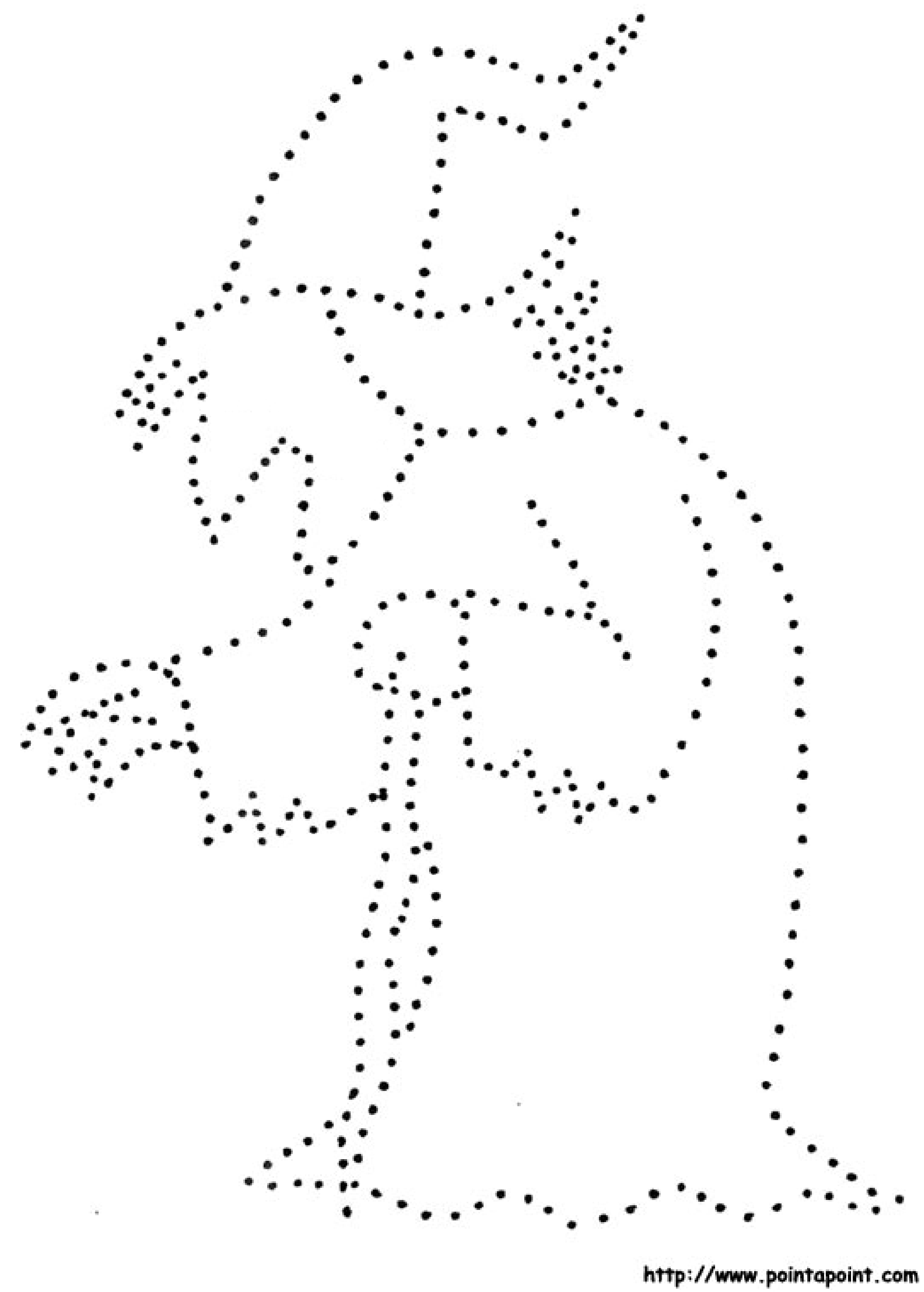
Steve Oudot

ÉCOLE POLYTECHNIQUE

Reconstruction Paradigm

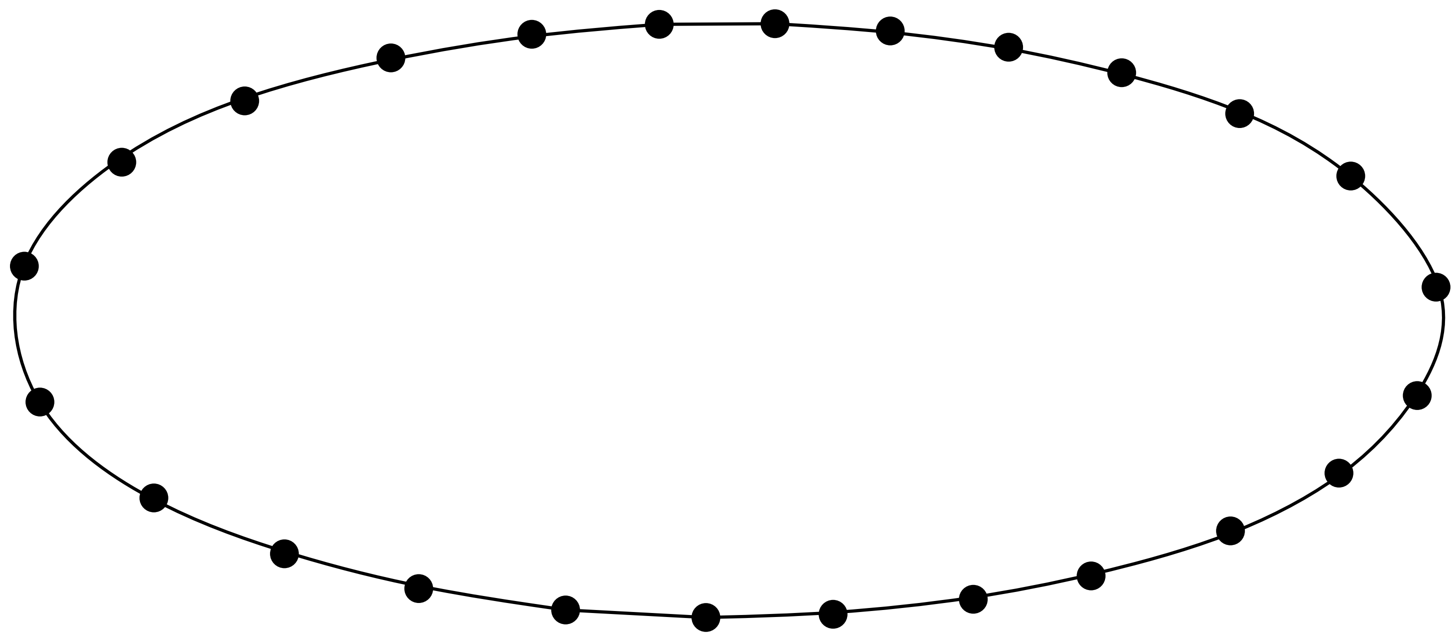
Q What do you see?

Why?



Reconstruction Paradigm

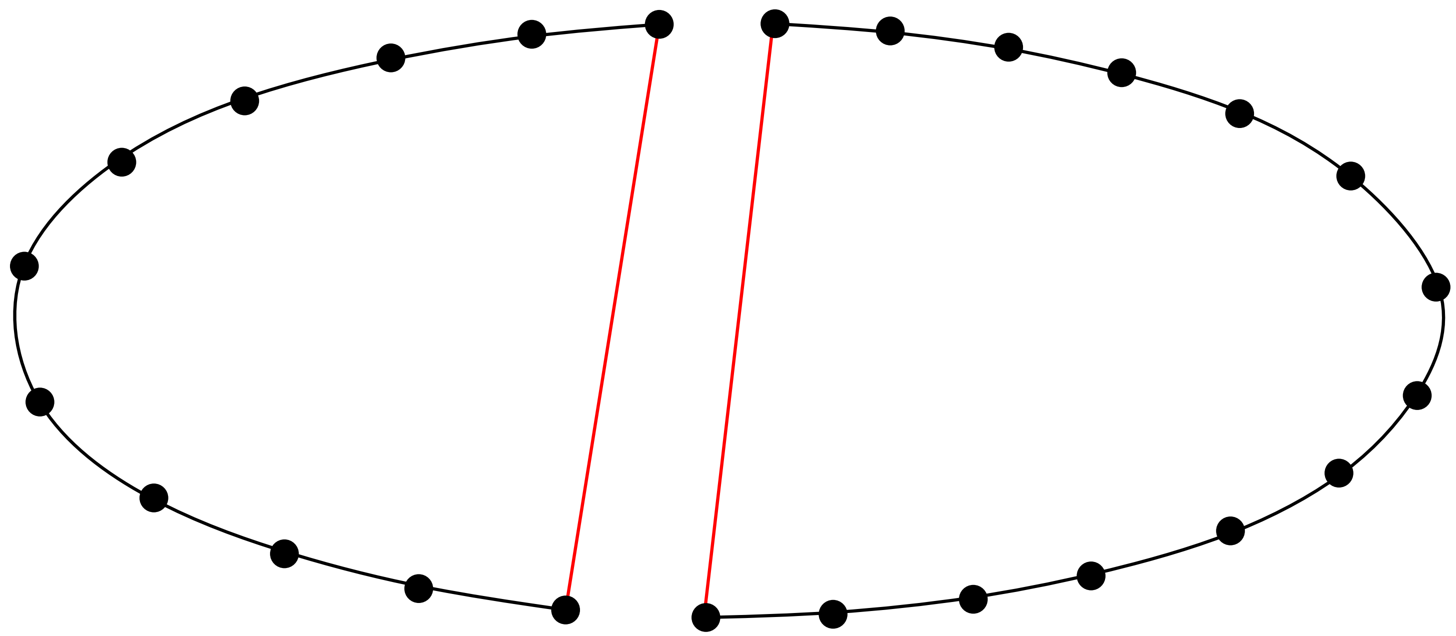
Input: point cloud $P \subset \mathbb{R}^d$ *finite*



Prior: points of P are sampled along some *unknown shape* M (manifold, compact set etc.), according to some *unknown measure* μ.

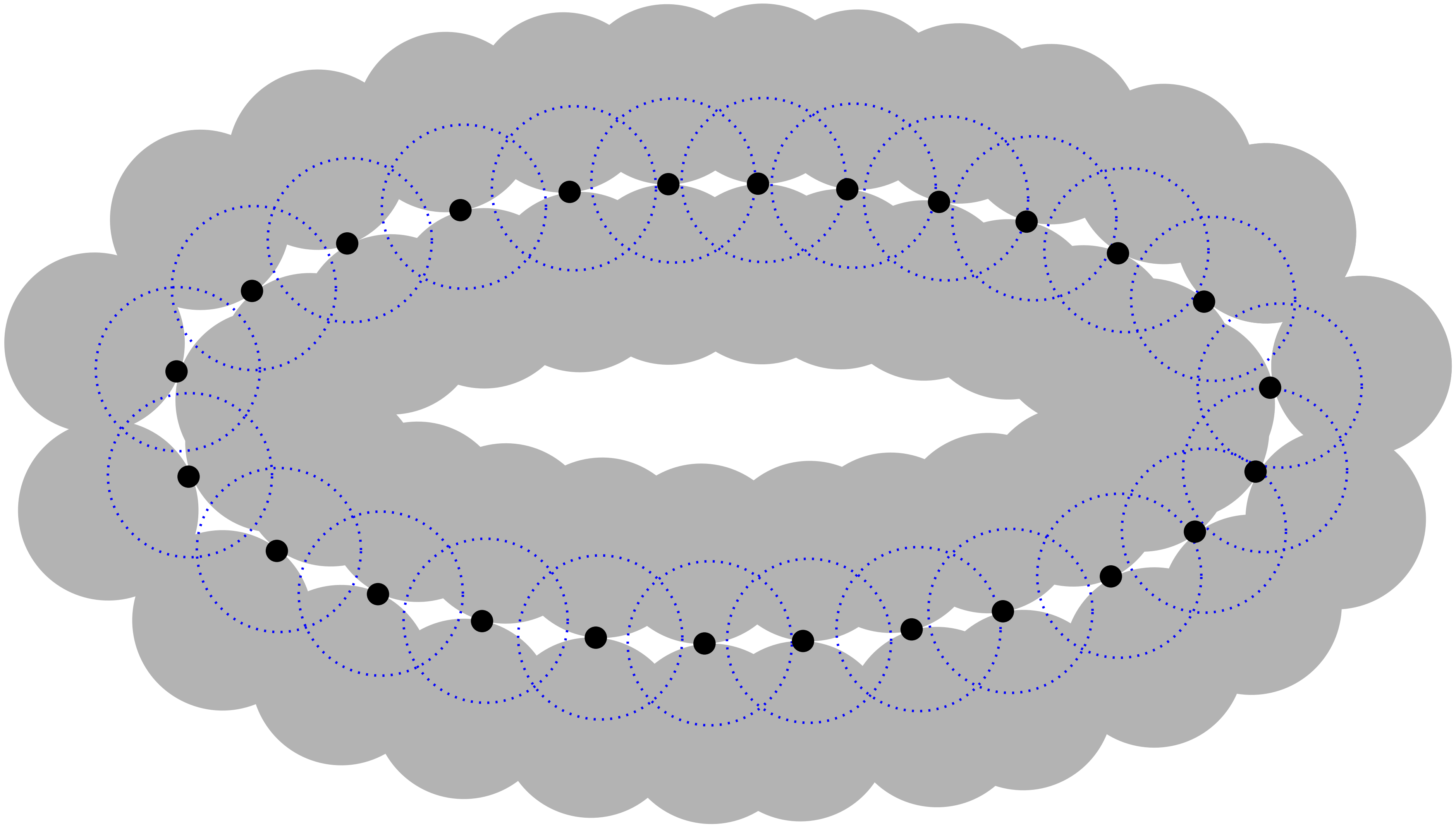
Goal: (**support estimation**) build an *approximation* (implicit, PL, simplicial, etc.) that is *structurally faithful* (homotopic, homeomorphic, isotopic, etc.) and *close* (in Hausdorff distance, in ℓ^2-distance, etc.) to M.

Reconstruction Paradigm



Reconstruction problem is ill-posed by nature.

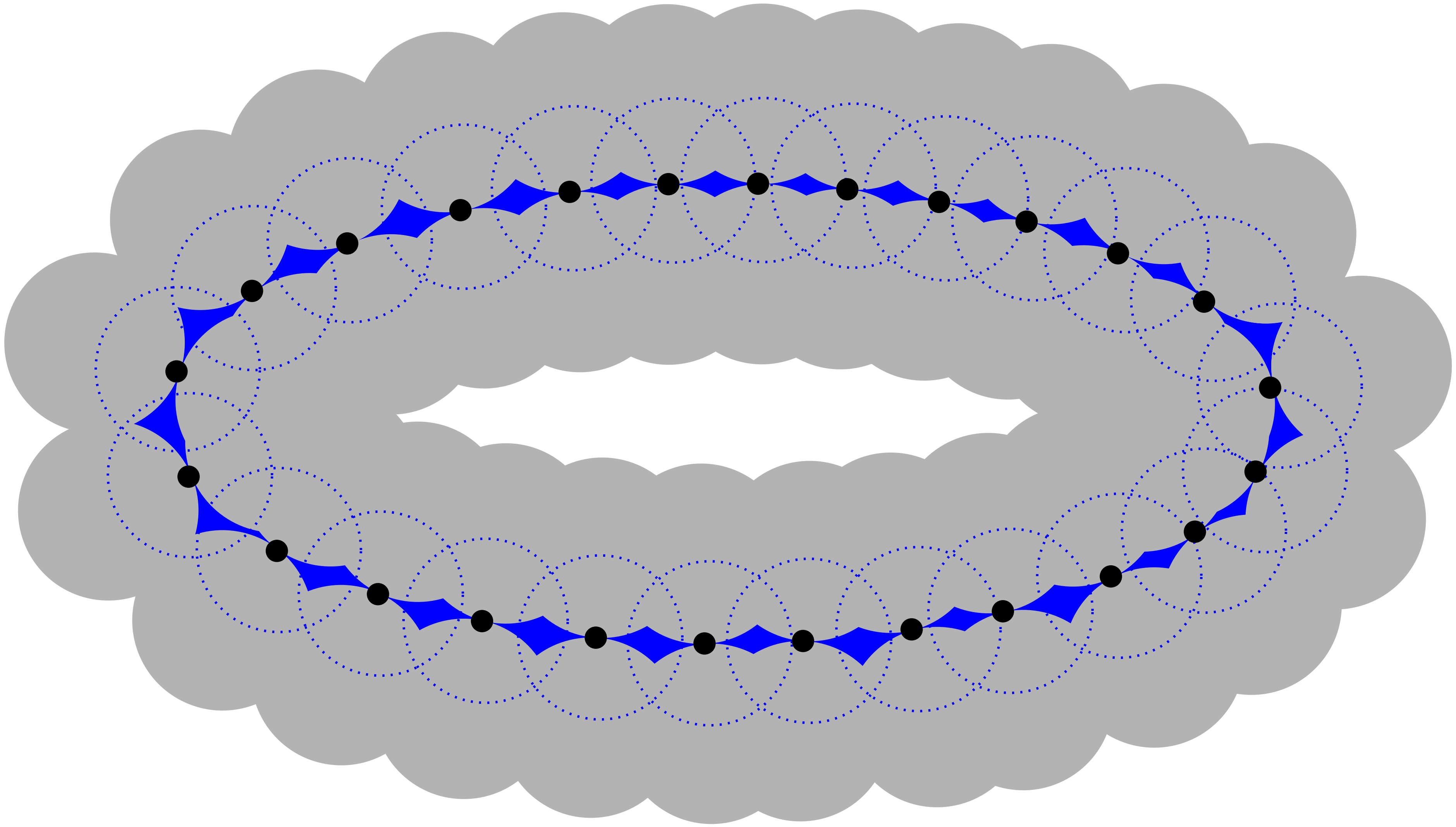
Reconstruction Paradigm



Reconstruction problem is ill-posed by nature.

$\rightarrow$ make *regularity assumptions* on M (fixed dimension, topological type, differentiability, etc.) and *sampling assumptions* (uniform measure, growth rate, etc.)

Reconstruction Paradigm



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→ make *regularity assumptions* on M (fixed dimension, topological type, differentiability, etc.) and *sampling assumptions* (uniform measure, growth rate, etc.)

→ for a suitable choice of hypotheses, the solution becomes unique **up to a set of deformations** (solution never unique!)

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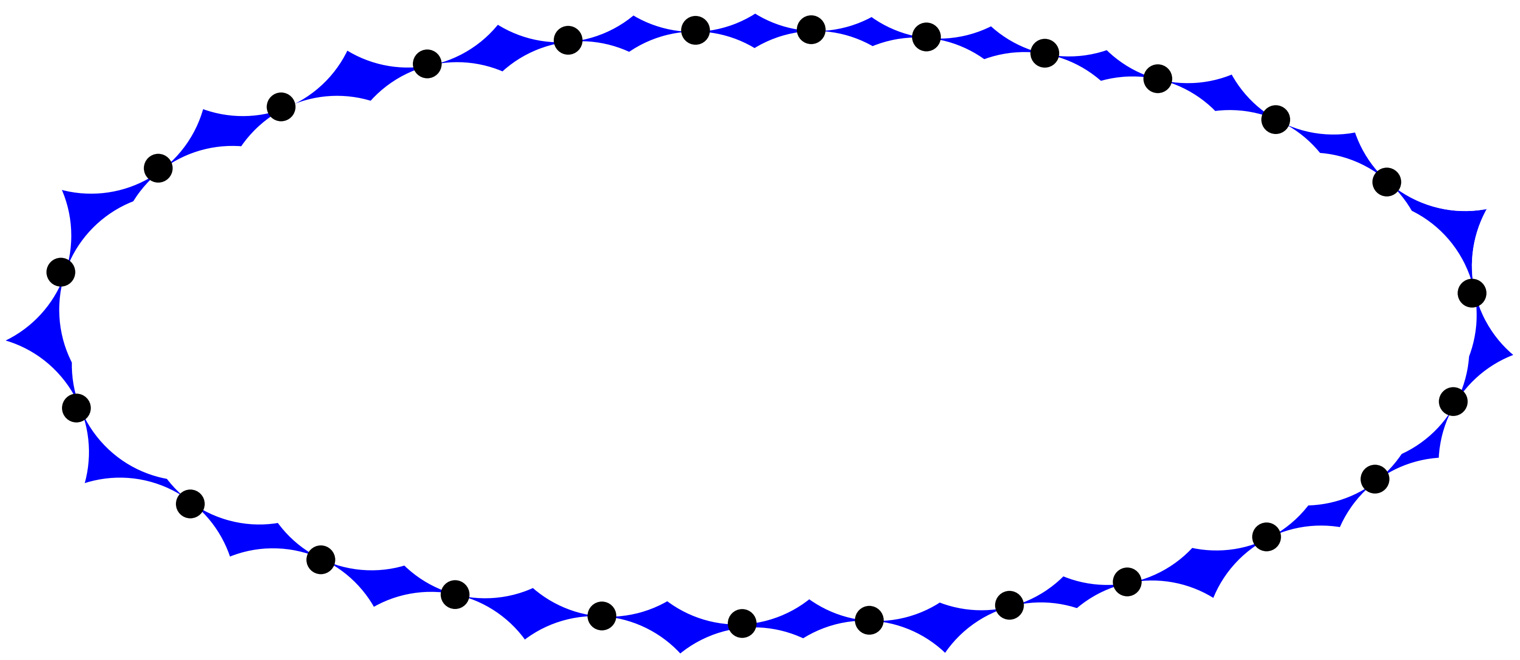

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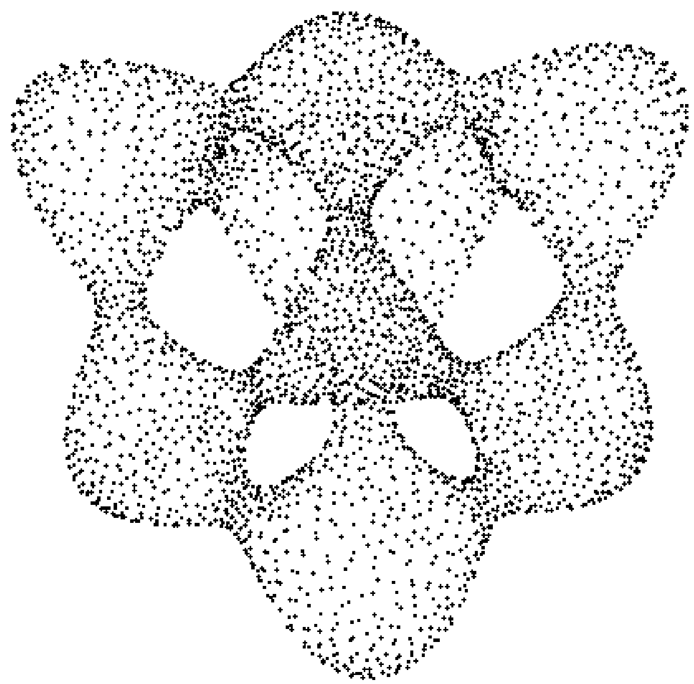
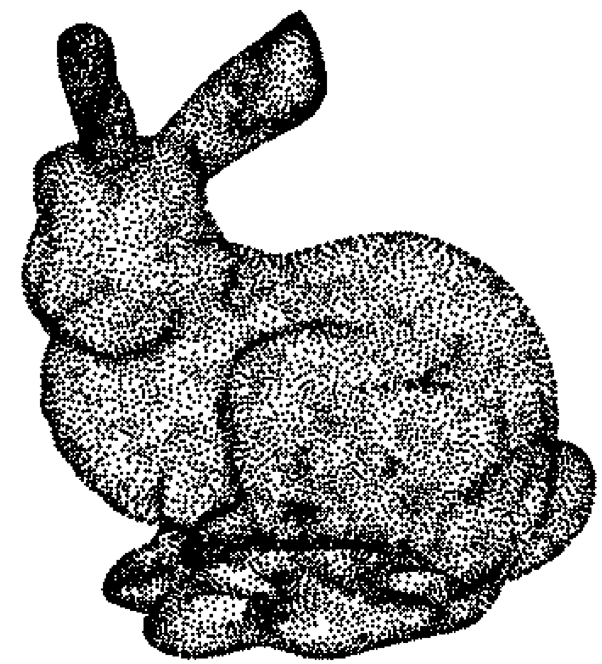
Various forms of inference





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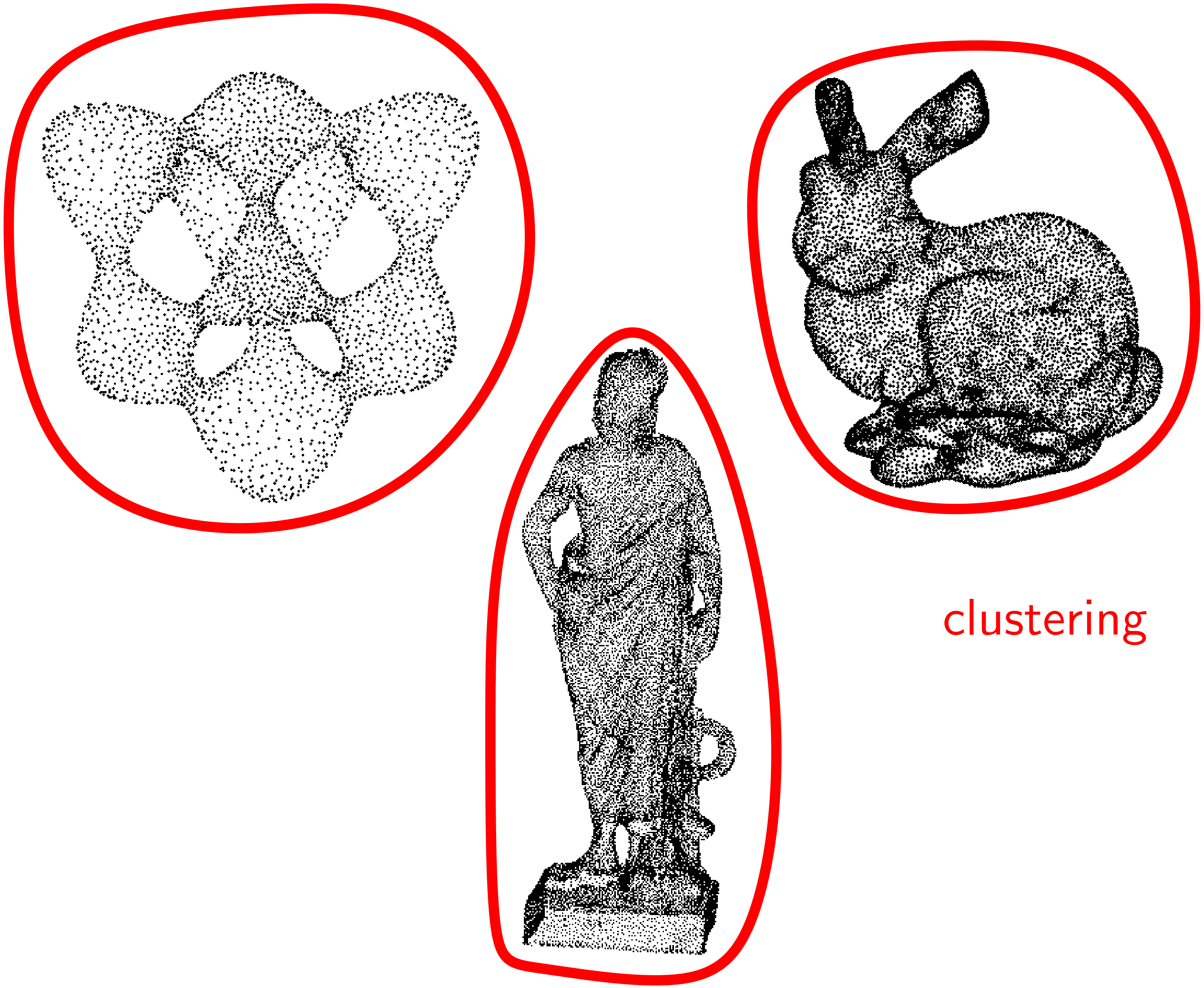
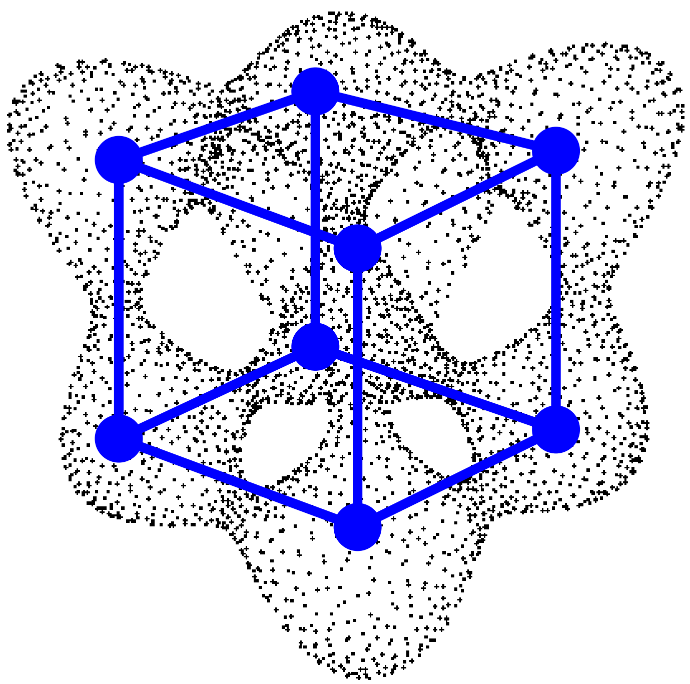
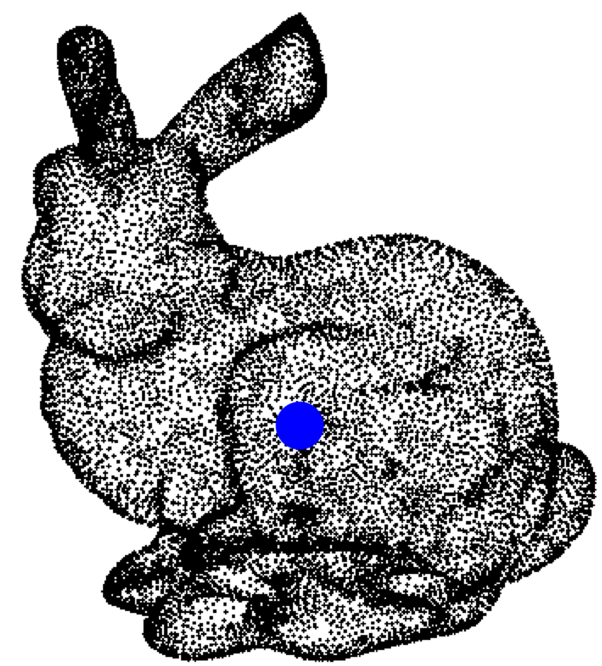
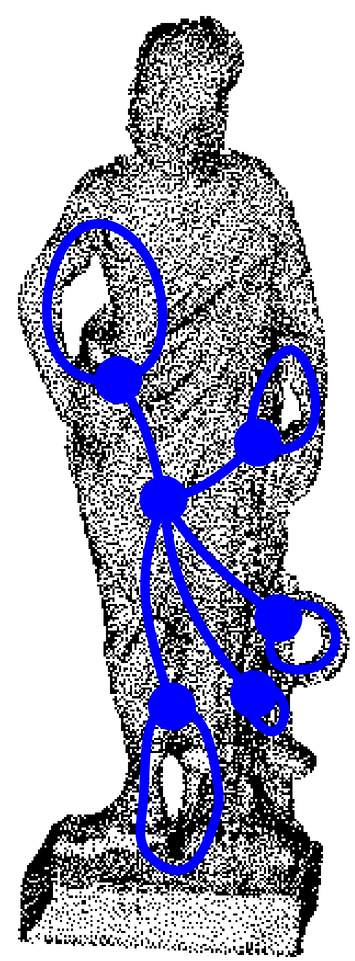


clustering

Various forms of inference



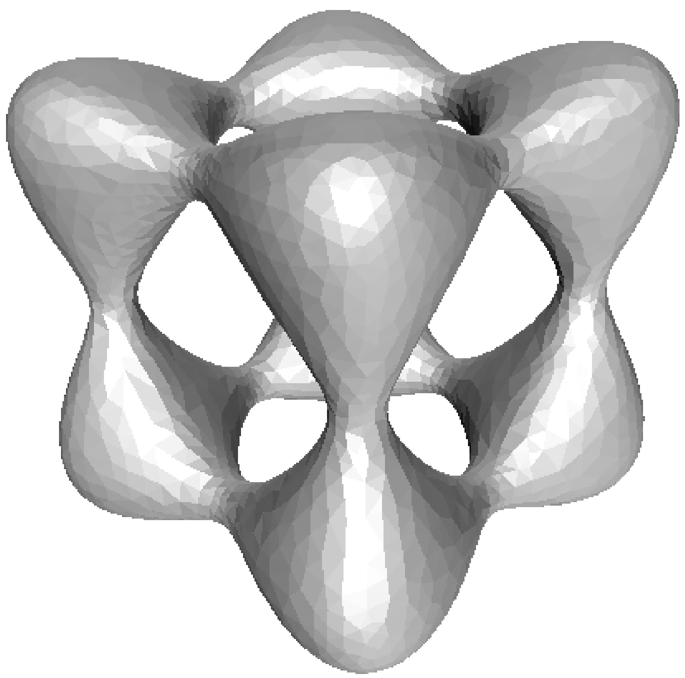


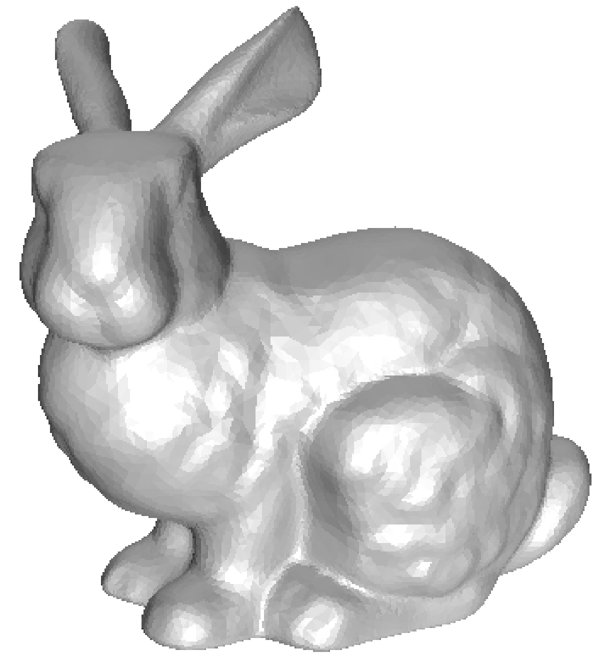


clustering

topological inference

Various forms of inference





clustering

topological inference

reconstruction

Where do the data come from?

3D scans

Sources

LASER

stereo vision

mechanical sensor

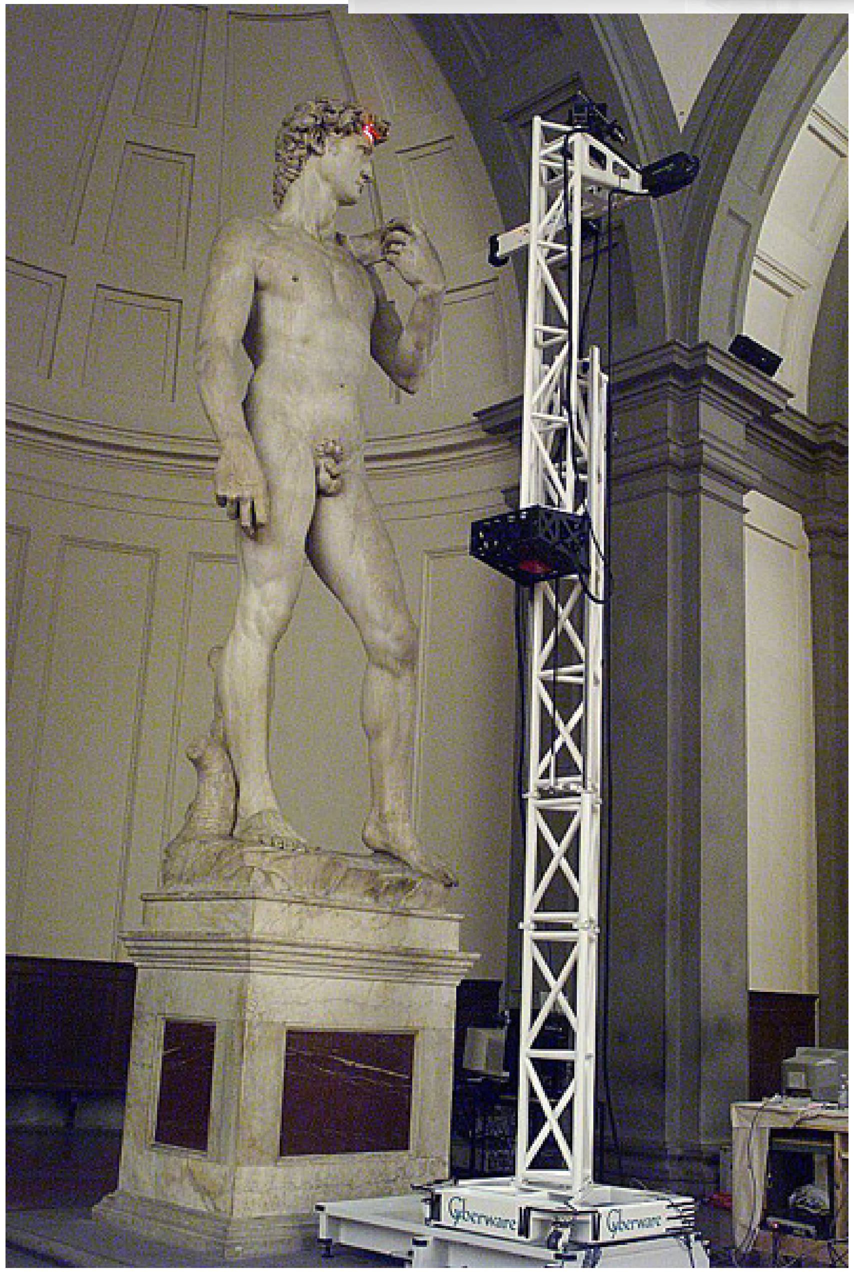
Applications

Reverse engineering

Prototyping

Quality control

Cultural heritage



Stanford Michelangelo Project

(raw data with 2 billion polygons, sampling with a precision of 0.25 mm)

Where do the data come from?

Medical Imaging

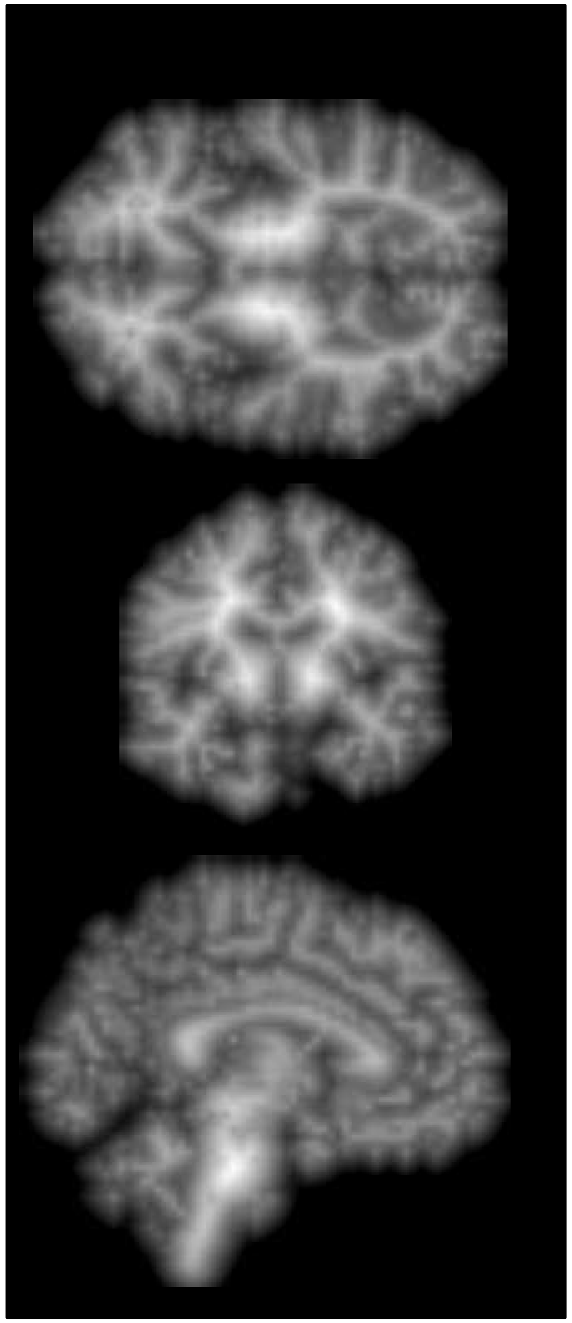
Sources

MRI scan

echograph

...

Intraoral 3d scanner



Applications

Diagnostic

Endoscopy simulation

Chirurgical intervention planning

Where do the data come from?

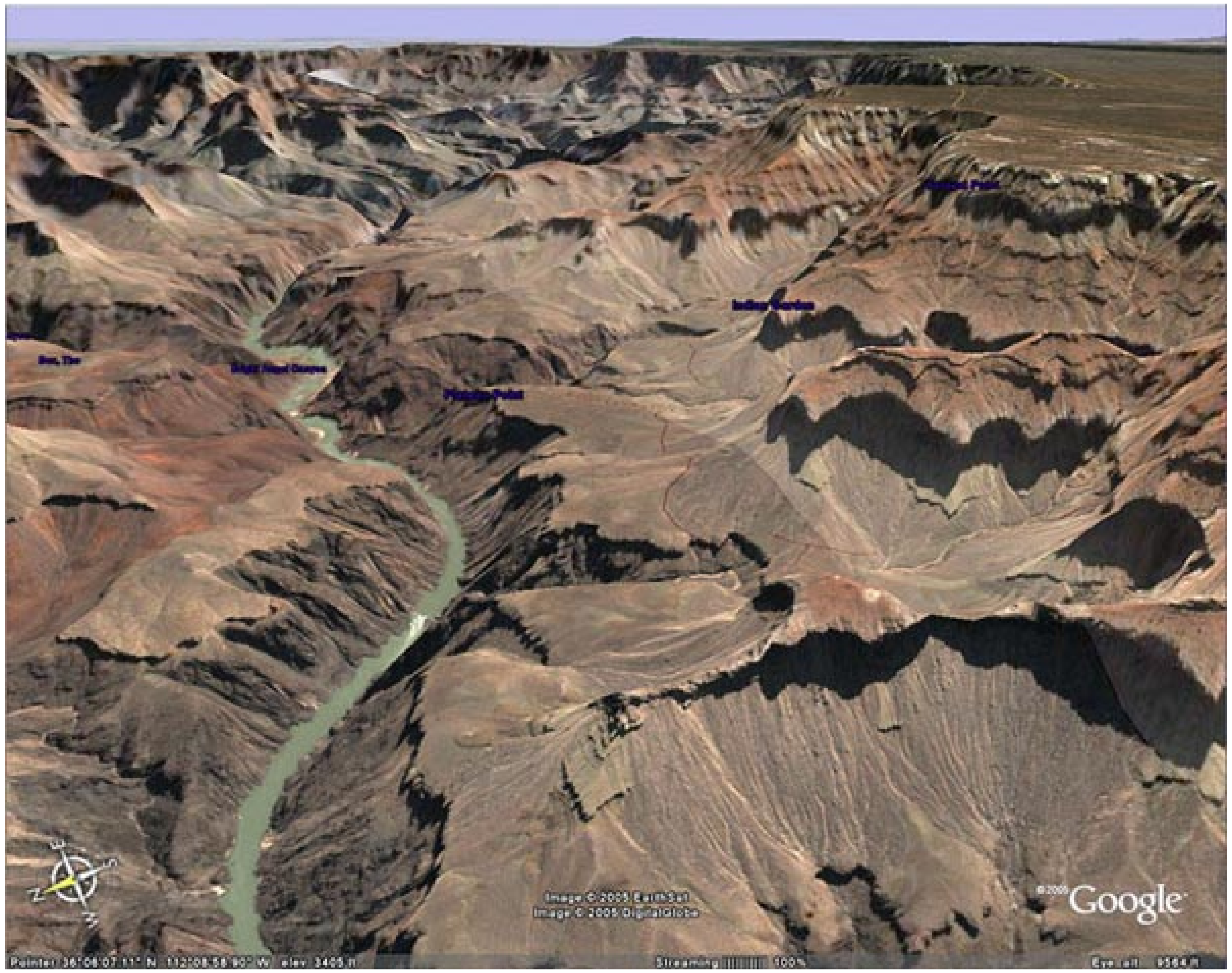
Geography, Geology

Sources

satellite/aerial images

ground probing

seismograph

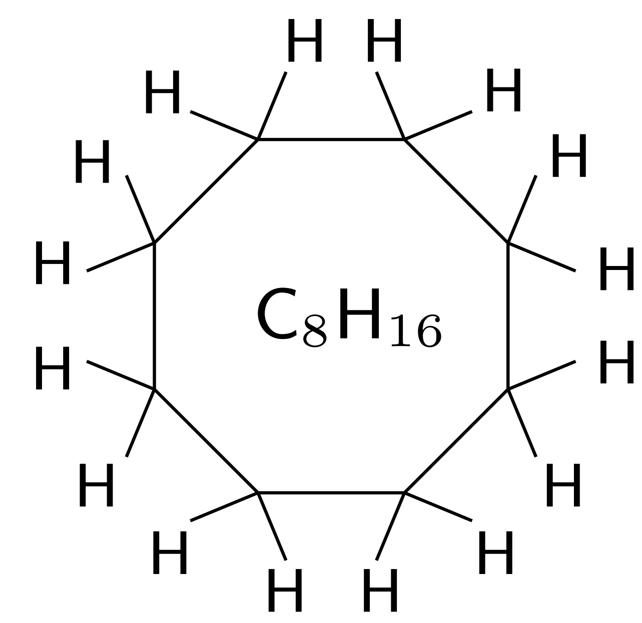


Applications

Maps making / Terrain modeling

Prospection (tunnels, oil)

Where do the data come from?



Higher-Dimensions

Sources

Data bases

Simulations

conformation space of cyclo-octane

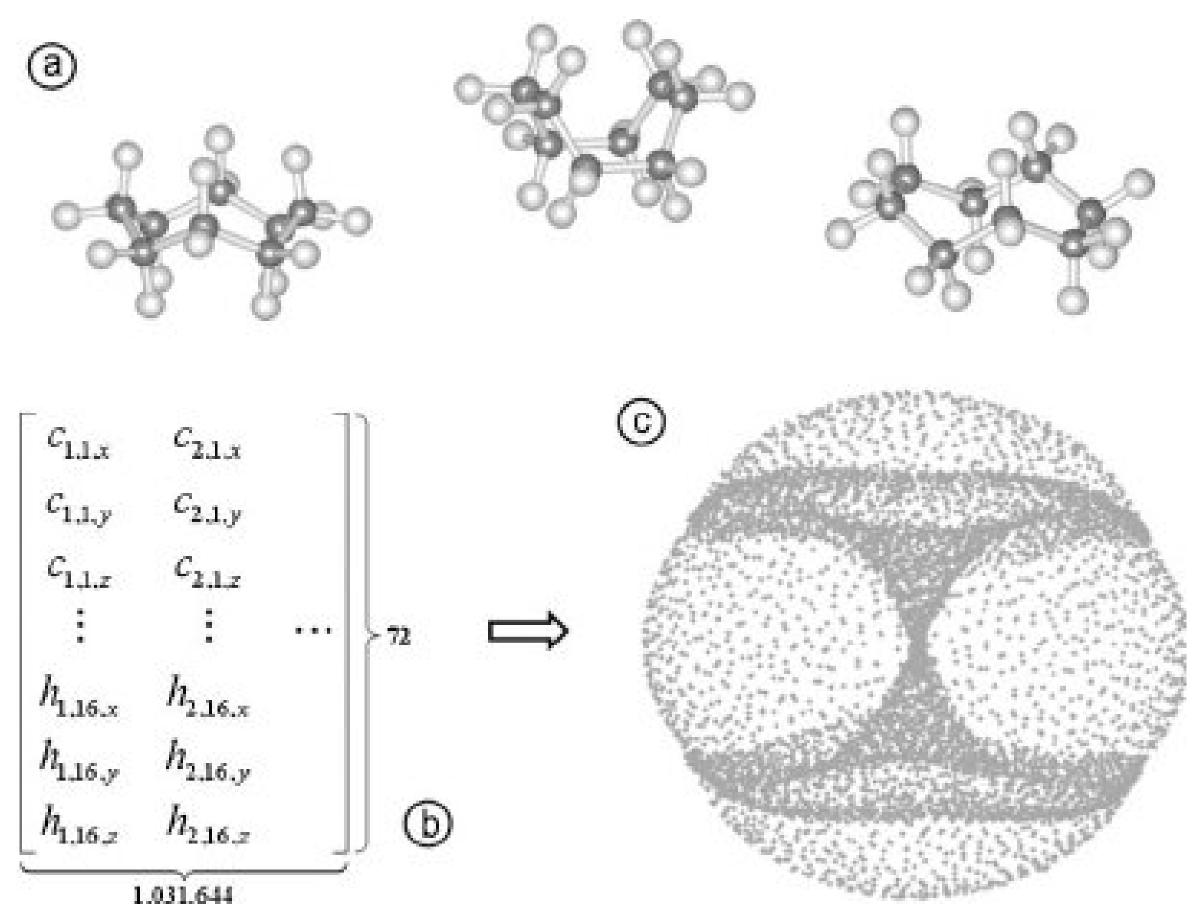
Applications

Machine Learning

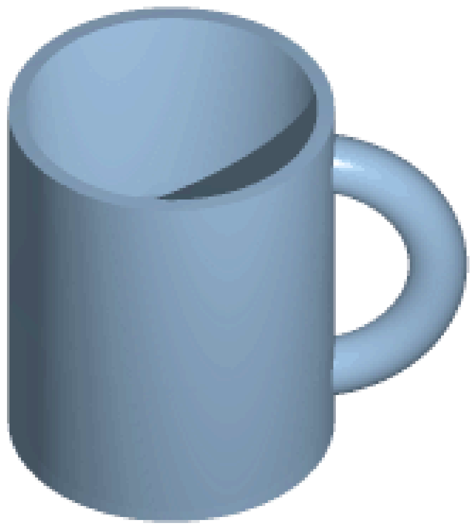
Robotics

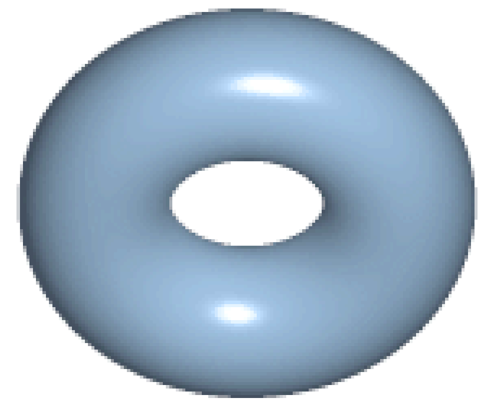
Image processing

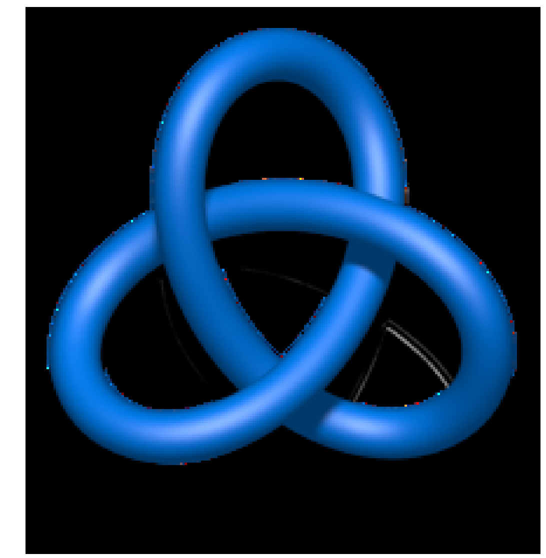
Biocomputing



Topological Criteria



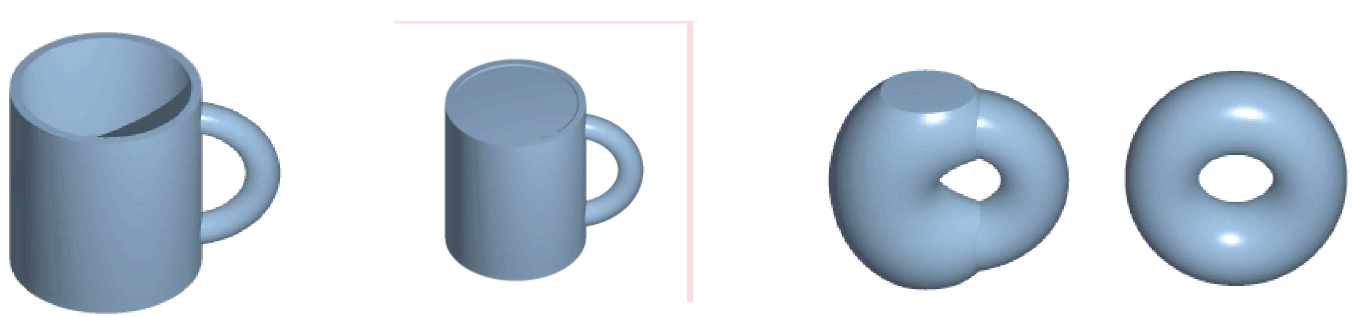




These three surfaces are *homeomorphic* (they all have genus 1)

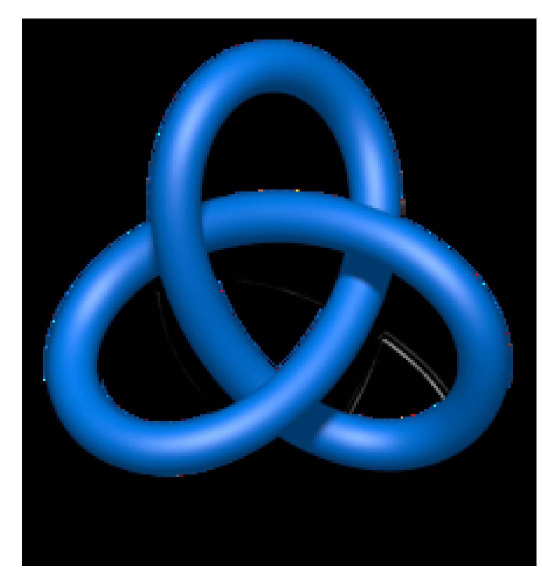
There exists a continuous bijection between surfaces, whose inverse is also continuous (formal definition given on the board)

isotopic surfaces (unknotted torus)



There exists a family of homeomorphisms which continuously transform the surfaces

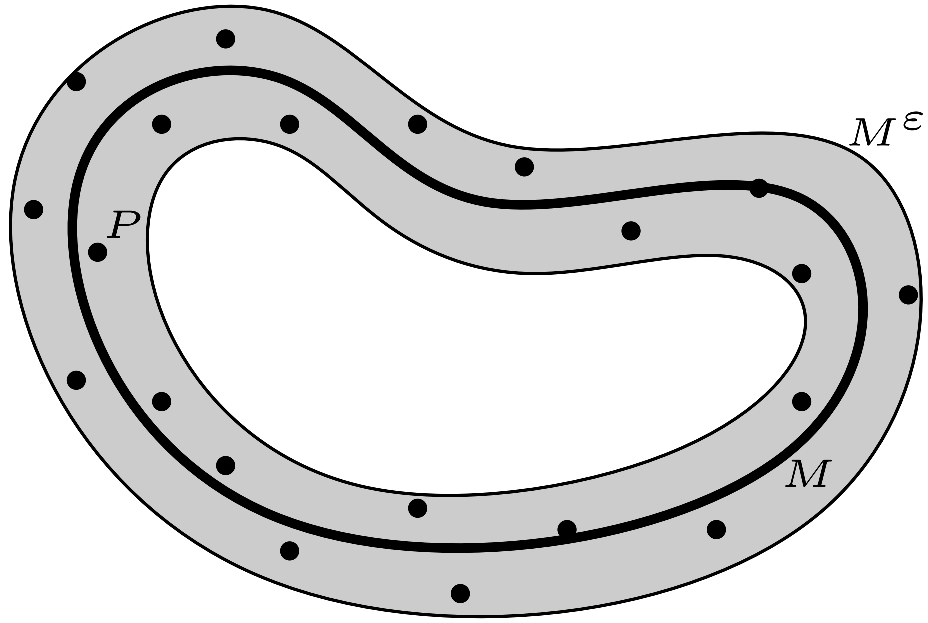
knotted torus



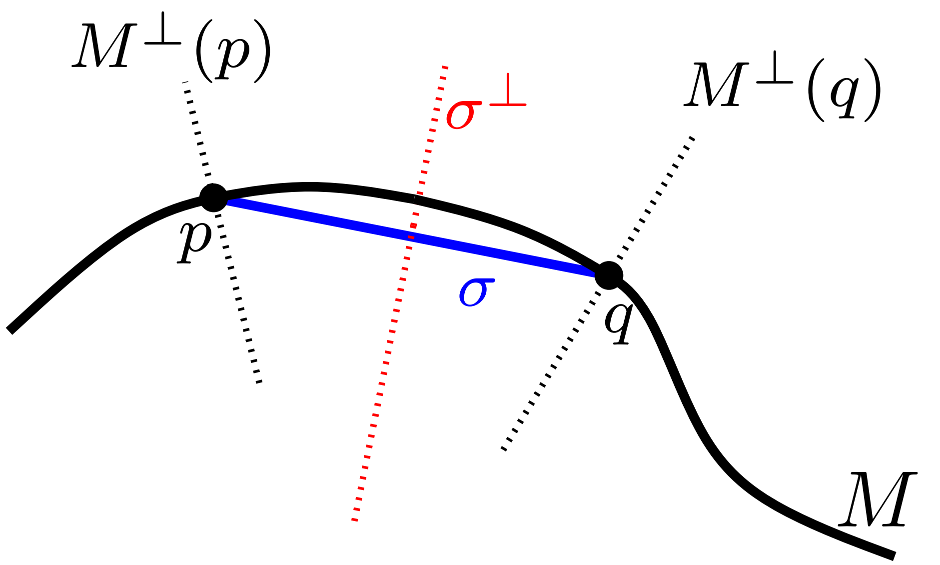
Geometric Criteria

Hausdorff distance (order 0):

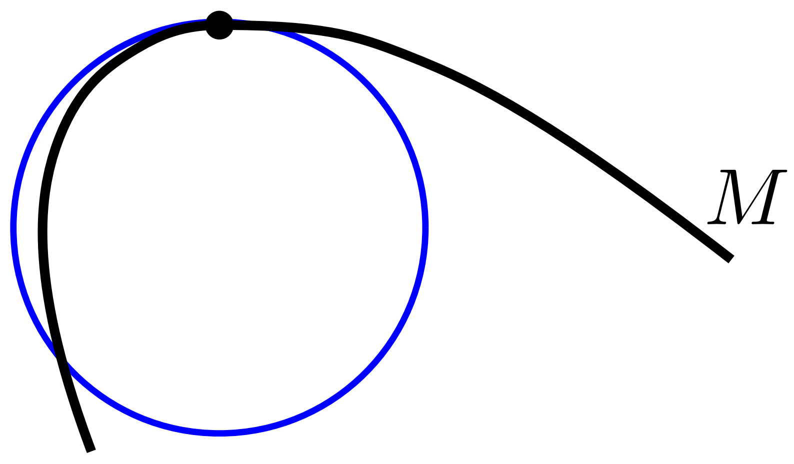
$$\mathrm{d_H}(P, M) = \inf\{\varepsilon \mid P^\varepsilon \supseteq M \text{ and } M^\varepsilon \supseteq P\}$$



Normals (order 1):



Curvature (order 2):



Geometric simplicial complexes

vertex set: $V = \{v_0, v_1, \ldots, v_{n-1}\} \subset \mathbb{R}^d$

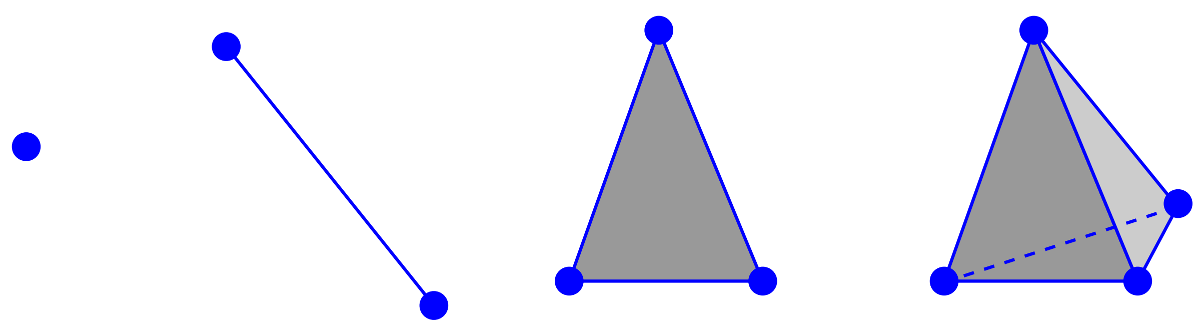
k-simplex: $\sigma = \mathrm{CH}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$

inclusion property (τ *face* of σ):

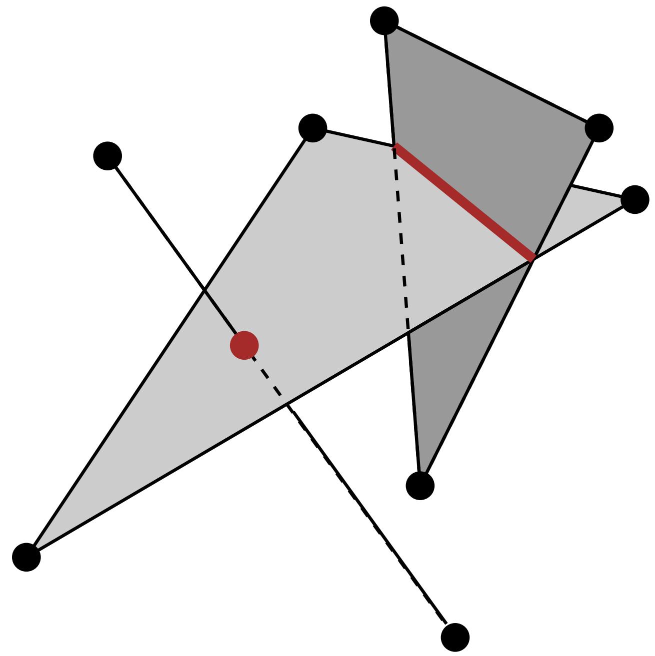
$$\sigma \in K \text{ and } V(\tau) \subseteq V(\sigma) \Longrightarrow \tau \in K$$

intersection property:

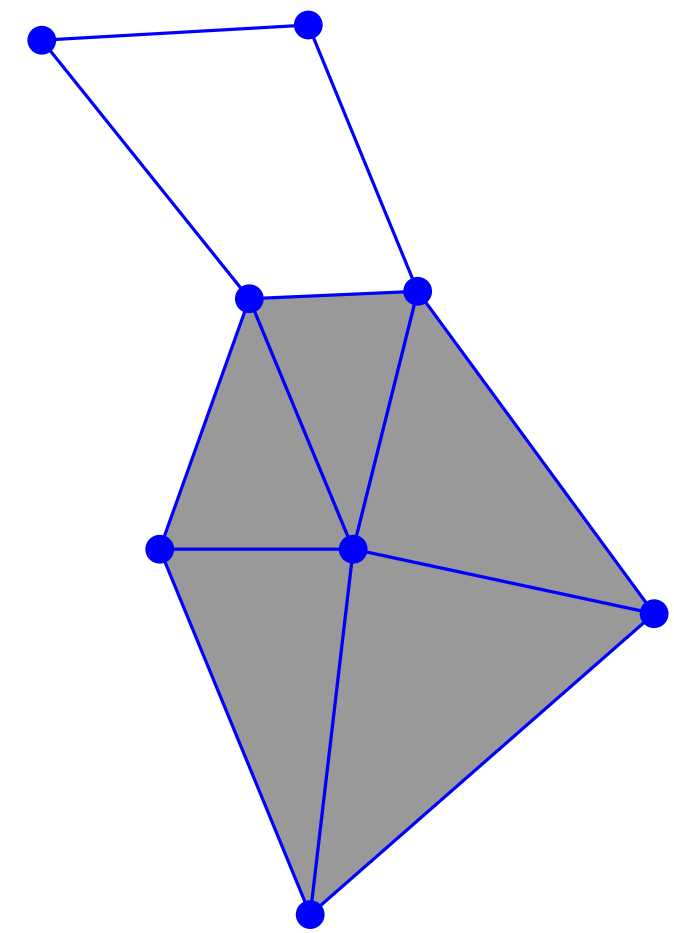
$$\sigma_1, \sigma_2 \in K \text{ and } \sigma_1 \cap \sigma_2 \neq \emptyset \Longrightarrow$$
$$\sigma_1 \cap \sigma_2 \in K \text{ and is a face of both}$$



0-simplex 1-simplex 2-simplex 3-simplex



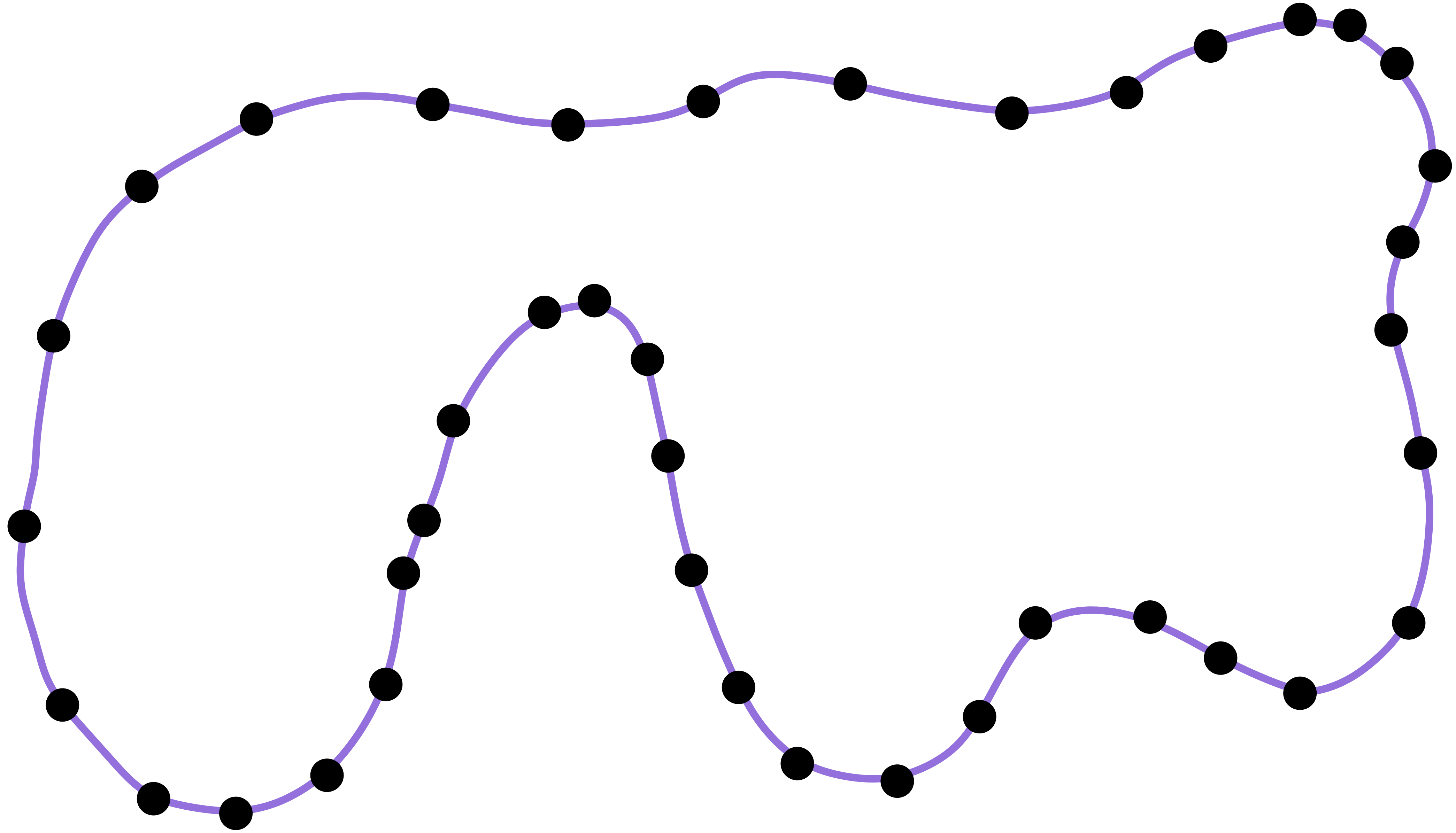
invalid simplicial complex



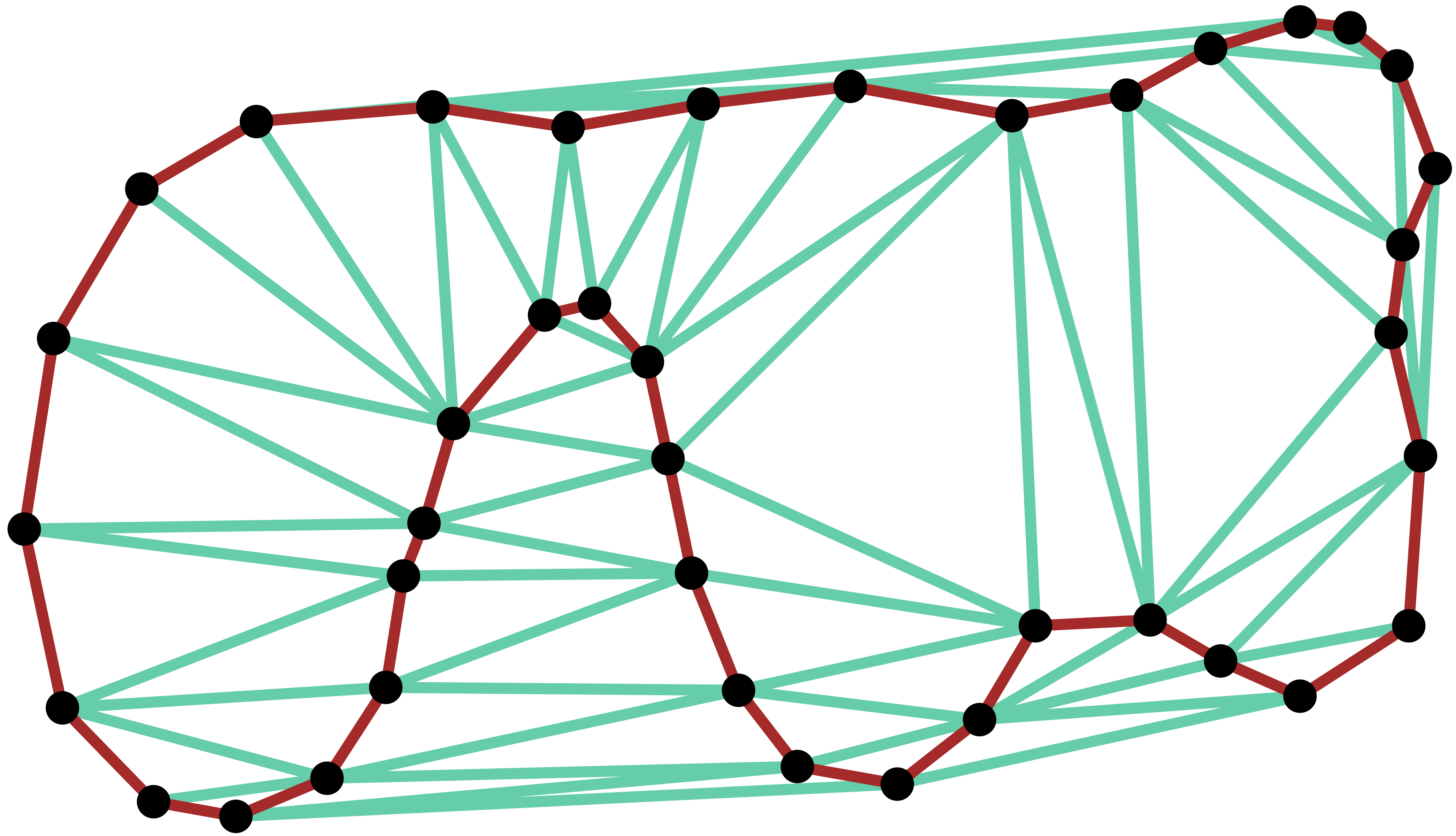
valid simplicial complex

Reconstruction using Delaunay

What Delaunay has to do with reconstruction



What Delaunay has to do with reconstruction



$\rightarrow$ faithful approximation of the curve appears as a subcomplex of the Delaunay

$\rightarrow$ should hold whenever the point cloud is sufficiently densely sampled

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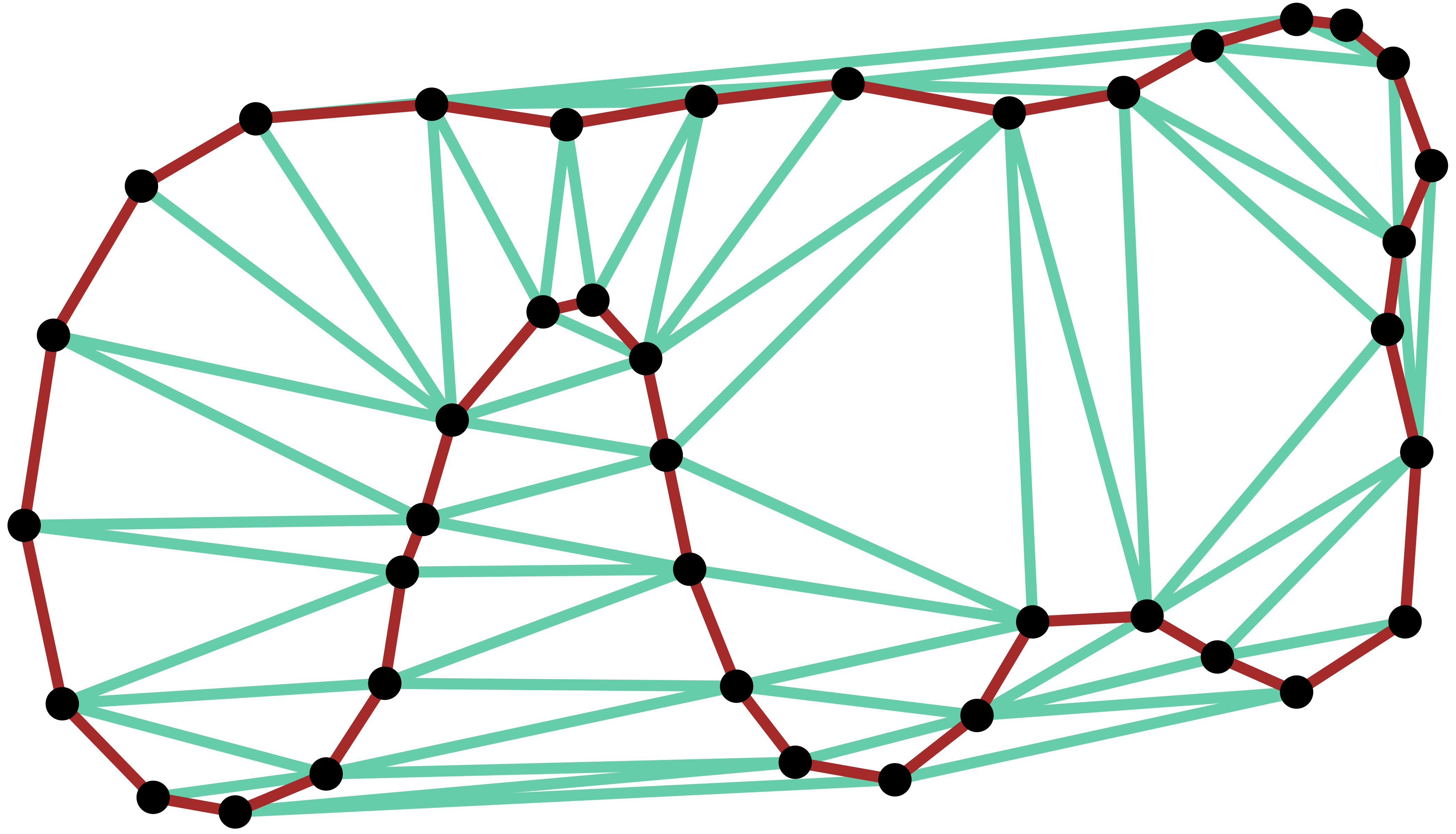
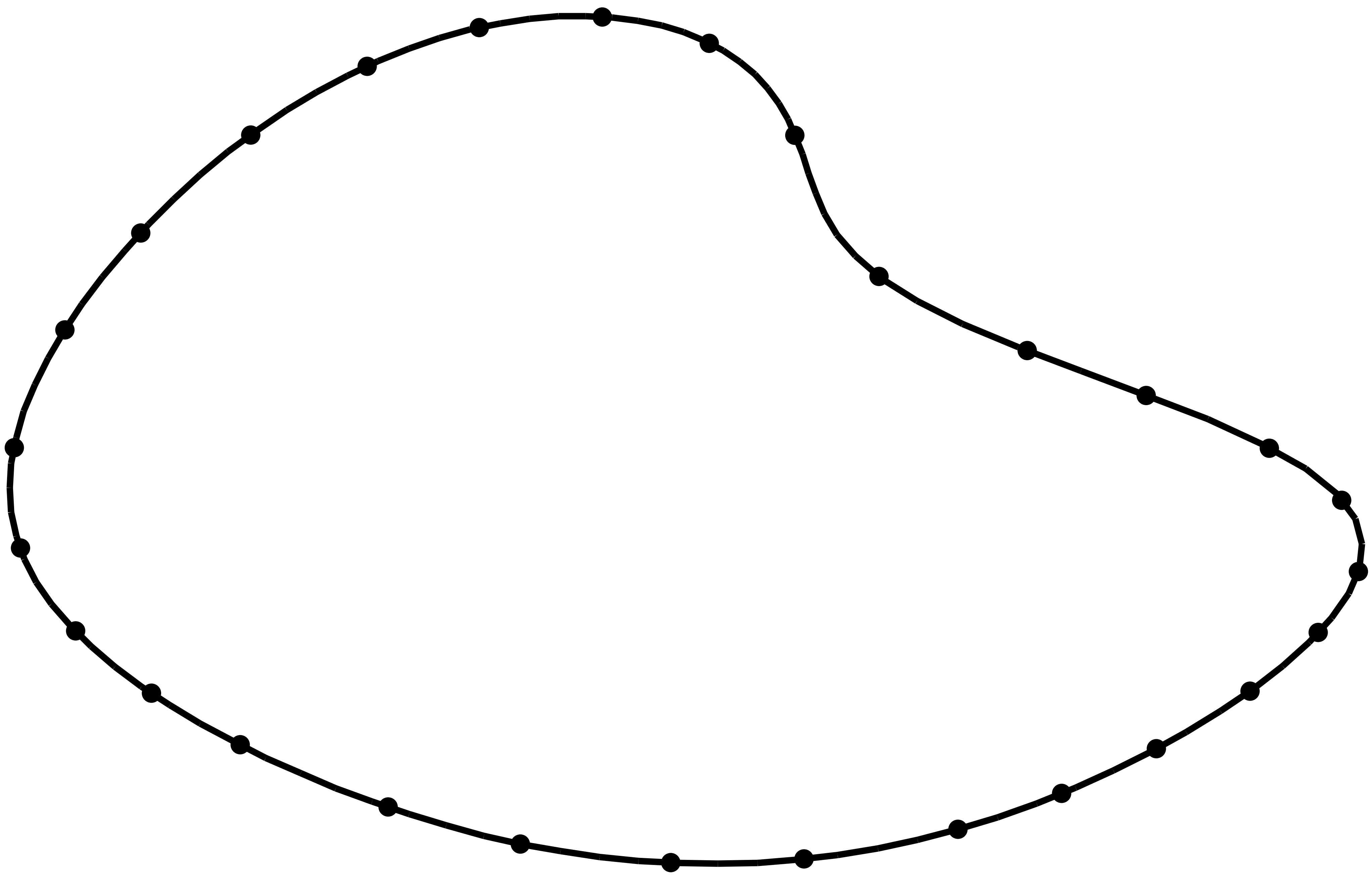


→ faithful approximation of the curve appears as a subcomplex of the Delaunay

→ should hold whenever the point cloud is sufficiently densely sampled

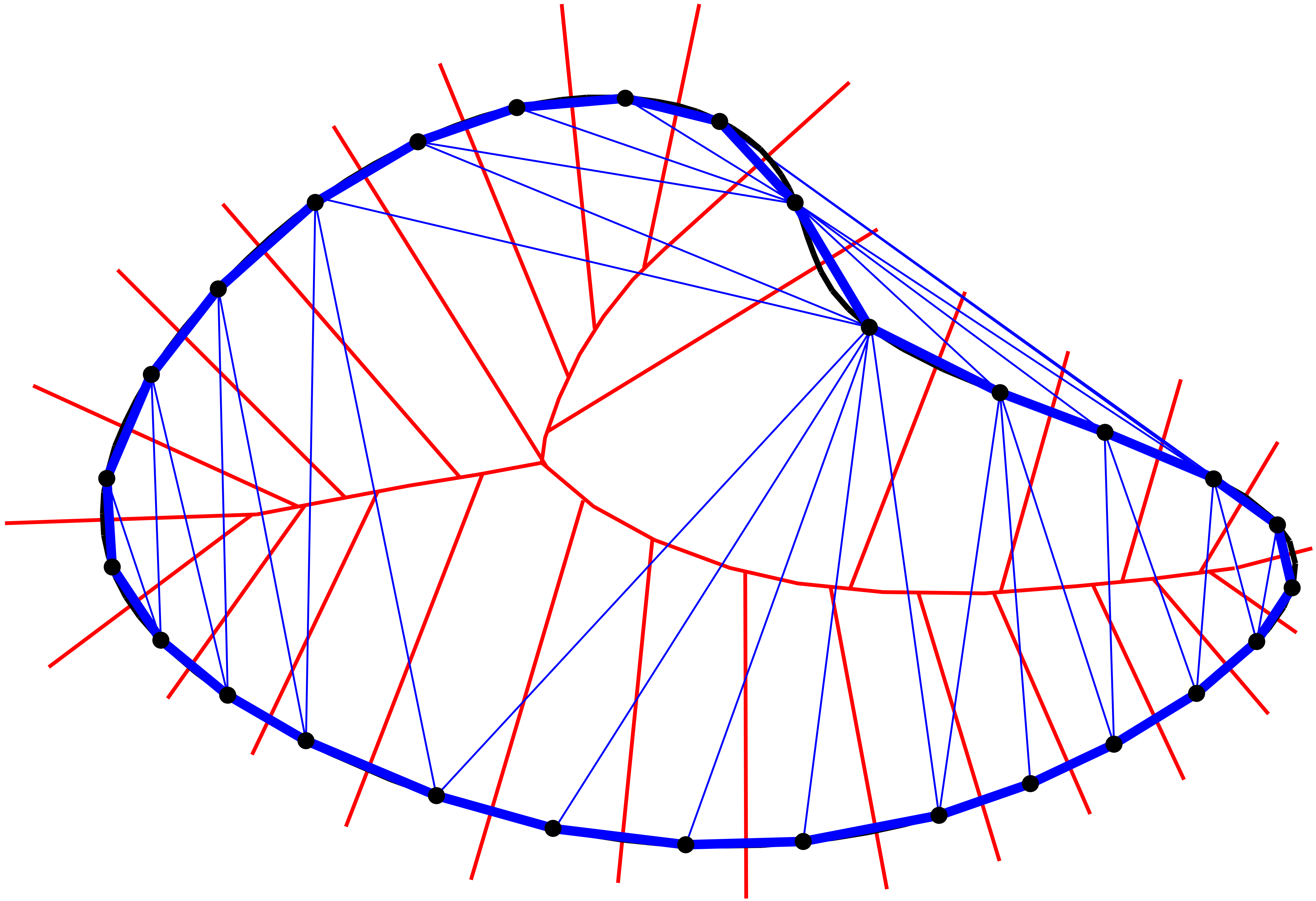
Q What is this *good* subcomplex? Can it be defined in some canonical way?

Restricted Delaunay triangulation



Restricted Delaunay triangulation

Def: $\mathcal{D}^M(P) := \{\sigma \in \mathcal{D}(P) \mid \sigma^* \cap M \neq \emptyset\}$



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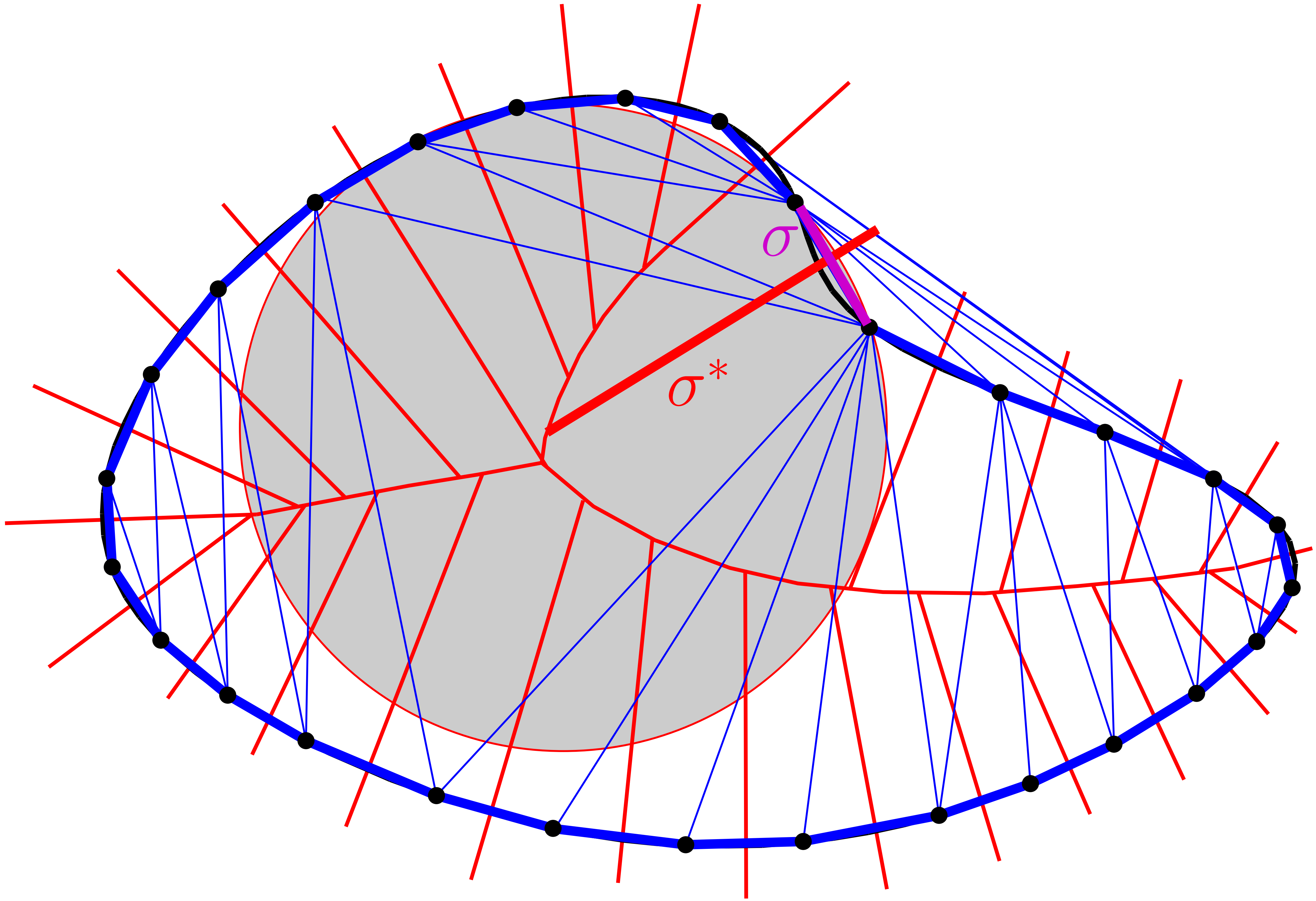

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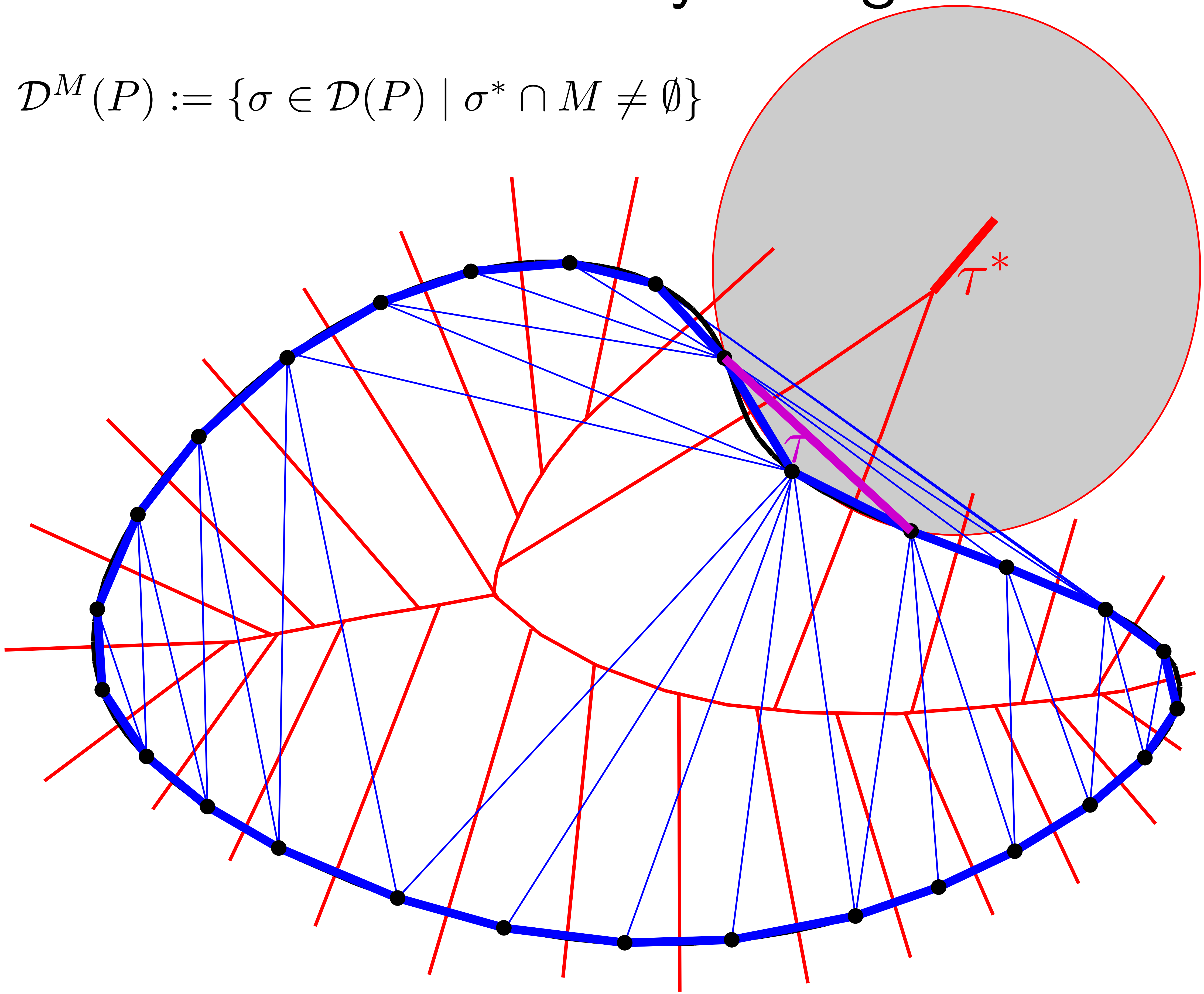
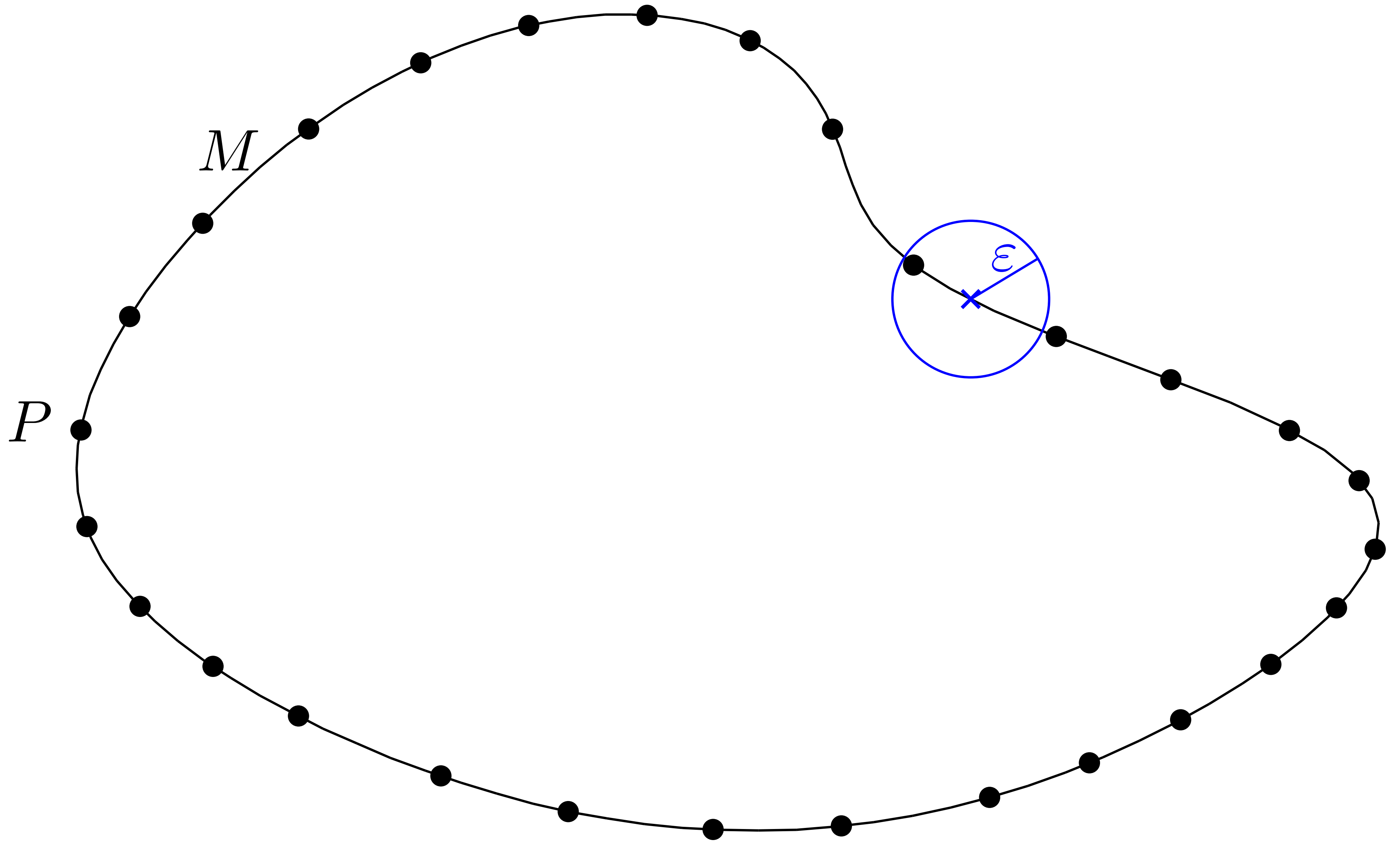


Sampling Condition

Def: P is an ε-**sample** of M if $\forall x \in M$, $\min\{\|x - p\| \mid p \in P\} \leq \varepsilon$.

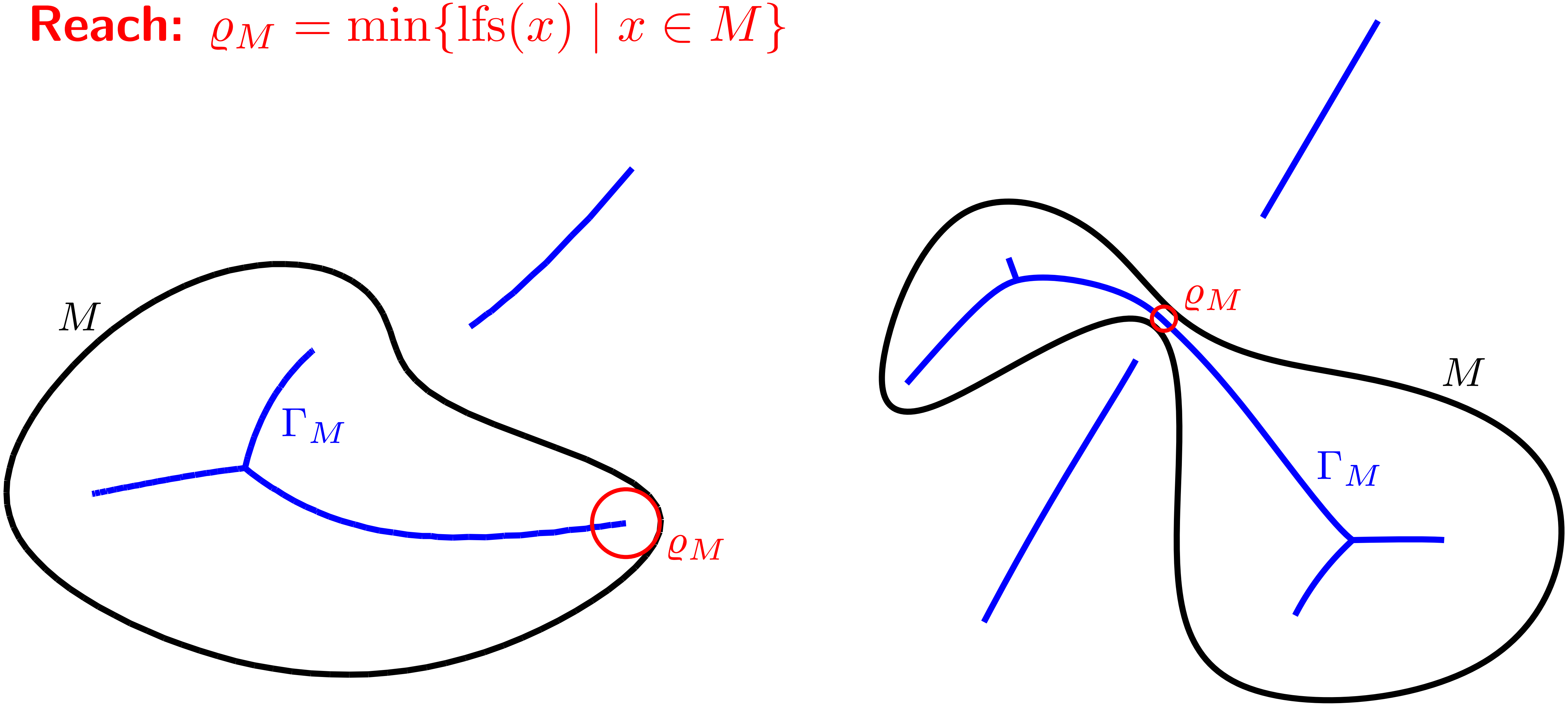


Regularity Condition

Medial axis: $\Gamma_M = \mathrm{cl}\{x \in \mathbb{R}^d \mid |\mathrm{NN}_M(x)| \geq 2\}$

Local feature size: $\forall x \in \mathbb{R}^d$, $\mathrm{lfs}(x) = \min\{\|x - m\| \mid m \in \Gamma_M\}$

Reach: $\varrho_M = \min\{\mathrm{lfs}(x) \mid x \in M\}$



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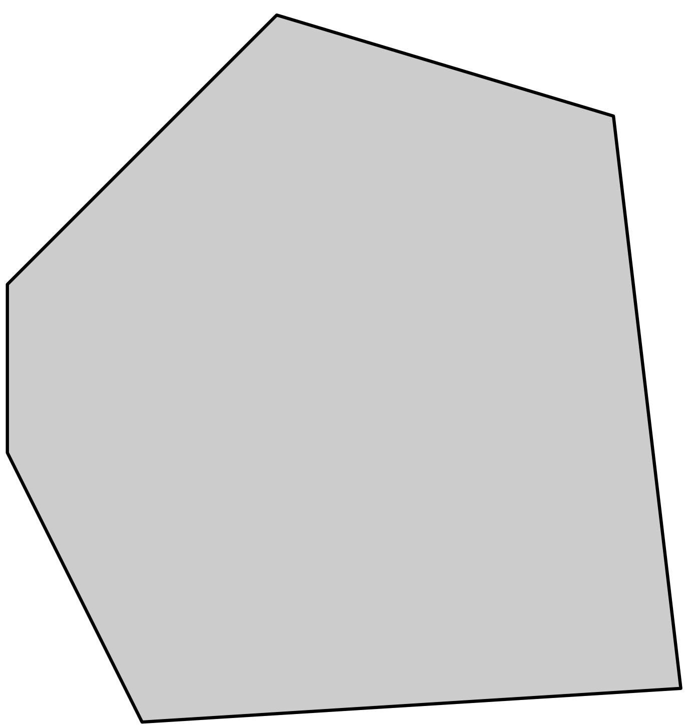
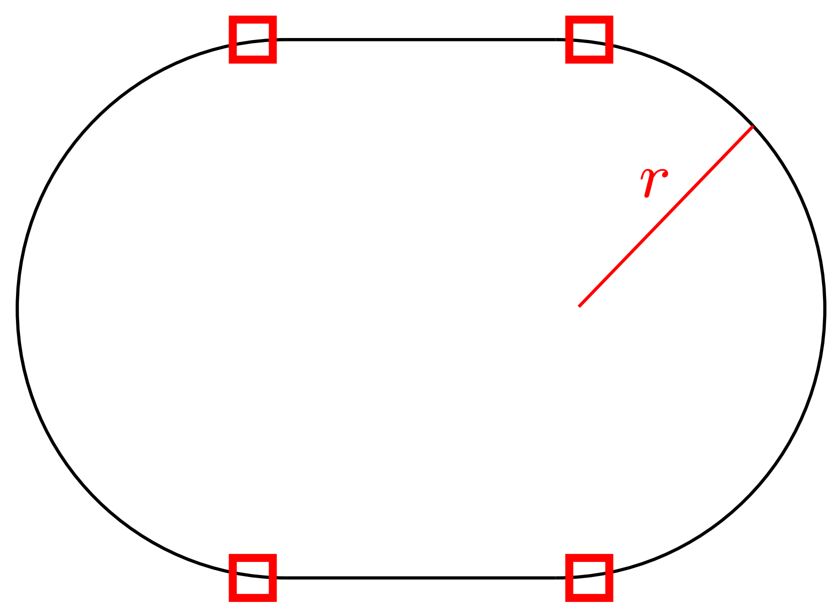
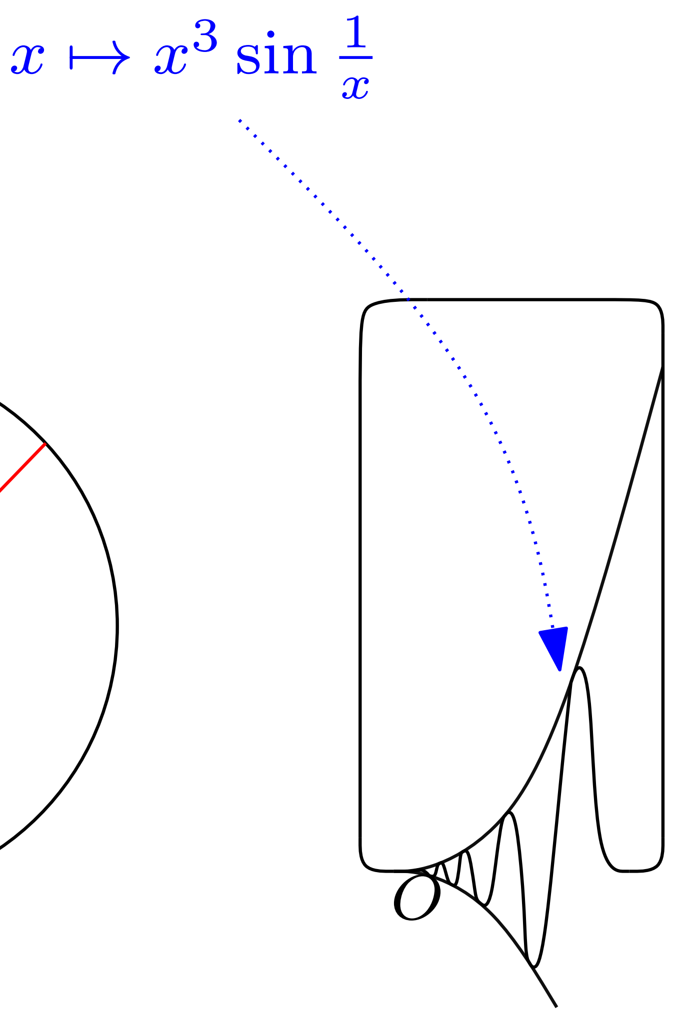


$\varrho_M = +\infty$
(convex)



$\varrho_M = r$
$C^{1,1}$ but not C^2)



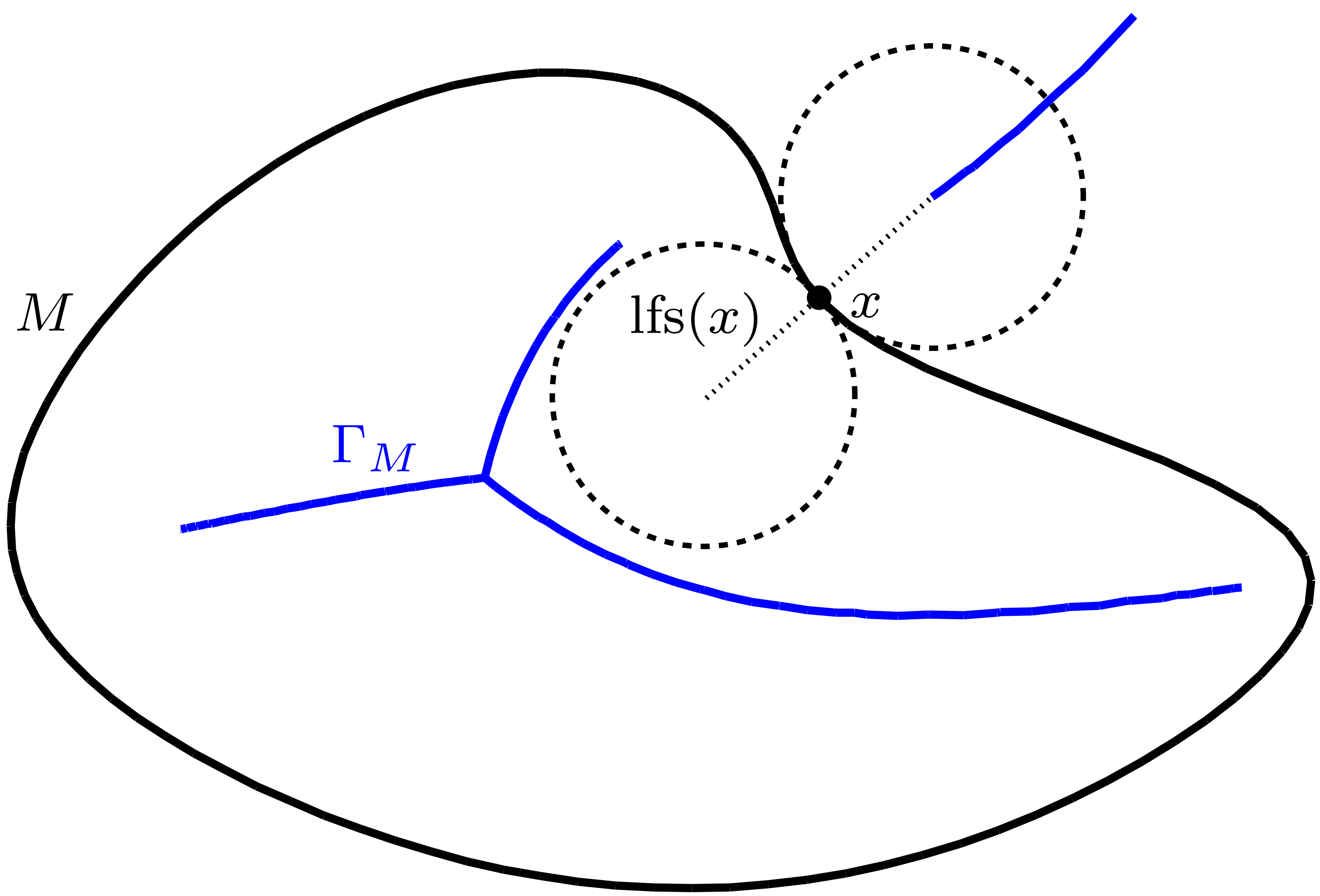
$\varrho_M = 0$
(C^1 but not $C^{1,1}$)

Regularity Condition

$\rightarrow$ Fundamental properties: (see [Federer 1958])

Tangent Ball Lemma: $\forall x \in M$, $\forall c \in M^{\perp}(x)$, $\|x - c\| < \mathrm{lfs}(x) \Rightarrow$

$$B^{o}(c, \|x - c\|) \cap M = \emptyset.$$



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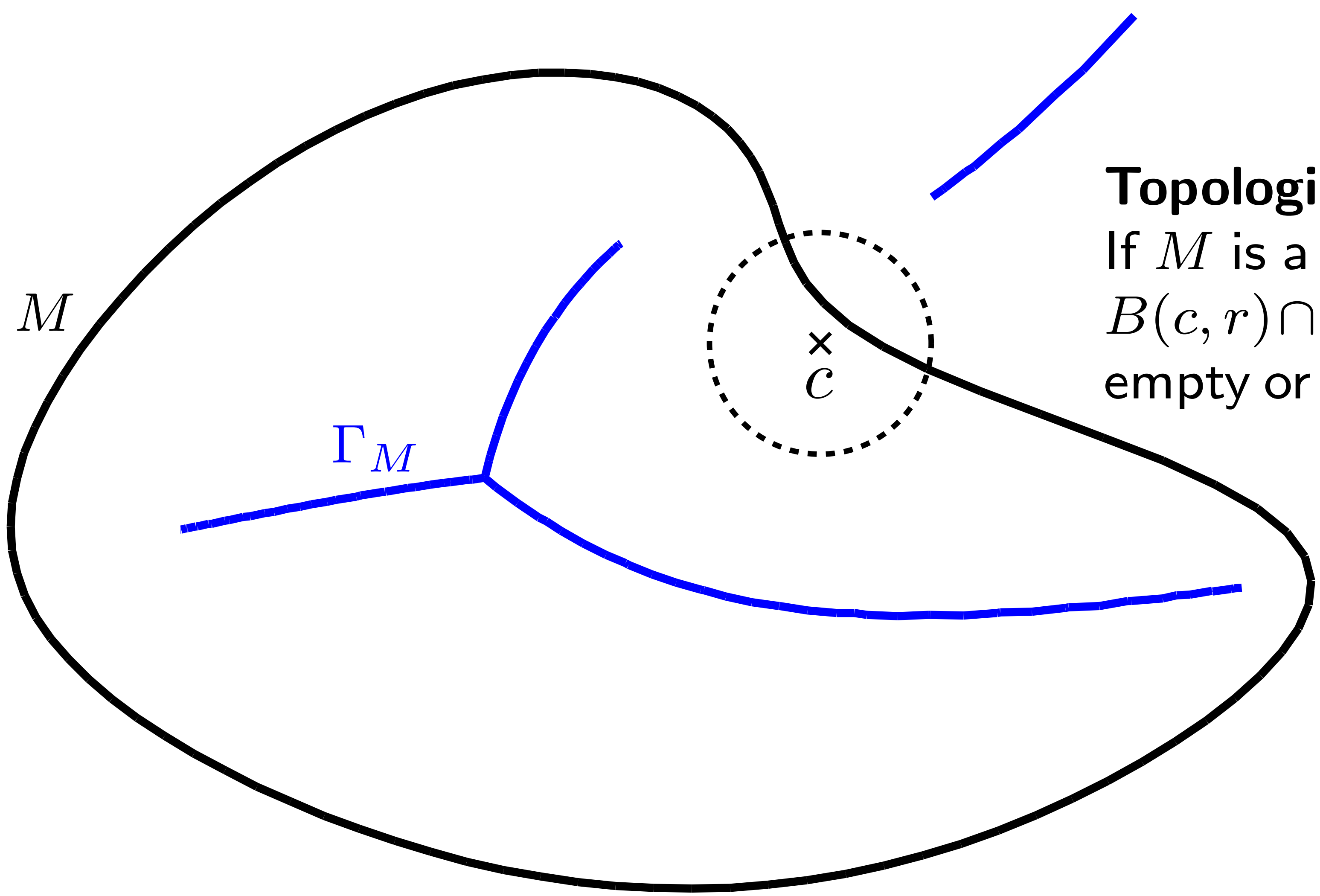
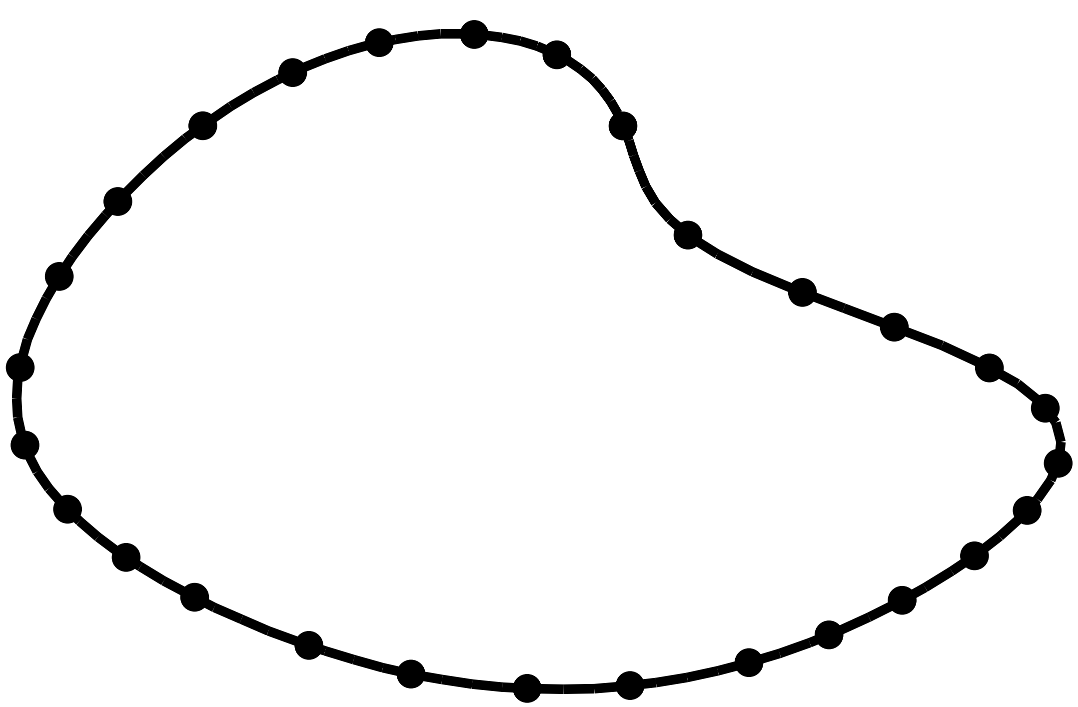


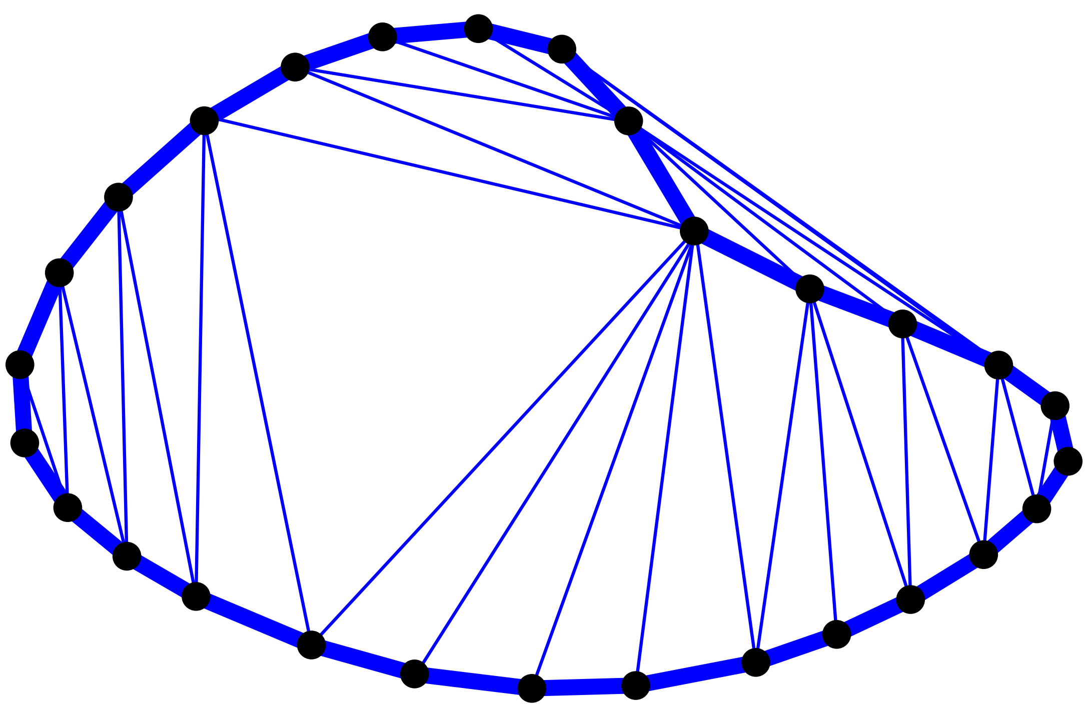
Topological Ball Lemma:
If M is a k-manifold, then $\forall B(c,r)$ s.t. $B(c,r) \cap \Gamma_M = \emptyset$, $B(c,r) \cap M$ is either empty or homeomorphic to the ball B^k.

Approximation via Restricted Delaunay

Theorem: [Amenta et al. 1998-99]
If M is a closed curve or surface with positive *reach* ϱ_M, and if P is an ε-*sample* of M with $\varepsilon < \varrho_M$ (curve) or $\varepsilon < 0.1\,\varrho_M$ (surface), then:

- $\mathcal{D}^M(P)$ is homeomorphic to M (denoted $\mathcal{D}^M(P) \simeq M$),
- $\mathrm{d_H}(\mathcal{D}^M(P), M) \in O(\varepsilon^2)$,
- $\forall \sigma \in \mathcal{D}^M(P)$, $\forall p \in V(\sigma)$, $\angle \sigma^\perp M^\perp(p) \in O(\varepsilon)$,
- $\cdots$ (similar areas, curvature estimation, etc.)





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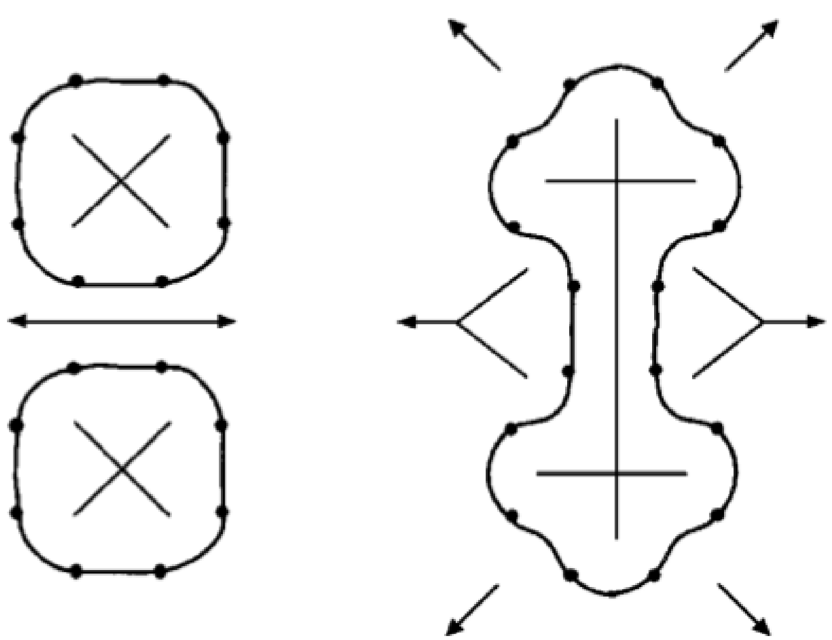
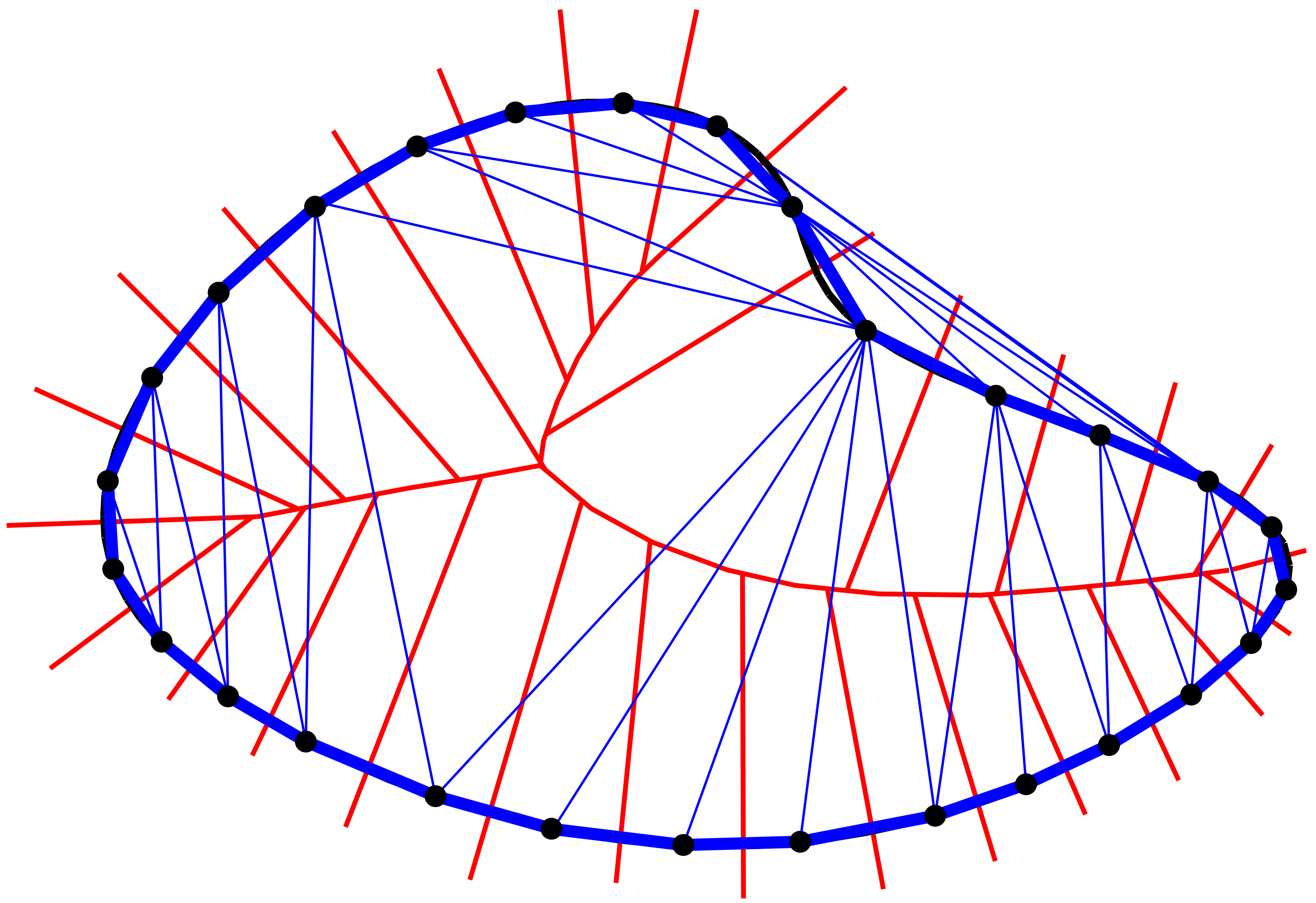


Reconstruction is uncertain if ε is not small enough compared to ϱ_M

Approximation via Restricted Delaunay

Proof for curves:

show that every edge of $\mathcal{D}^M(P)$ connects consecutive points of P along M, and vice-versa



Approximation via Restricted Delaunay

Proof for curves:

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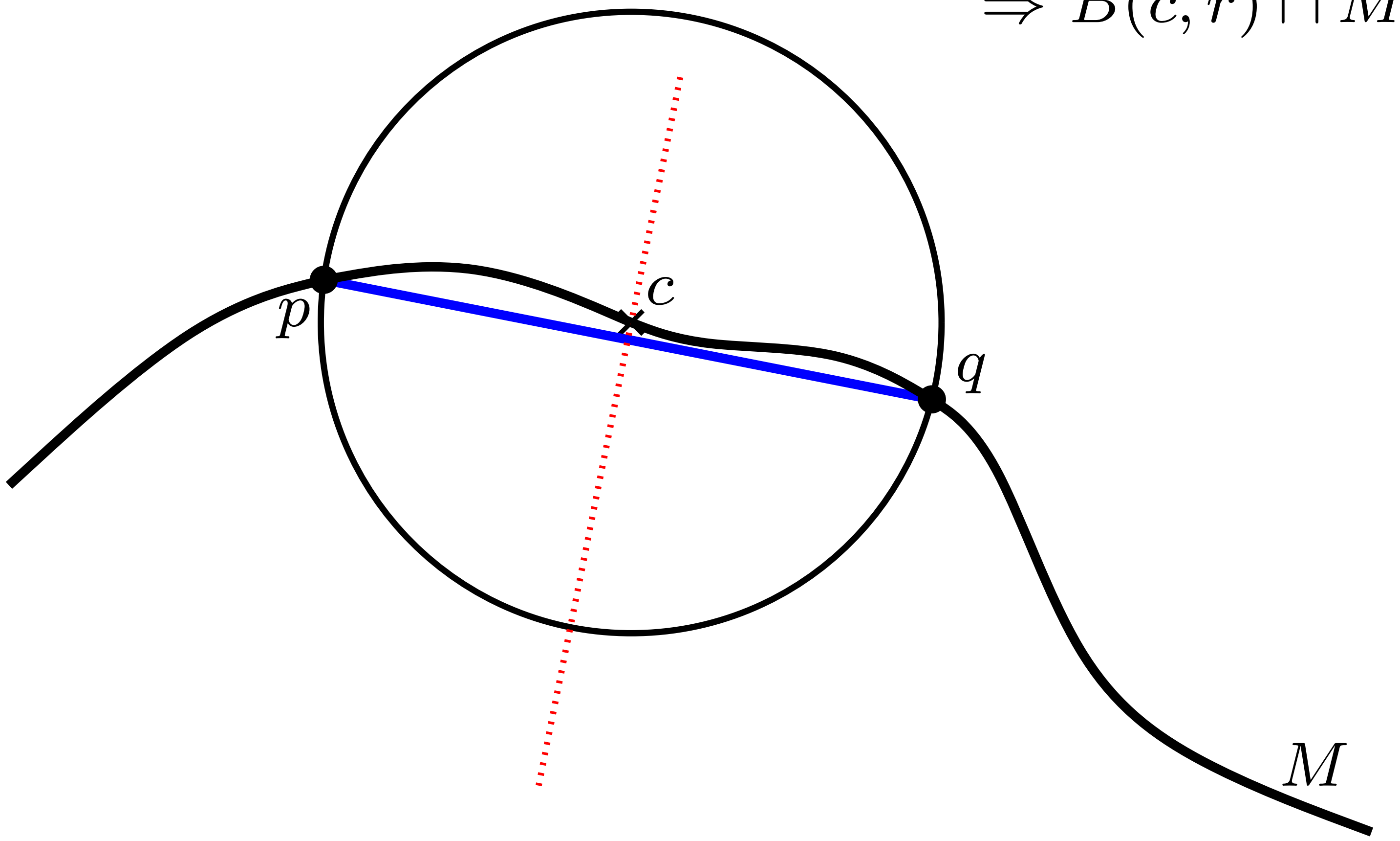
Let $c \in pq^* \cap M$.

$r = \|c - p\| = \|c - q\| = \mathsf{d}(c, P) \leq \varepsilon < \varrho_M \leq \mathrm{lfs}(c)$

$\Rightarrow B(c, r) \cap M$ is a topological arc



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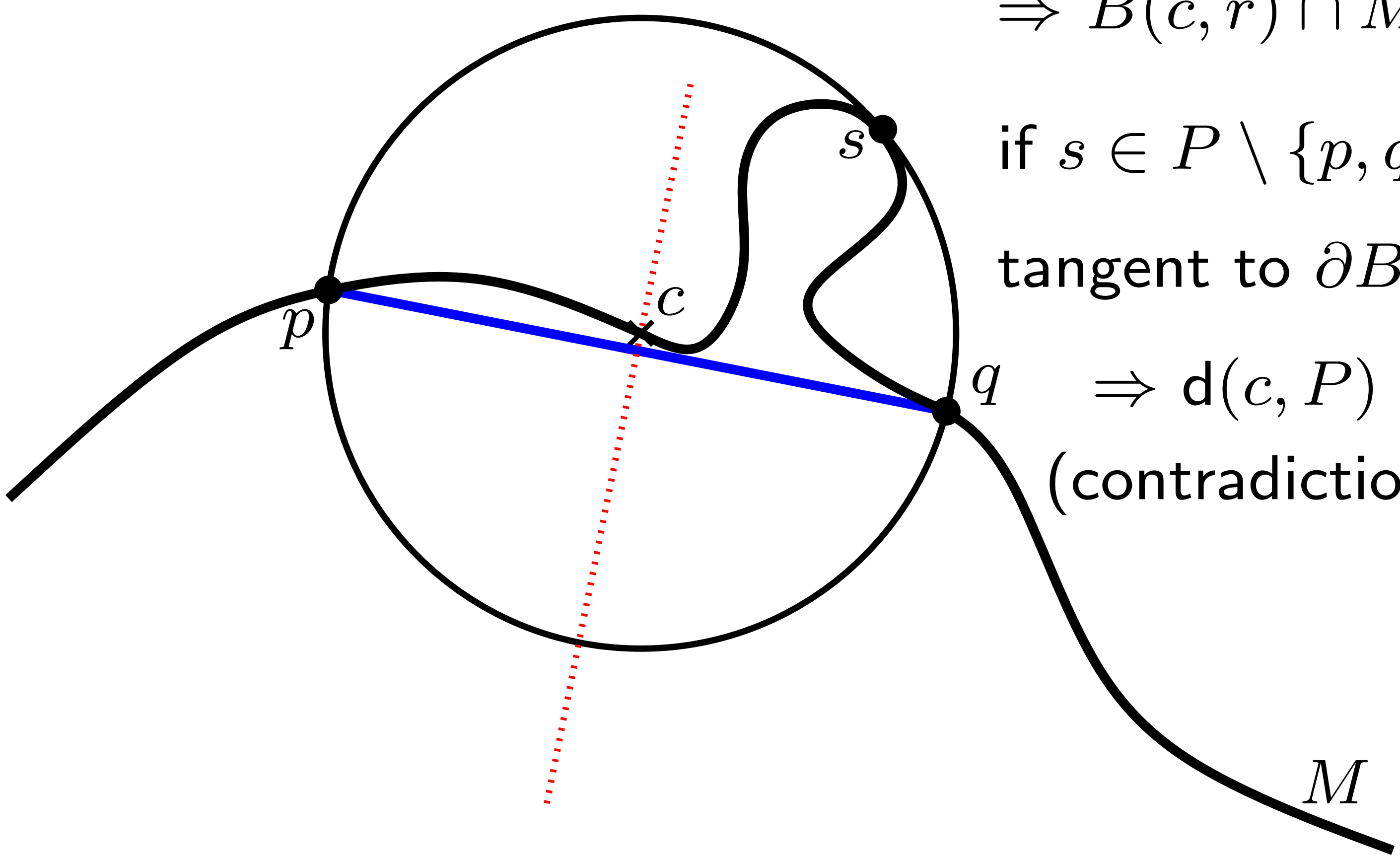


$\Rightarrow B(c, r) \cap M$ is a topological arc

if $s \in P \setminus \{p, q\}$ belongs to this arc, then the arc is tangent to $\partial B(c, r)$ in p, q or s (say s)

$\Rightarrow \mathsf{d}(c, P) = r = \|c - s\| \geq \mathrm{lfs}(s) > \varepsilon$.

(contradiction with the hypothesis of the theorem)

Approximation via Restricted Delaunay

Proof for curves:

show that every edge of $\mathcal{D}^M(P)$ connects consecutive points of P along M, and vice-versa

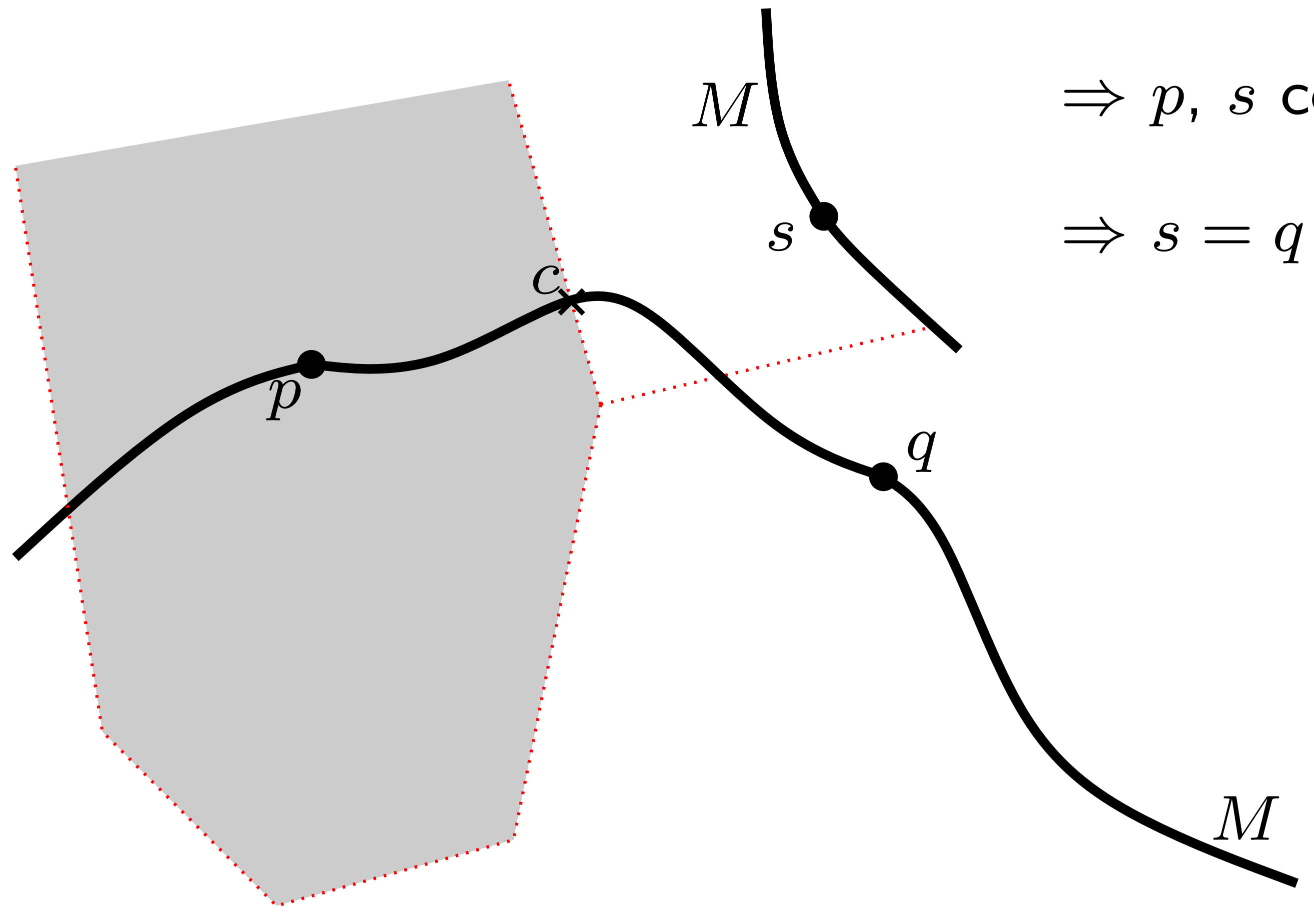
Let $c \in \mathrm{arc}_M(pq) \cap \partial p^*$. $\qquad c \in ps^*$ for some $s \in P \setminus \{p\}$

$\Rightarrow ps \in \mathcal{D}^M(P)$

$\Rightarrow p$, s consecutive along M, with $c \in \mathrm{arc}_M(ps)$

(by previous part of the proof)

$\Rightarrow s = q$

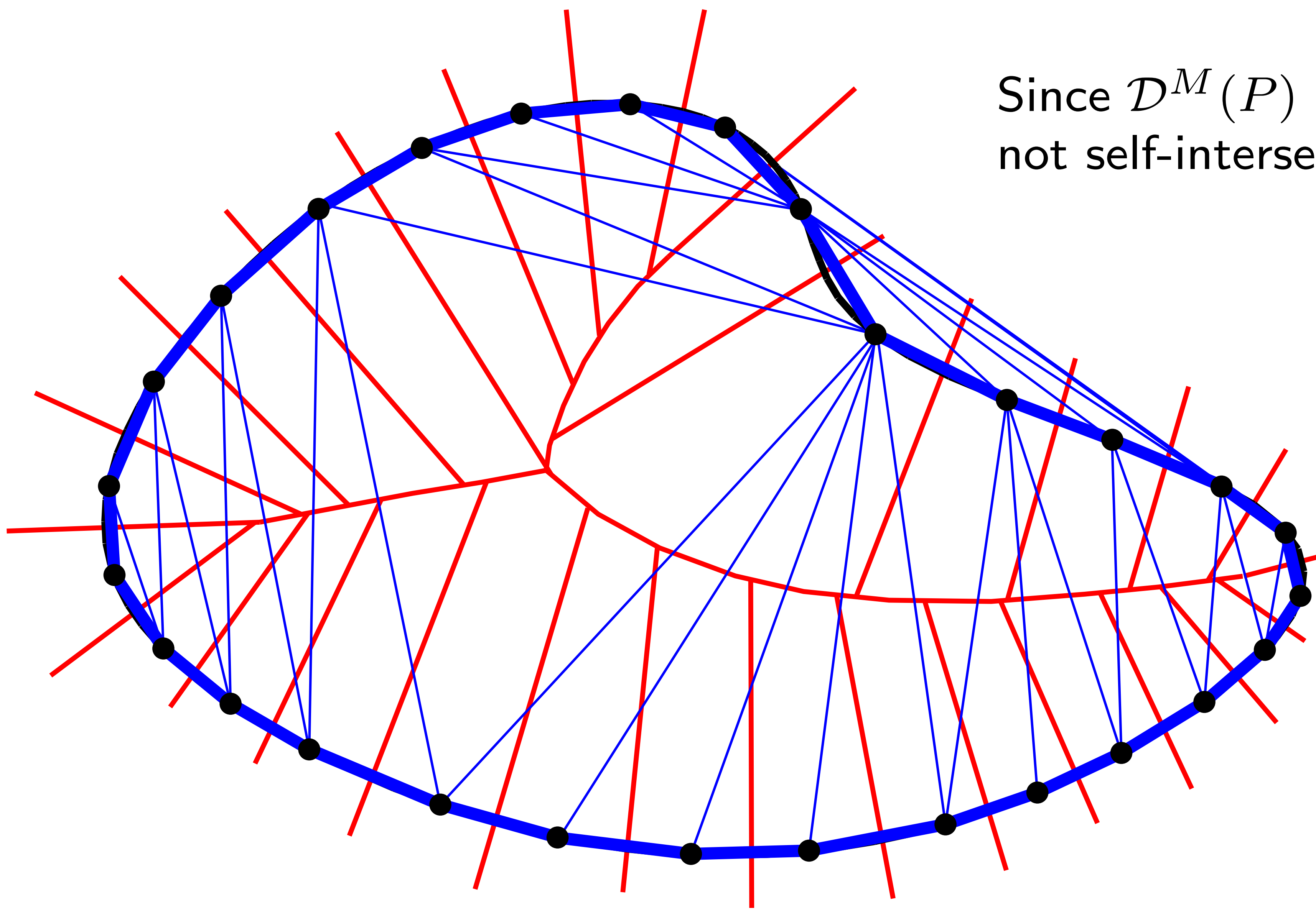


Approximation via Restricted Delaunay

Proof for curves:

show that every edge of $\mathcal{D}^M(P)$ connects consecutive points of P along M, and vice-versa

$\Rightarrow \mathcal{D}^M(P)$ is homeomorphic to M between each pair of consecutive points of P

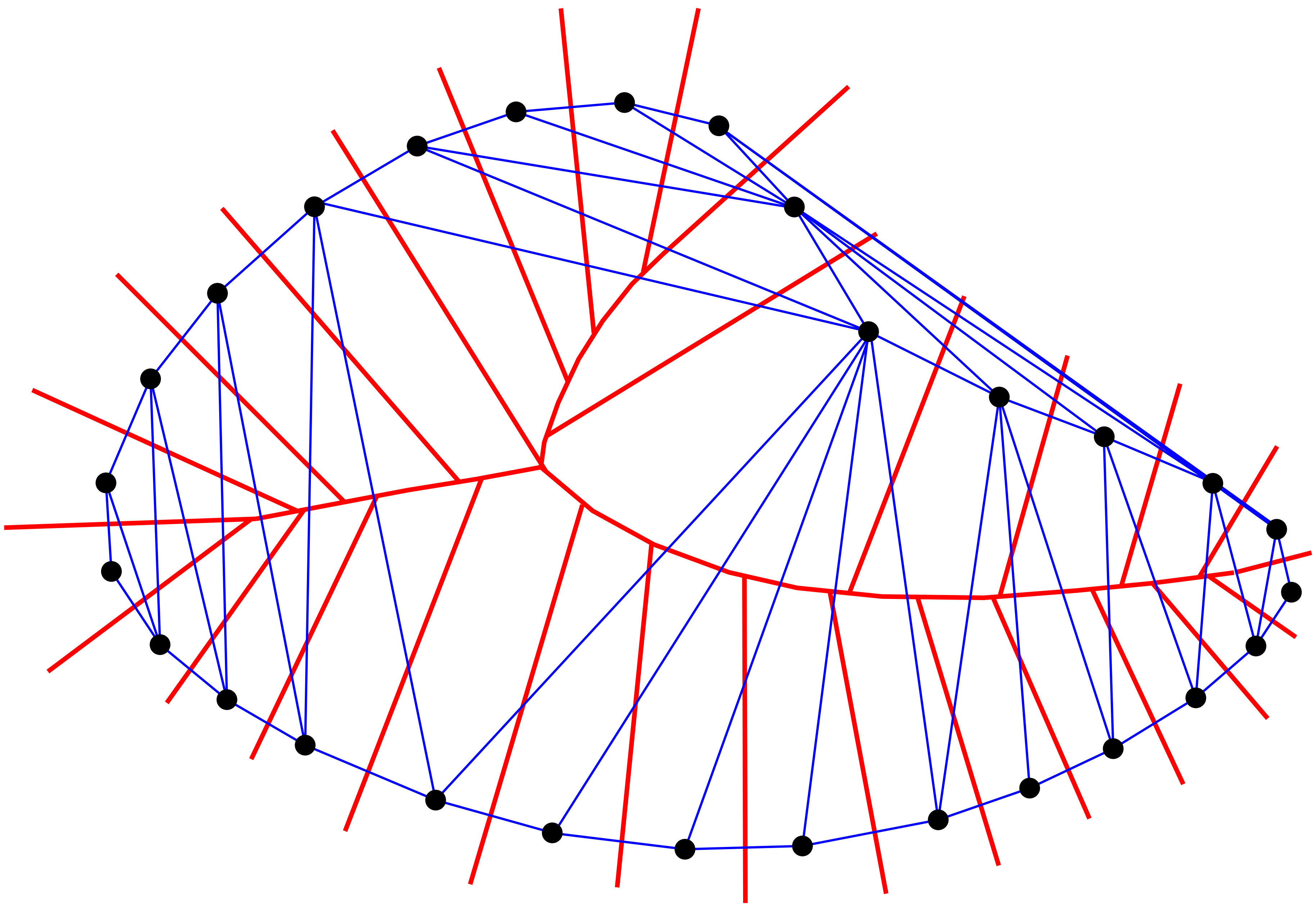


Since $\mathcal{D}^M(P)$ is embedded in $\mathcal{D}(P)$, it does not self-intersect $\Rightarrow$ global homeomorphism

Computing the Restricted Delaunay

Q How to compute $\mathcal{D}^M(P)$ when M is unknown?

$\rightarrow$ a whole family of algorithms use various Delaunay extraction criteria:

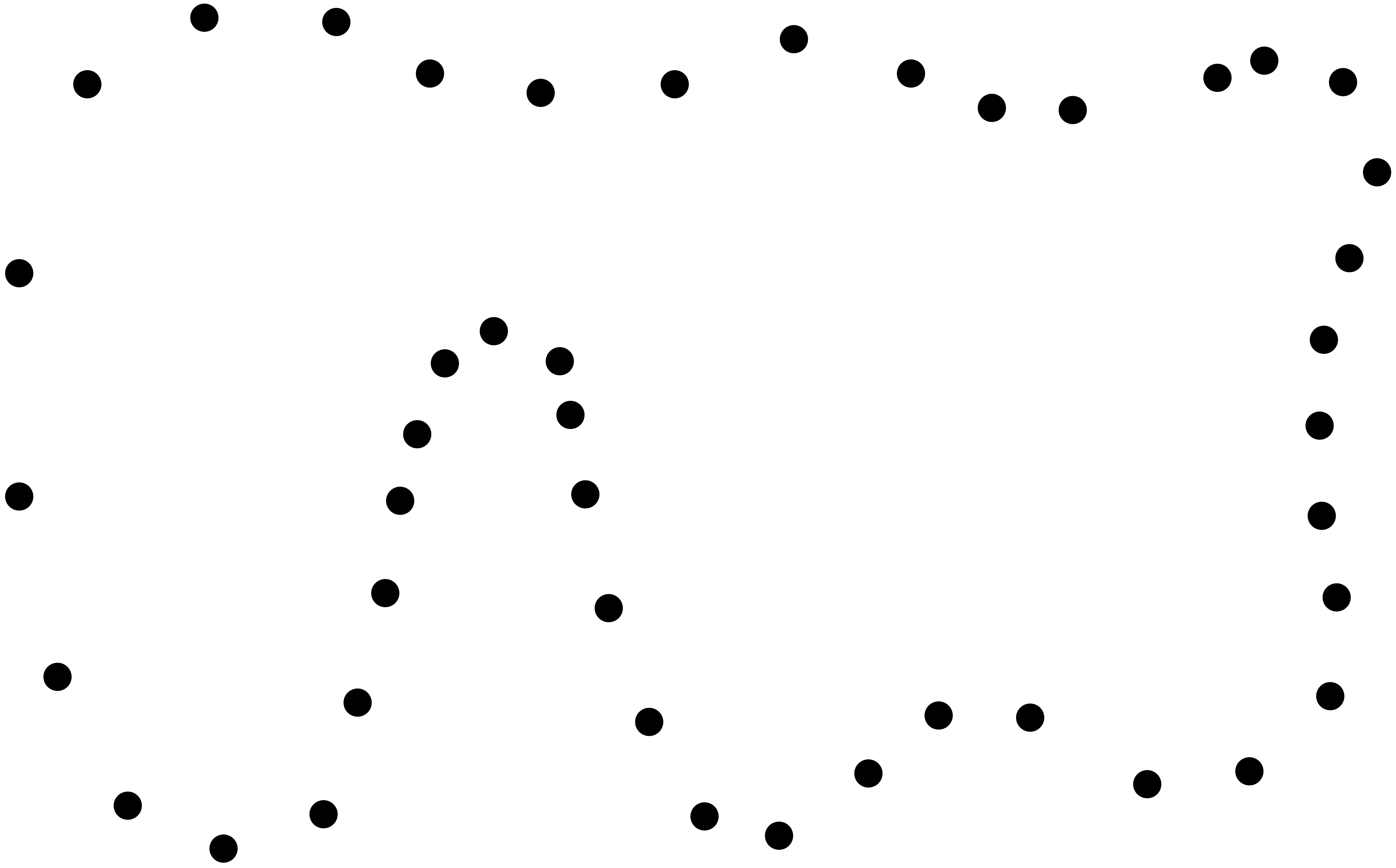


- crust
- power crust
- cocone
- tight cocone
- $\cdots$

Crust Algorithm

Crust algorithm

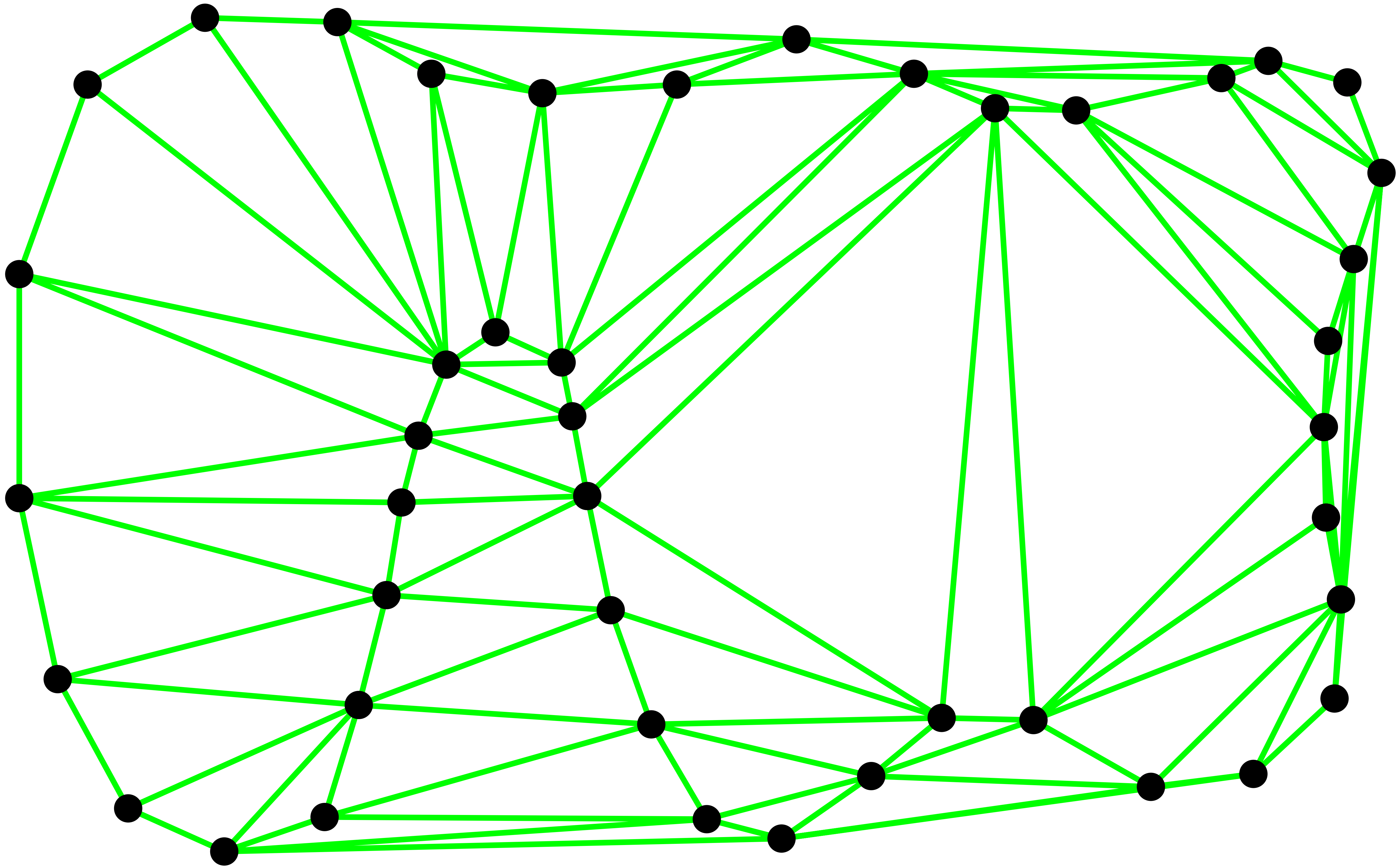
[Amenta et al. 1997-98]



Crust algorithm

[Amenta et al. 1997-98]

1. Compute Delaunay triangulation of P



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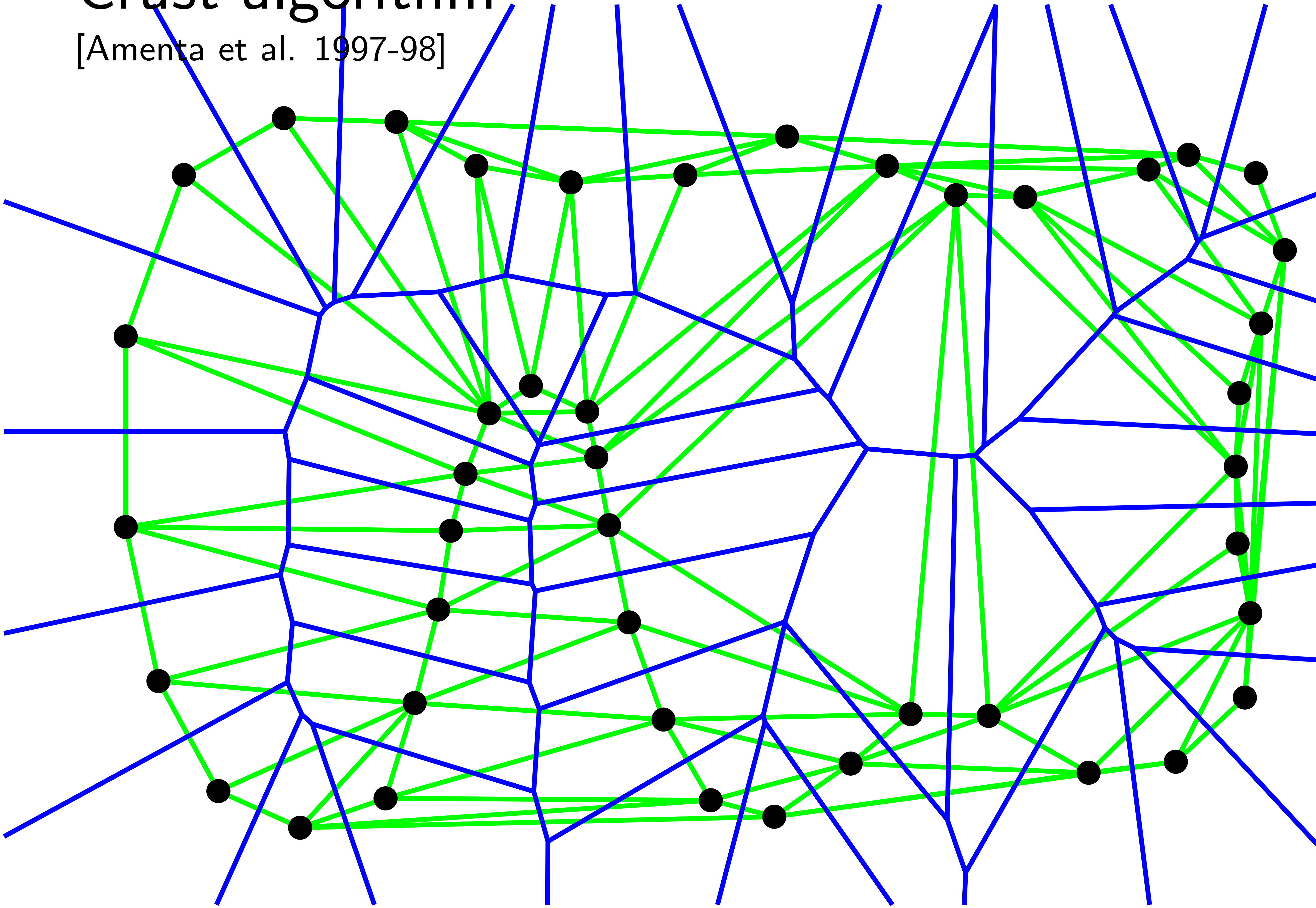
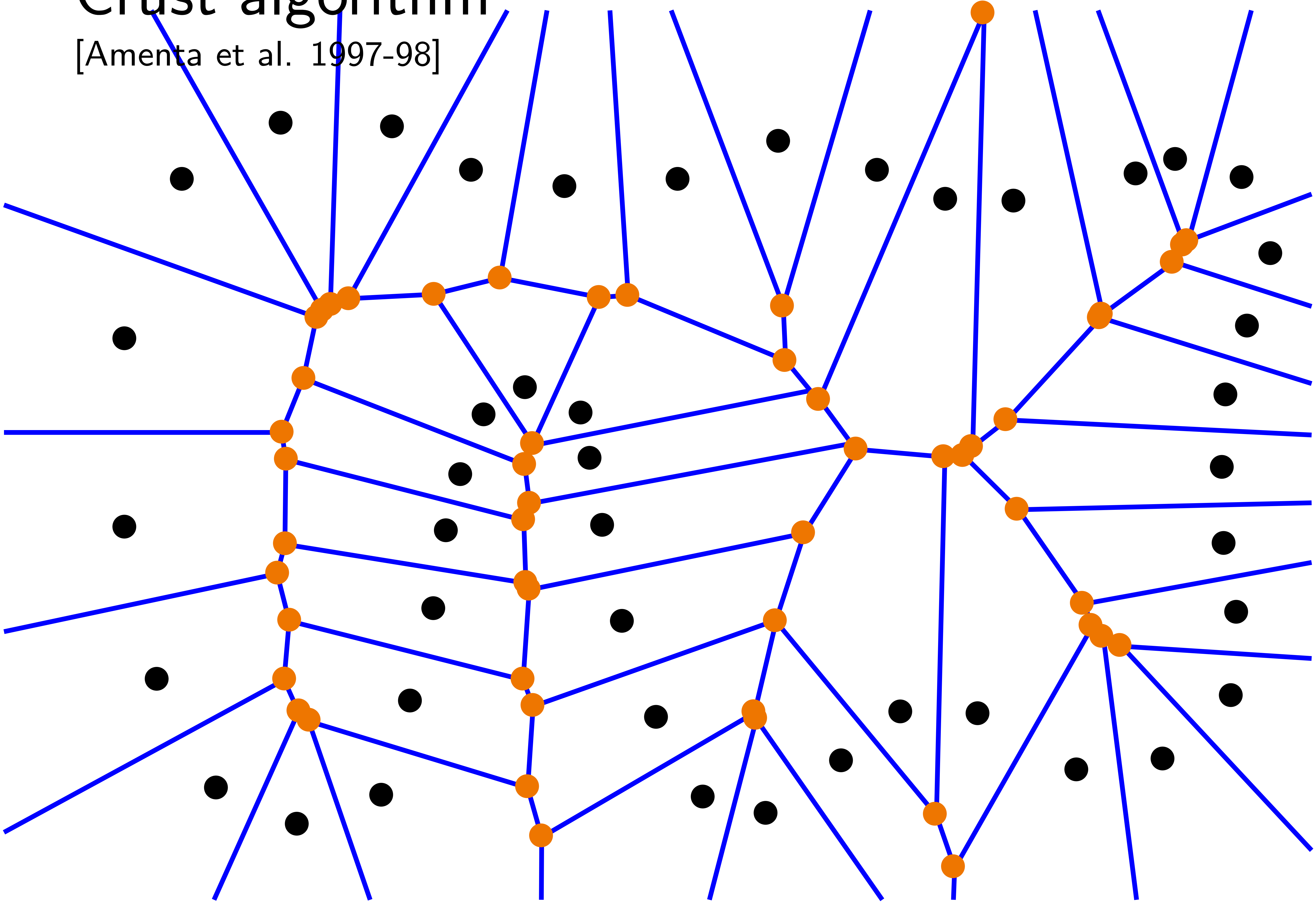


Crust algorithm

[Amenta et al. 1997-98]

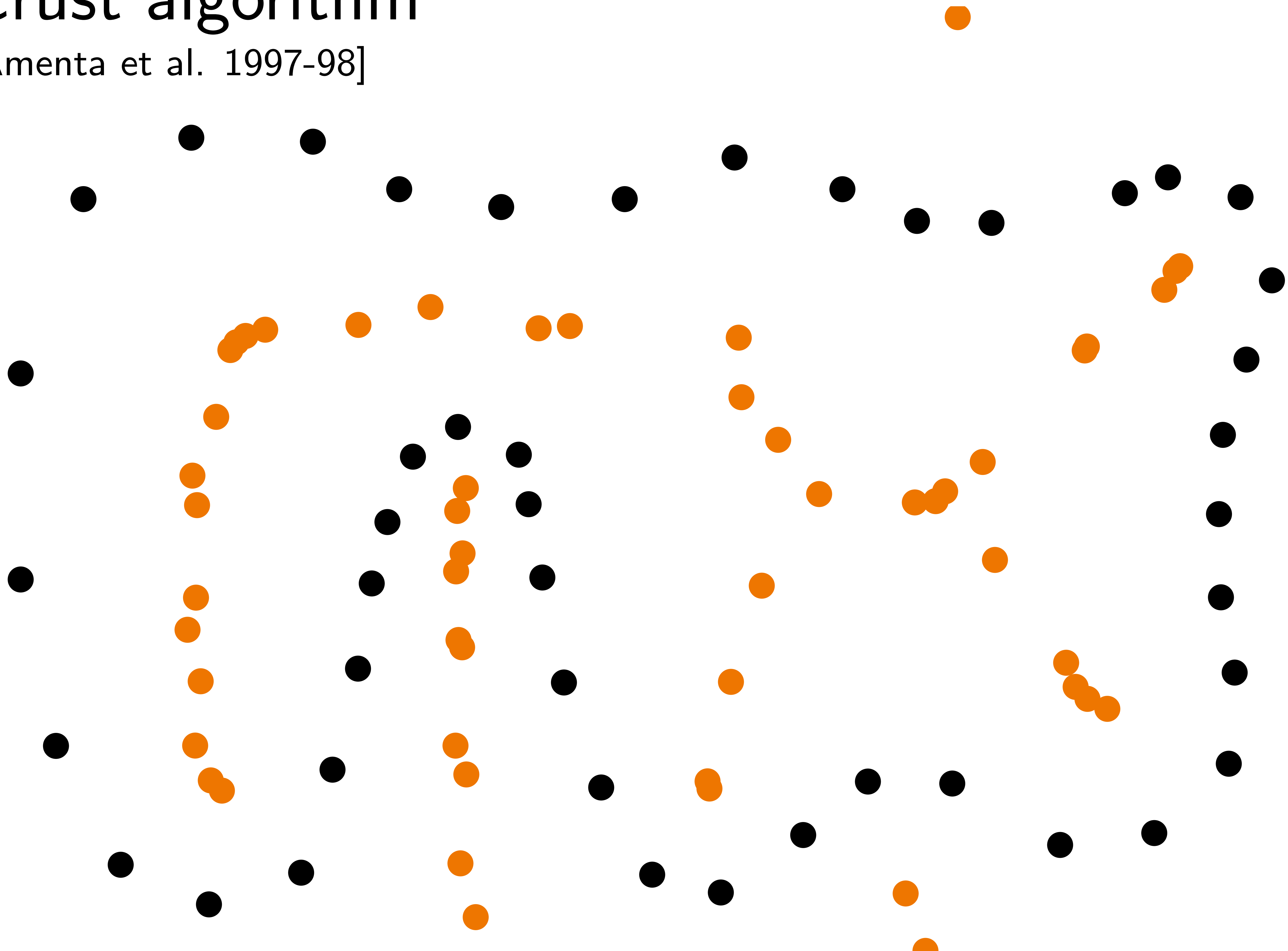
2. Compute *poles* (furthest Voronoi vertices)



Crust algorithm

[Amenta et al. 1997-98]

3. Add poles to the set of vertices



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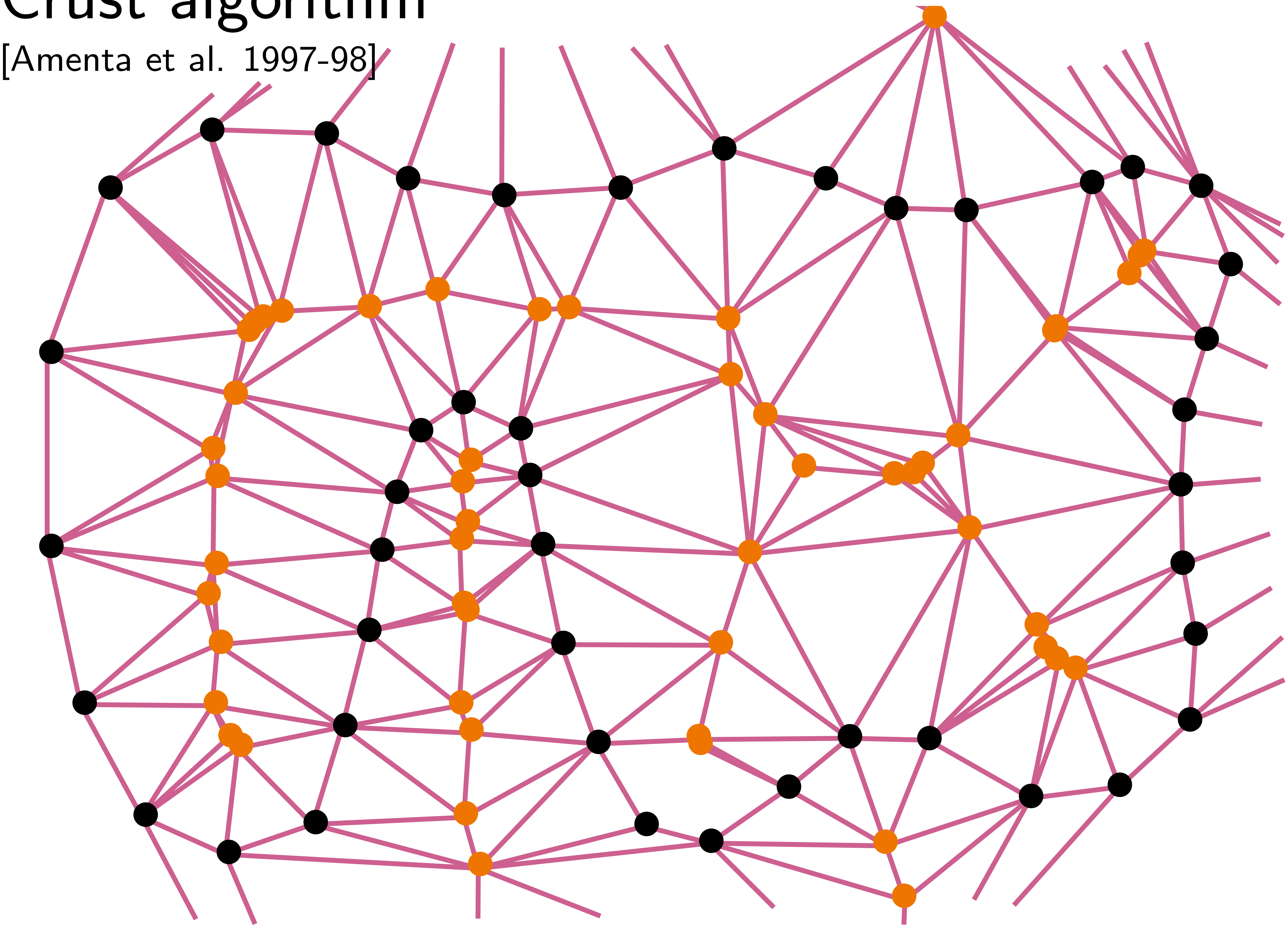
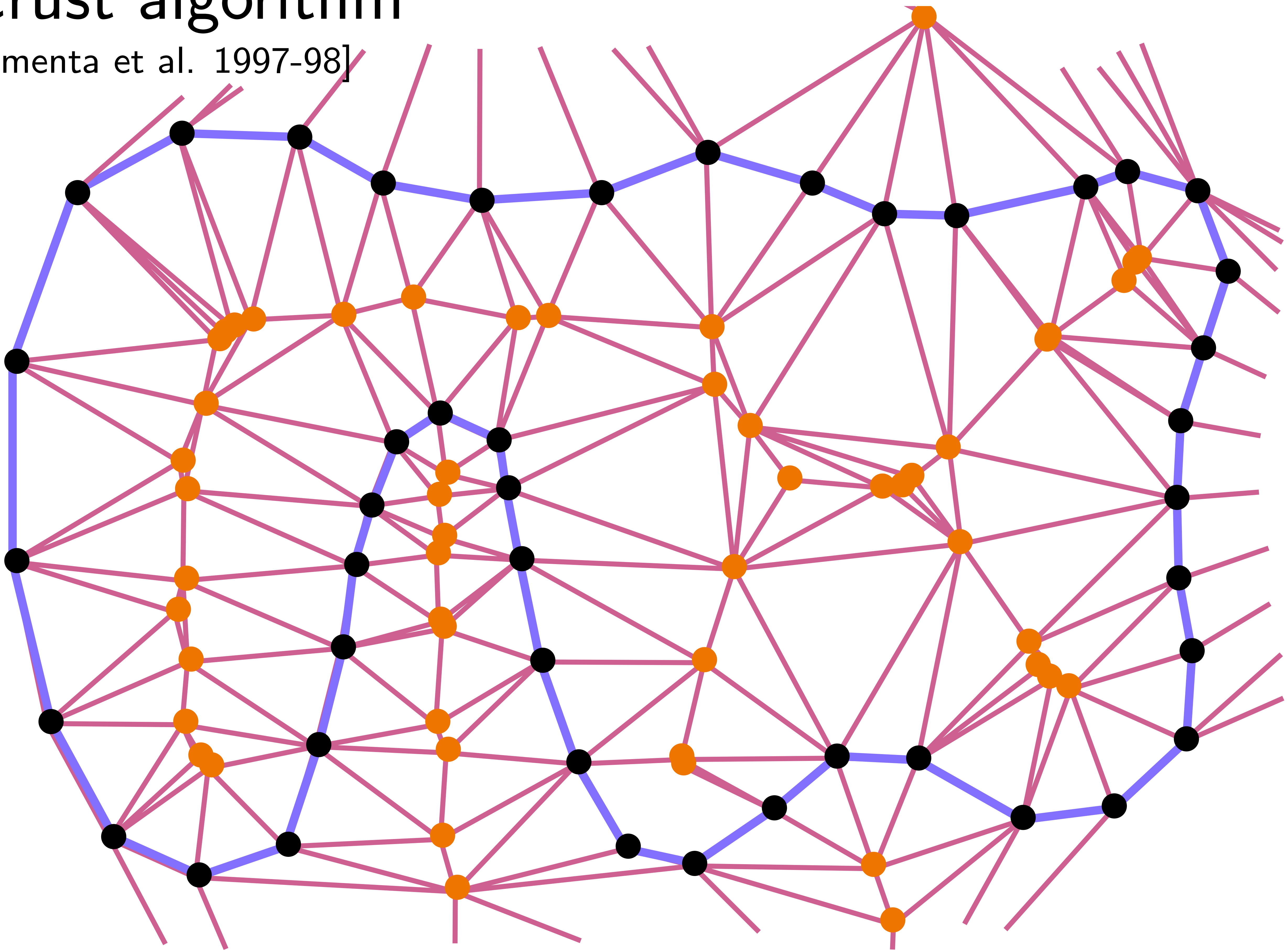


Crust algorithm

[Amenta et al. 1997-98]

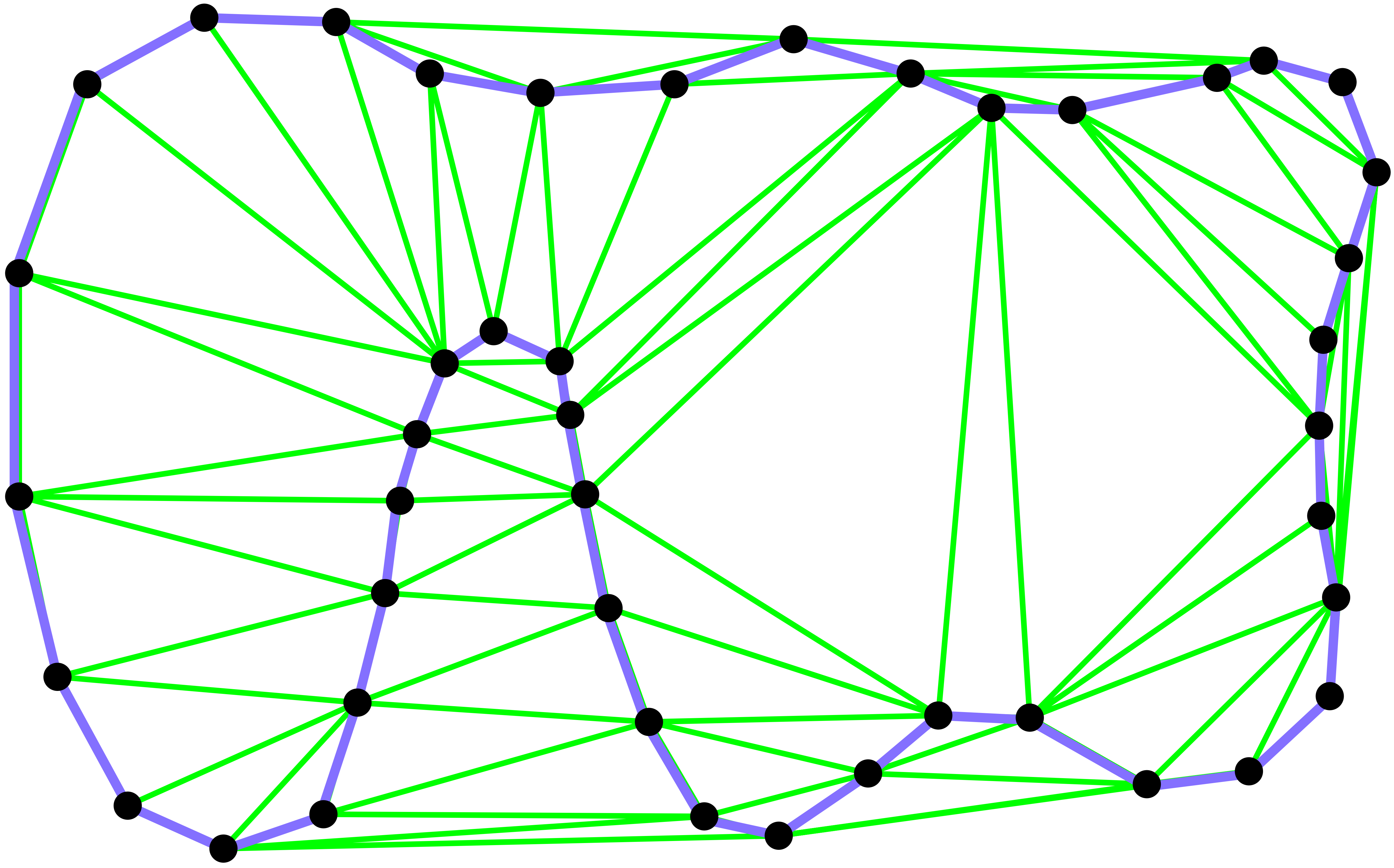
4. Keep Delaunay simplices whose vertices are in P



Crust algorithm

in 2-d, crust $= \mathcal{D}^M(P) \approx M$

[Amenta et al. 1997-98]



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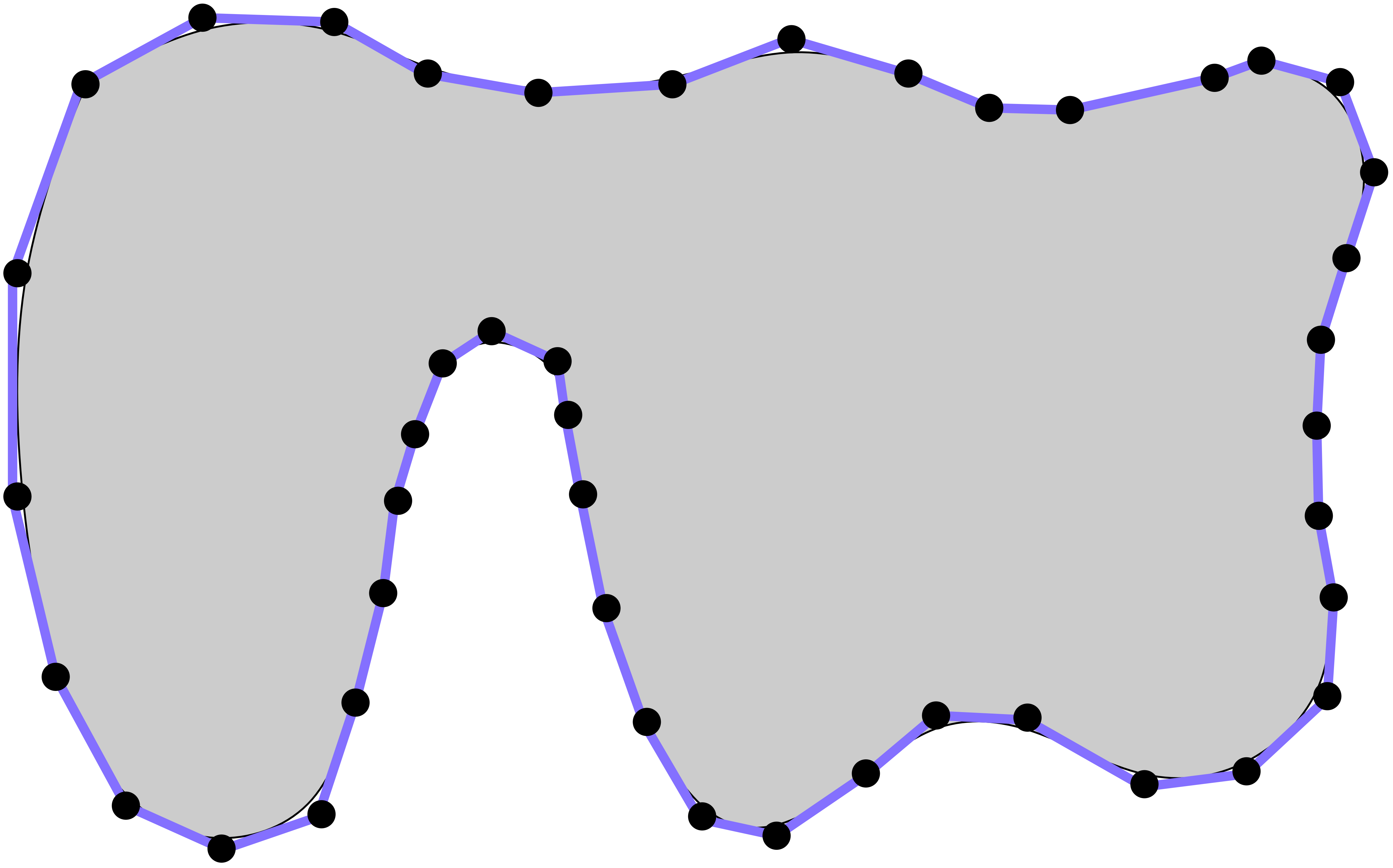

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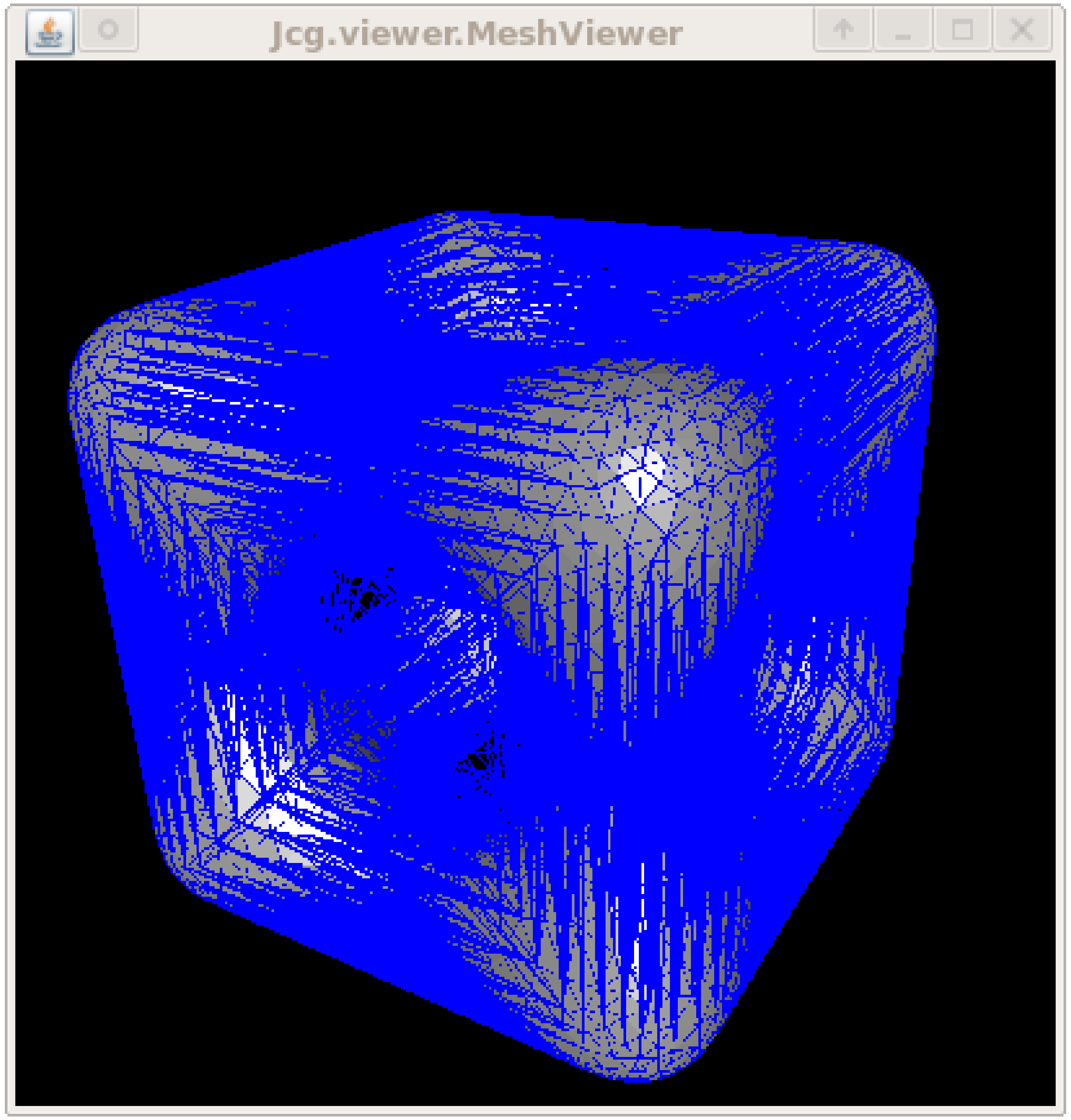
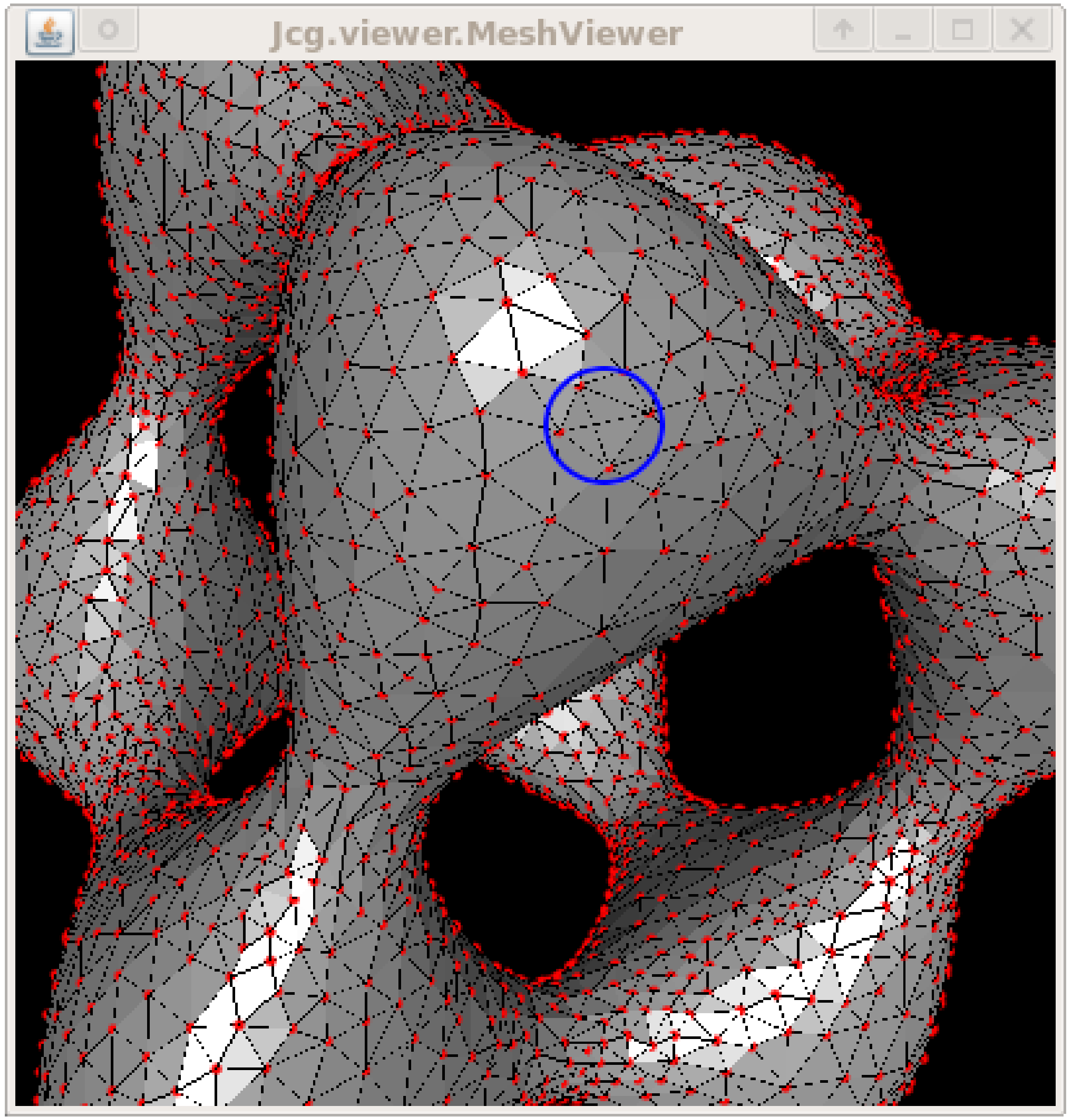
in 3-d, crust $\supseteq \mathcal{D}^M(P) \approx M$





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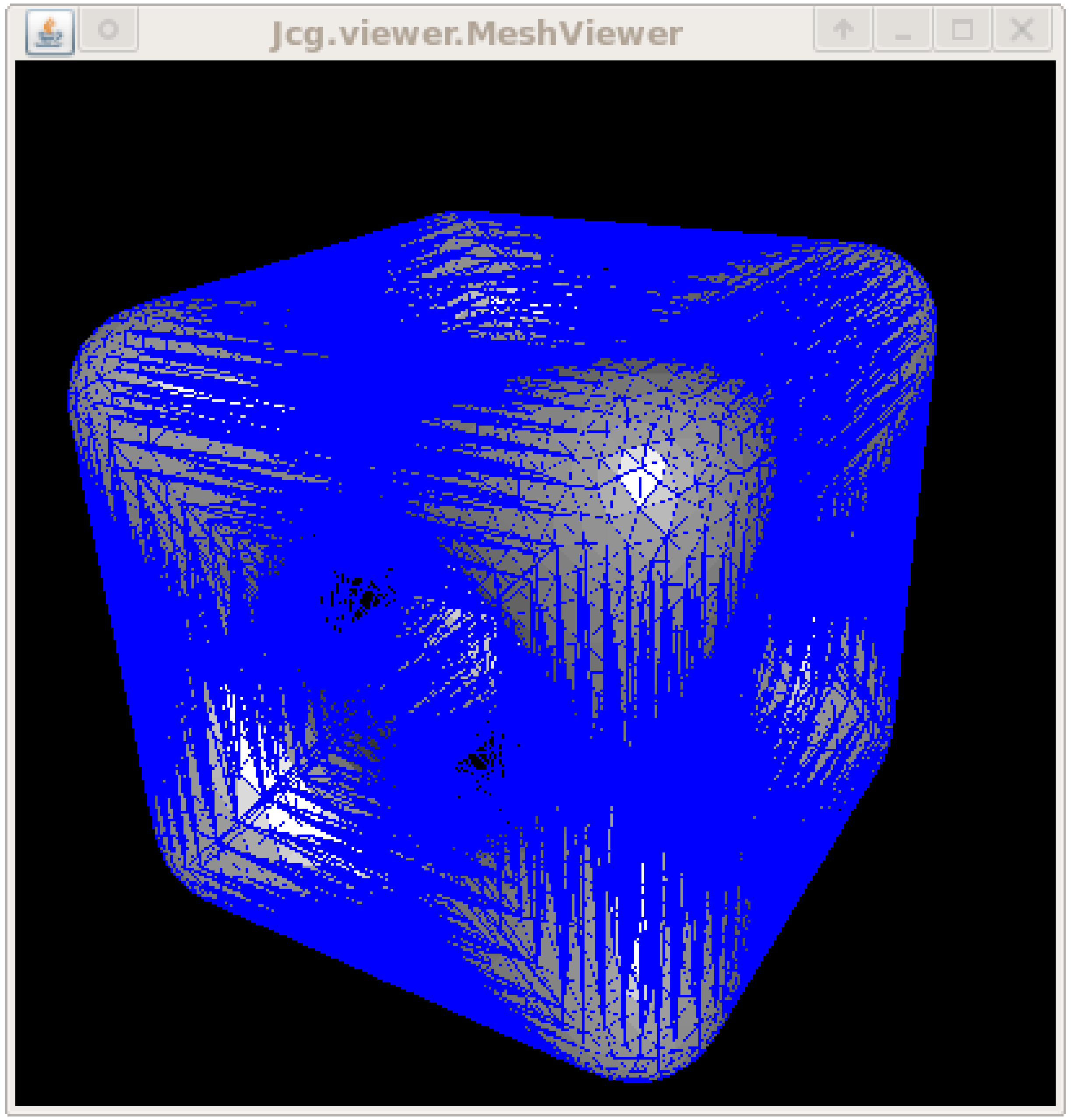
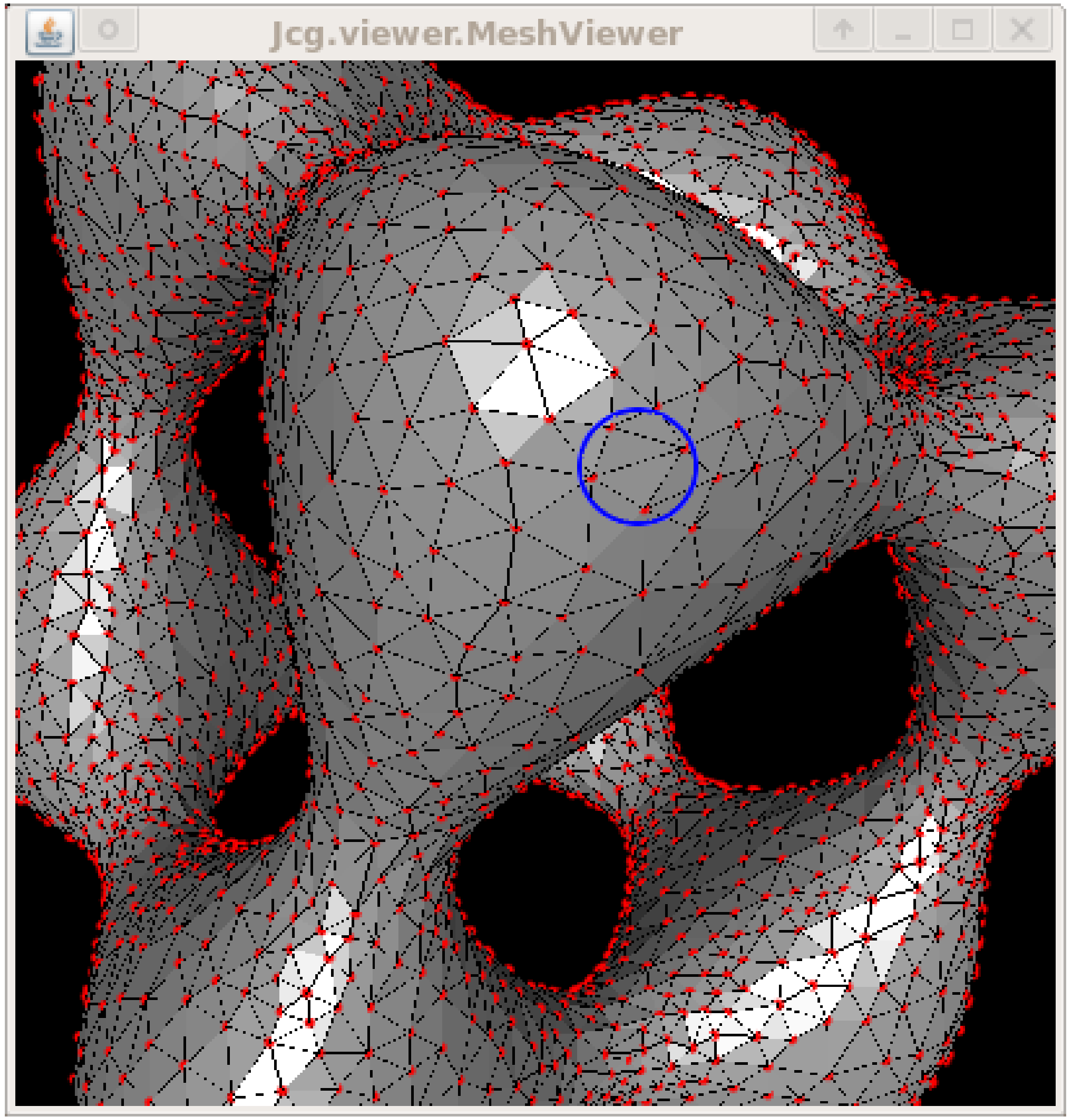
$\rightarrow$ manifold extraction step in post-processing

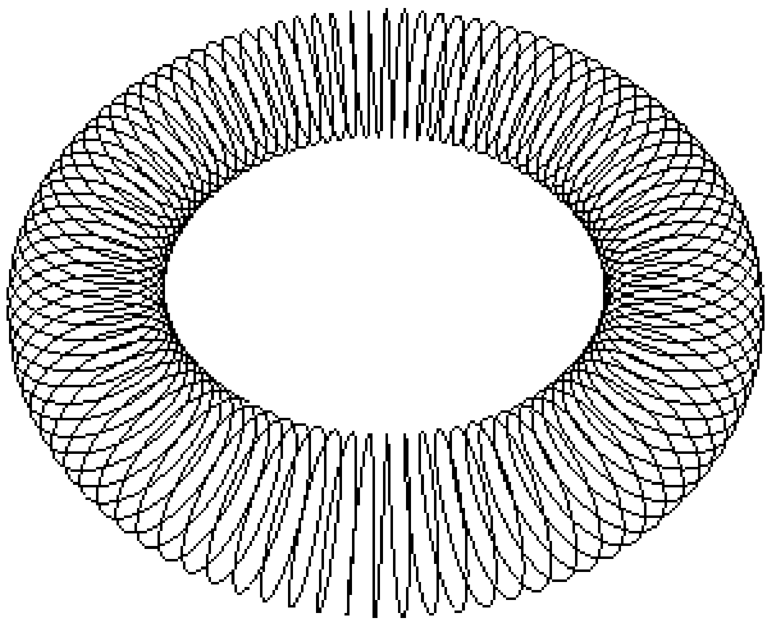


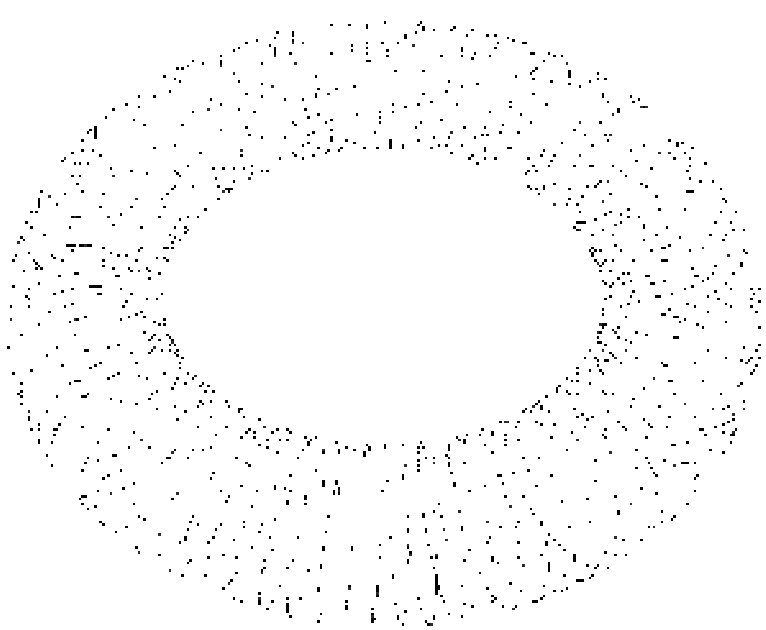


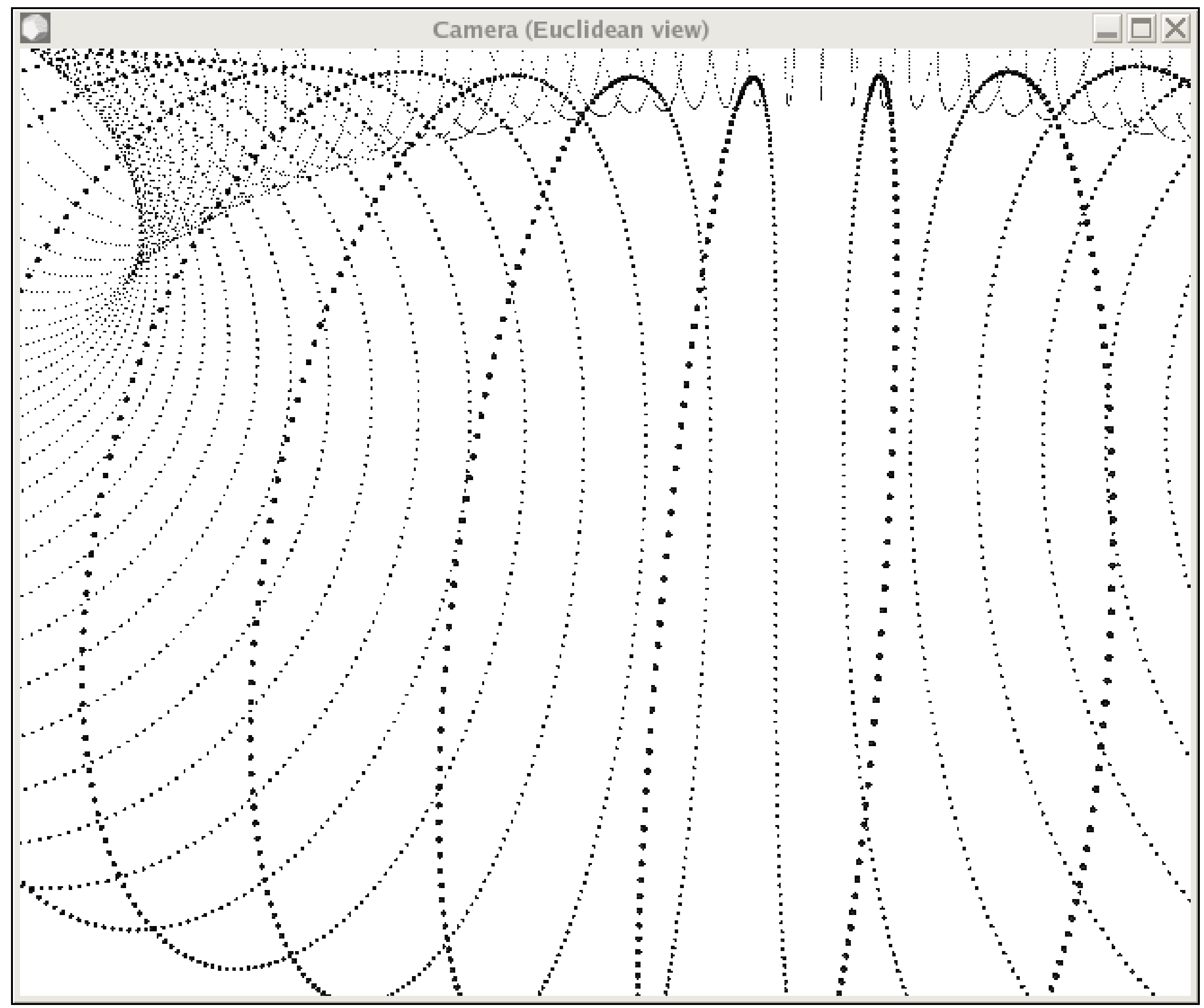
Witness Complex

Motivation: effect of scale / dimensionality

What is the reconstruction?





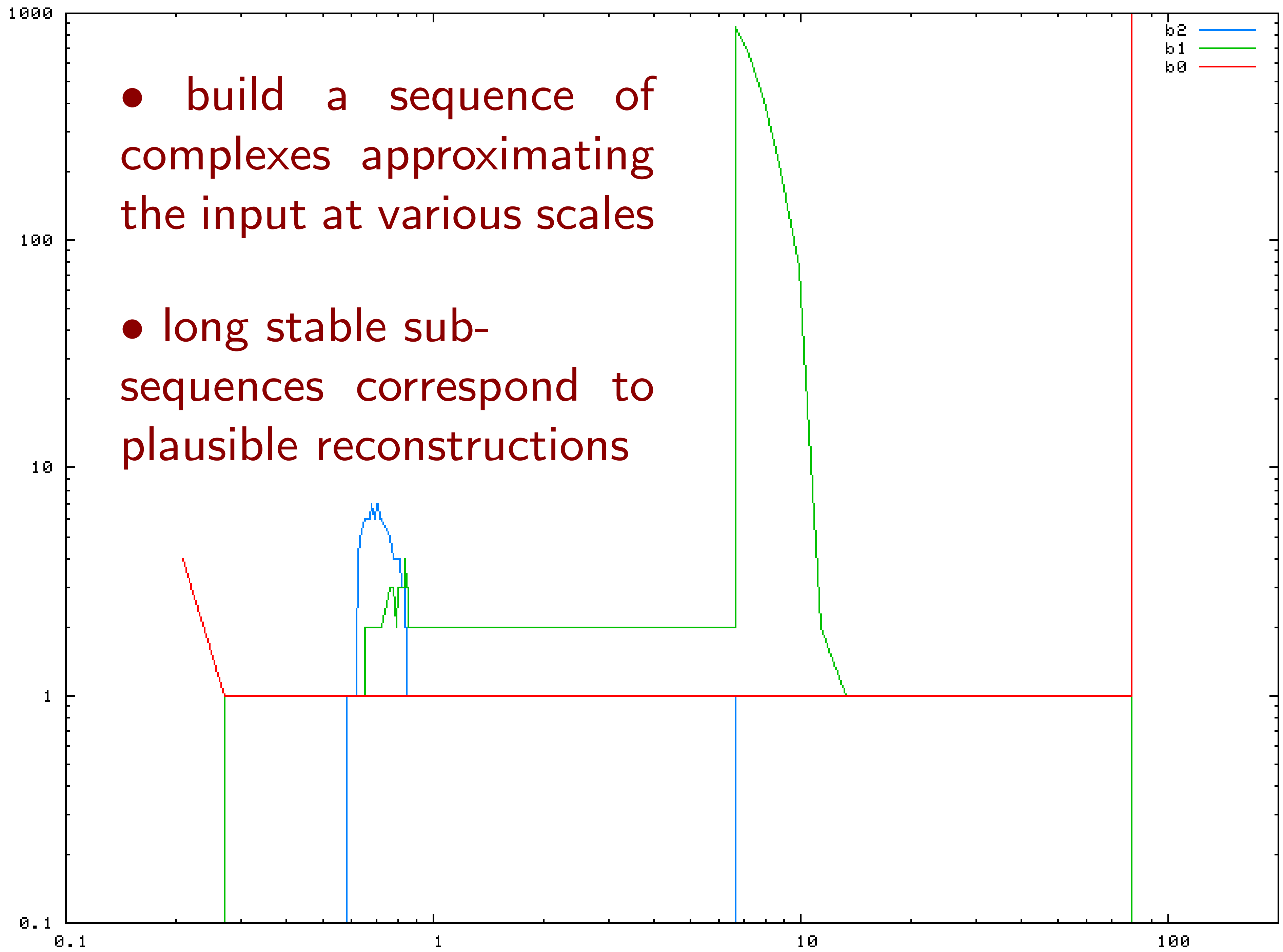


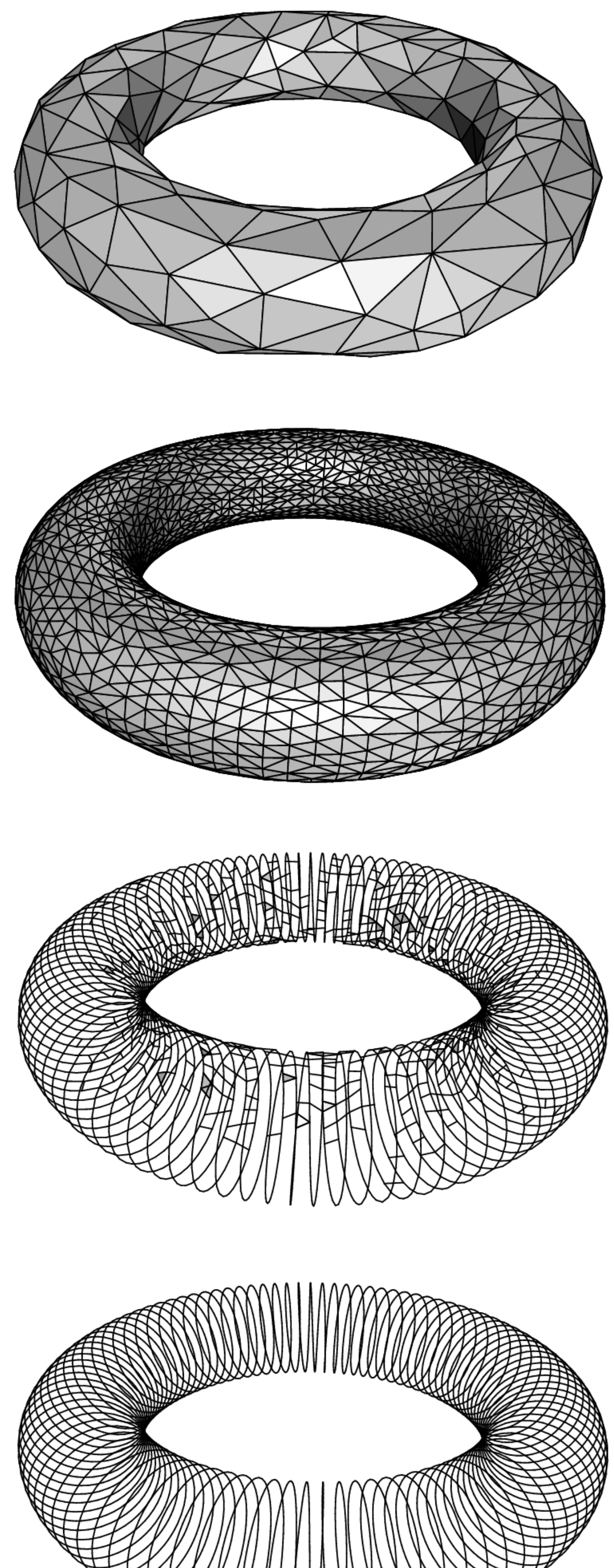
Multi-scale reconstruction

[Guibas, O. 07]

- build a sequence of complexes approximating the input at various scales

- long stable sub-sequences correspond to plausible reconstructions





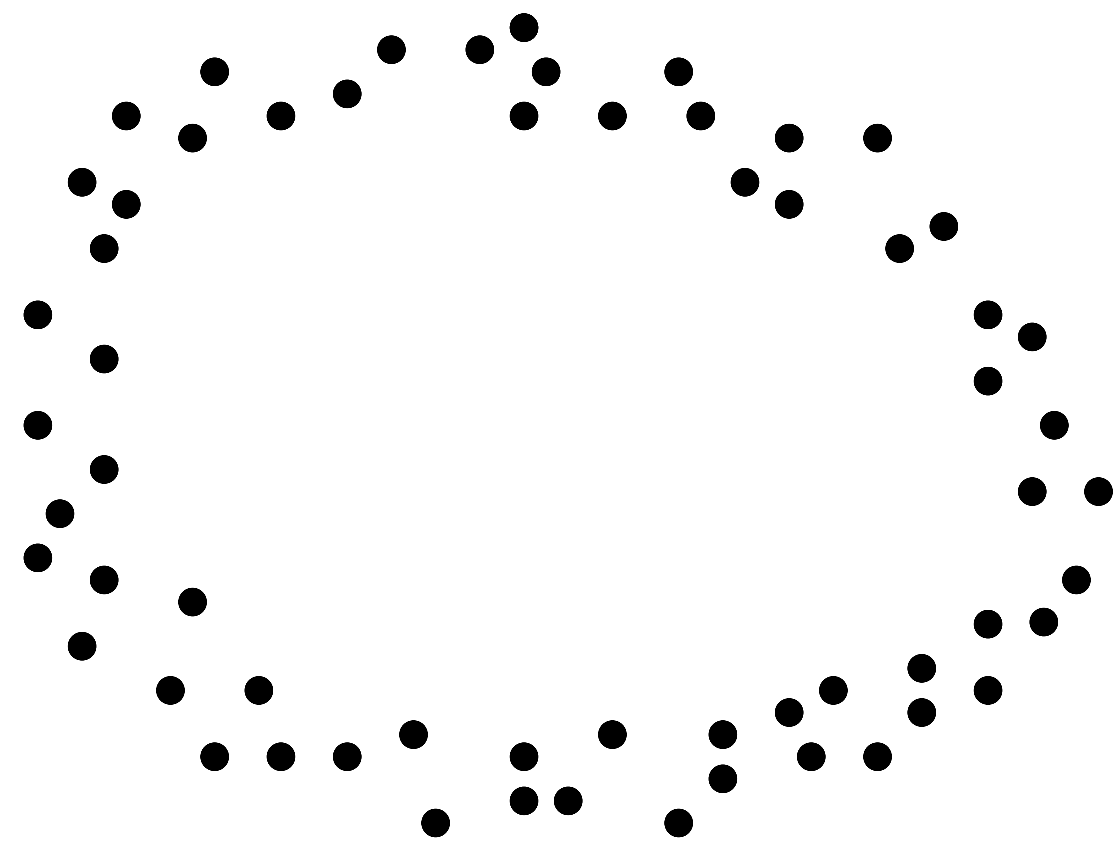
→ the witness complex enables the use of the Delaunay paradigm

Multi-scale reconstruction algorithm

[Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^n$

$\rightarrow$ resample W iteratively, and maintain a simplicial complex:



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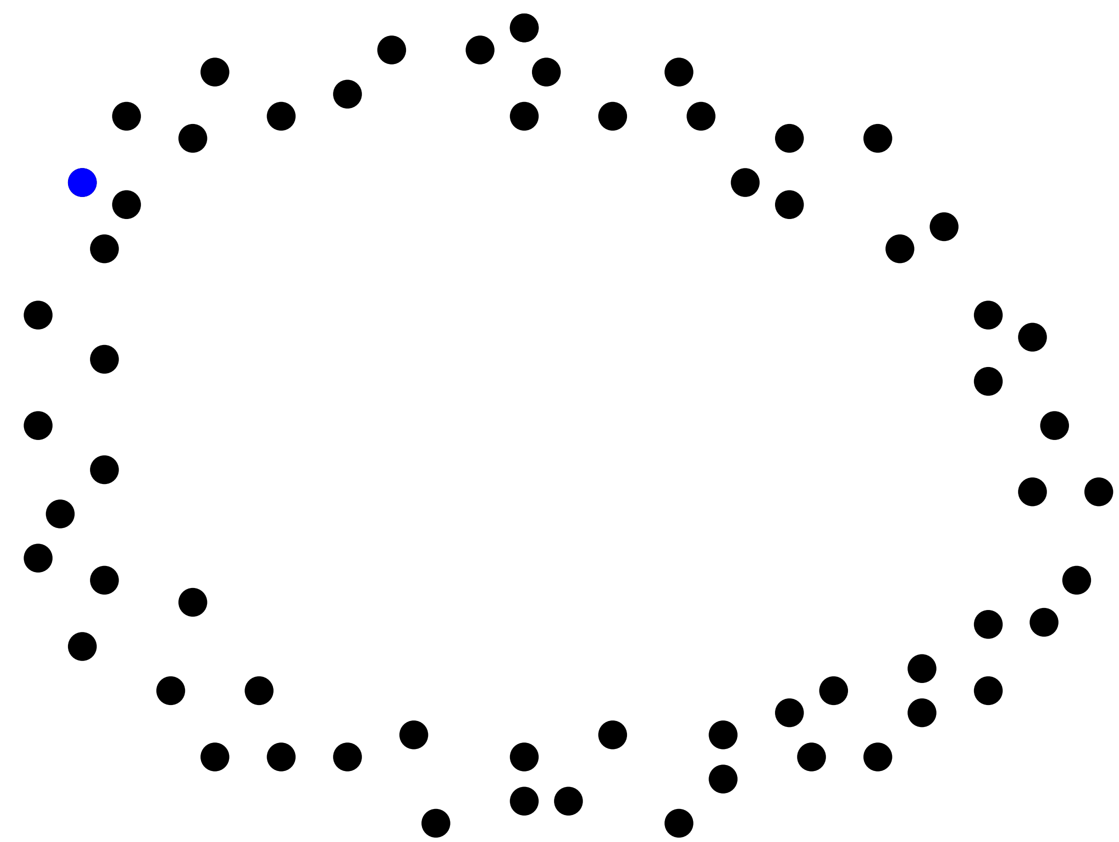

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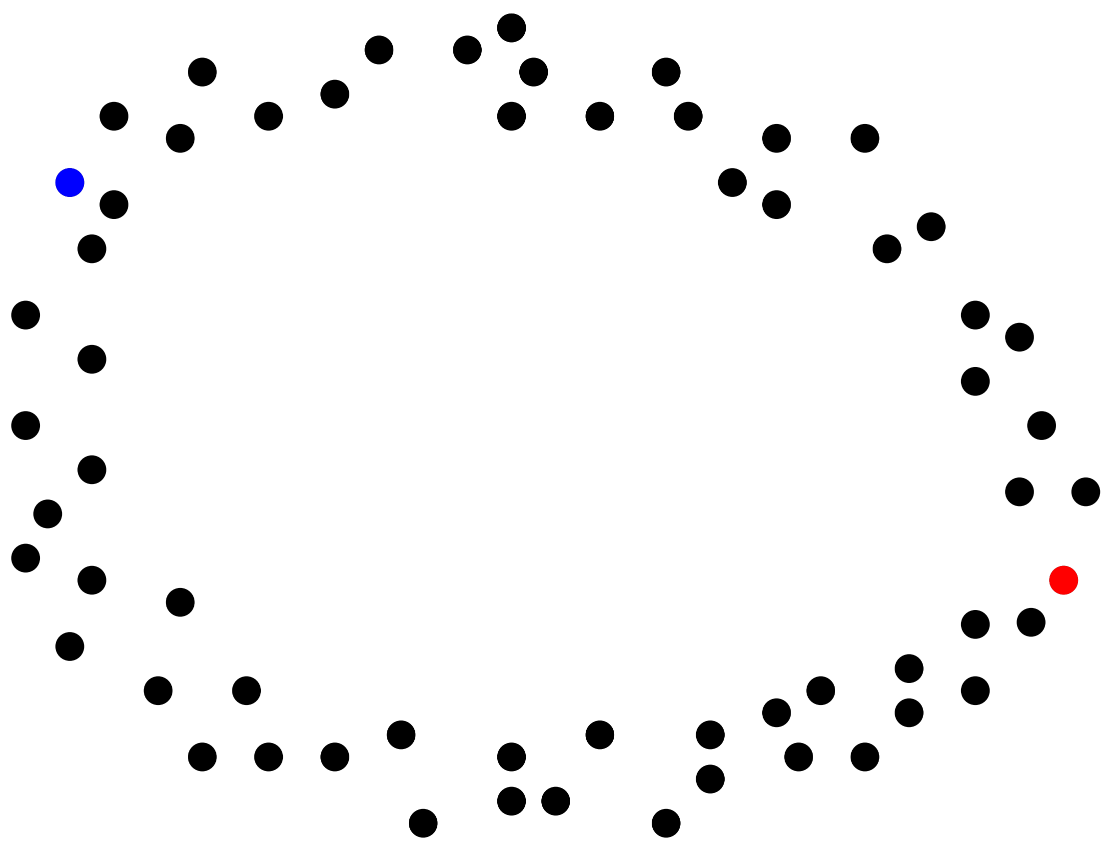

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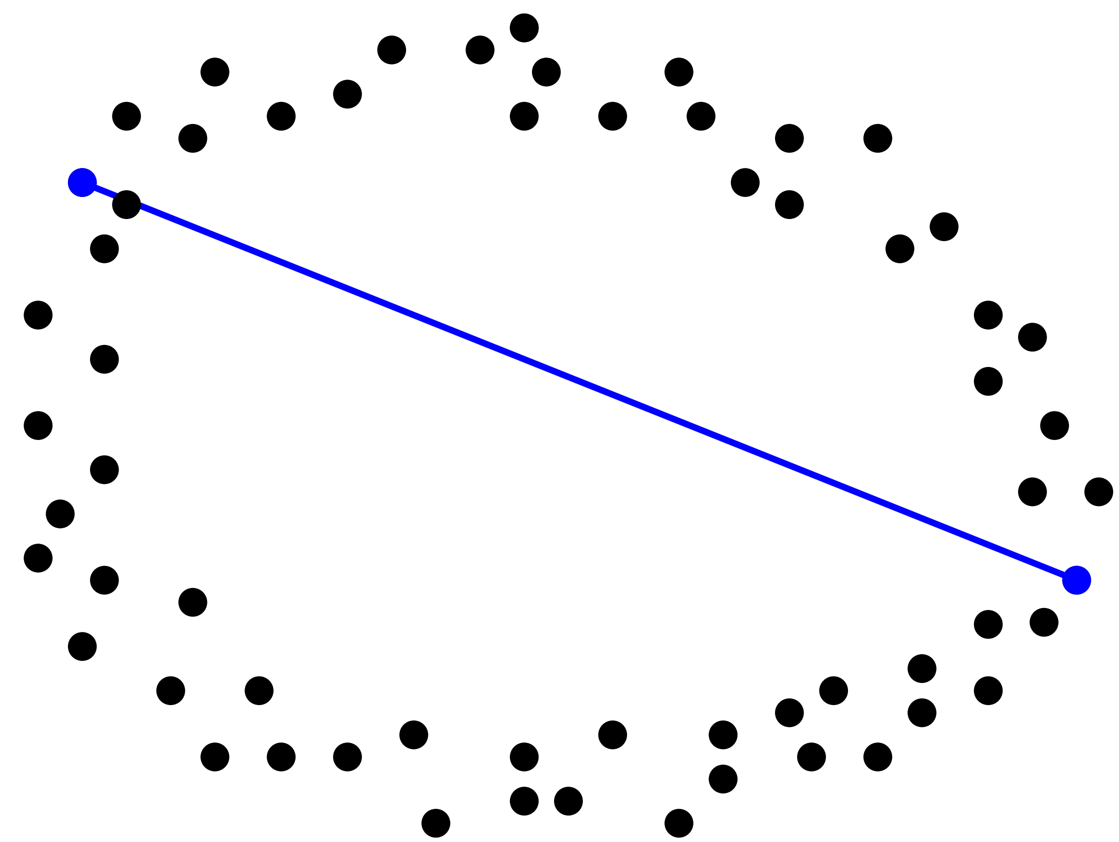
$L := L \cup \{q\}$;

update simplicial complex;

END_WHILE



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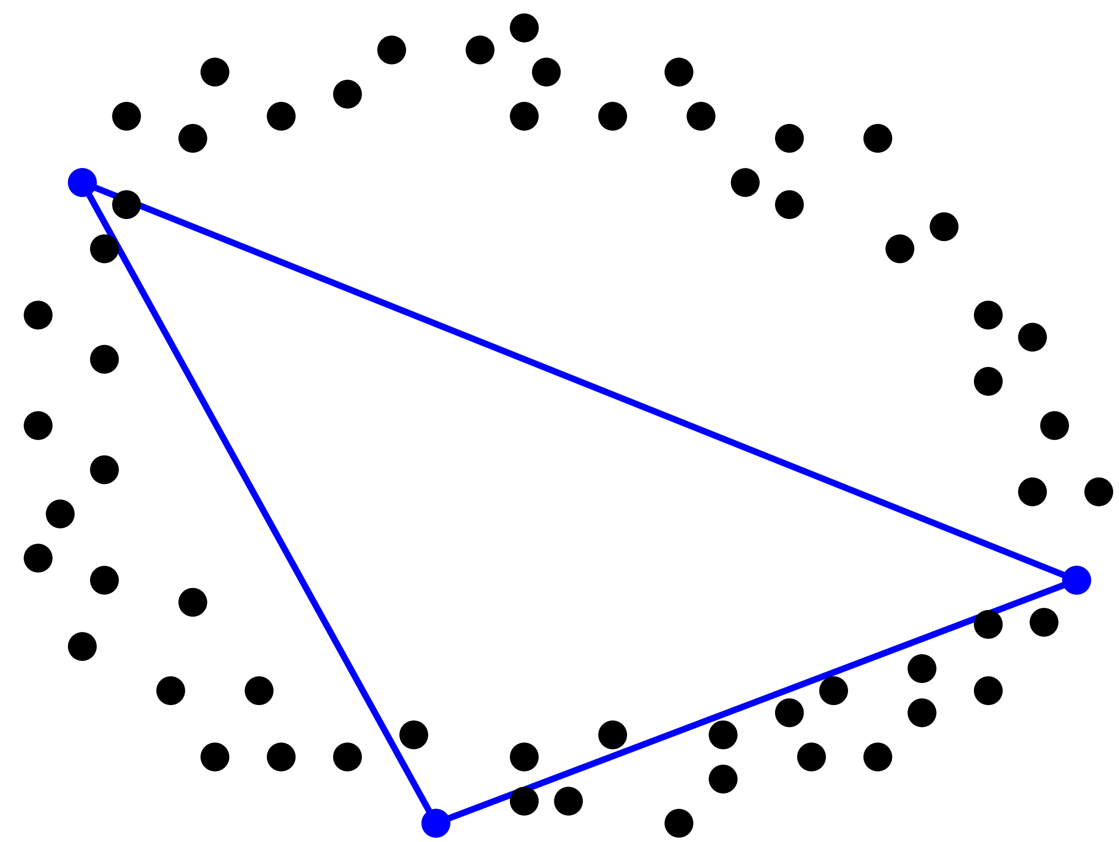

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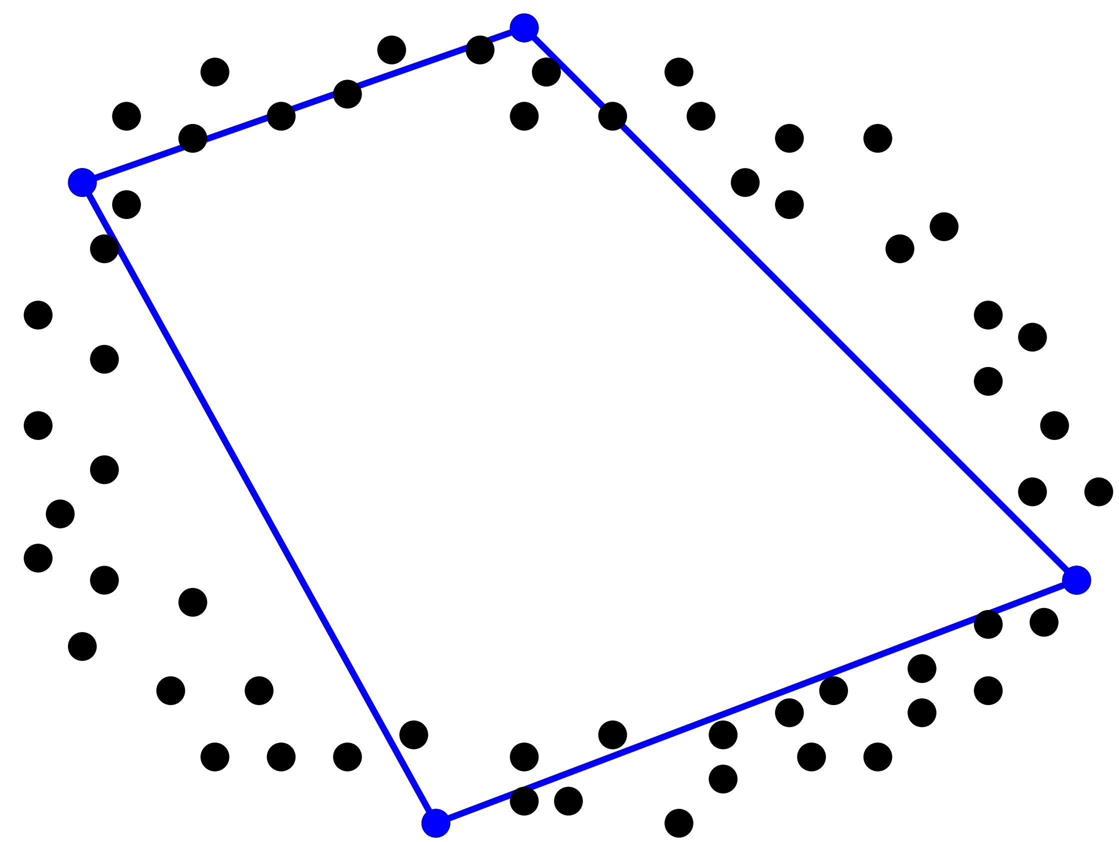

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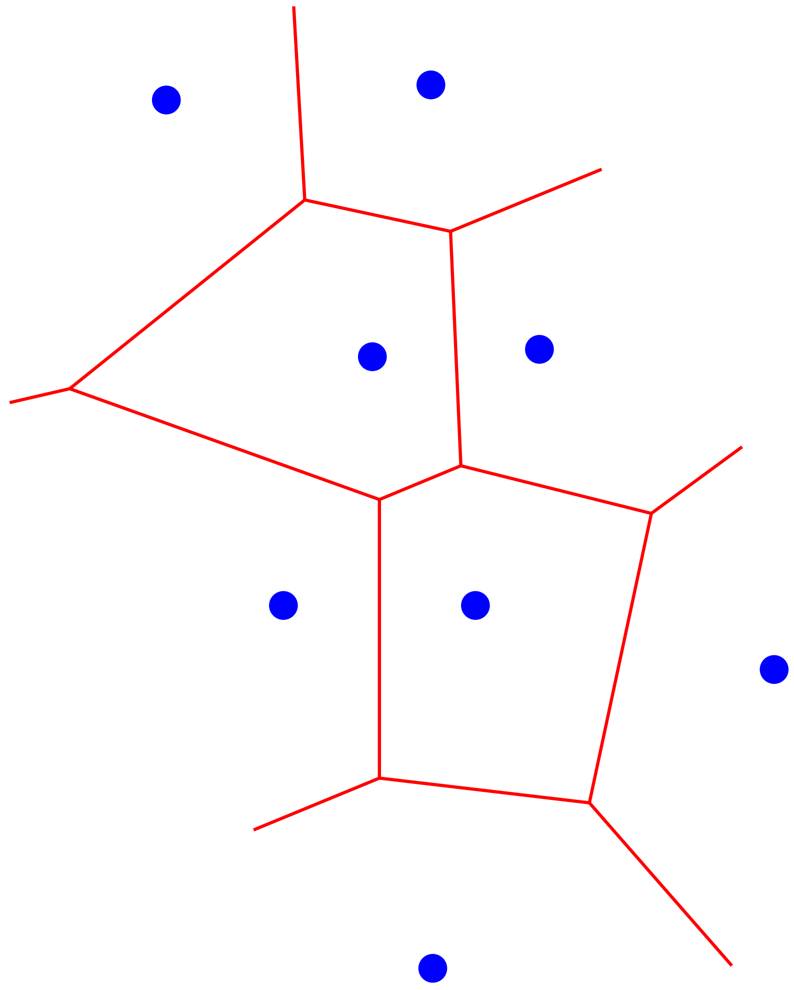
update simplicial complex;

END_WHILE

Output: the sequence of simplicial complexes

Witness complex
(definition)

Let $\color{blue}{L} \subseteq \mathbb{R}^d$ (landmarks) s.t. $|L| < +\infty$ and $W \subseteq \mathbb{R}^d$ (witnesses)



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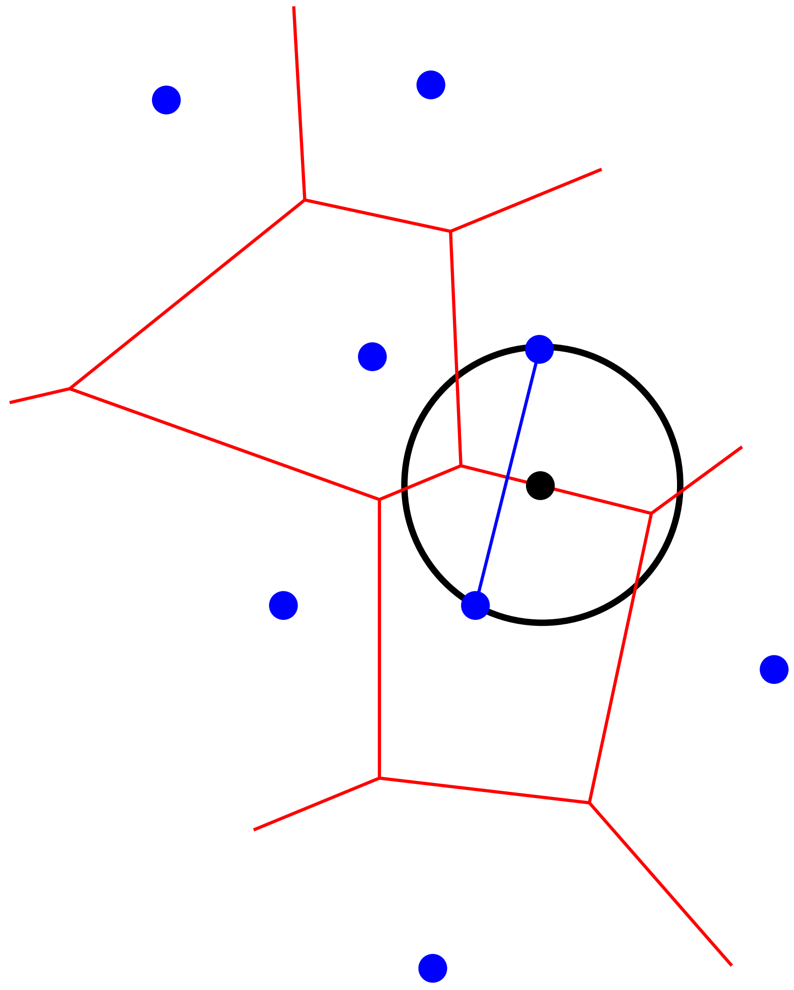
Let $L \subseteq \mathbb{R}^d$ (landmarks) s.t. $|L| < +\infty$ and $W \subseteq \mathbb{R}^d$ (witnesses)

Def. $w \in W$ *strongly witnesses* $[v_0, \cdots, v_k]$ if $\|w - v_i\| = \|w - v_j\| \leq \|w - u\|$ for all $i, j = 0, \cdots, k$ and all $u \in L \setminus \{v_0, \cdots, v_k\}$.



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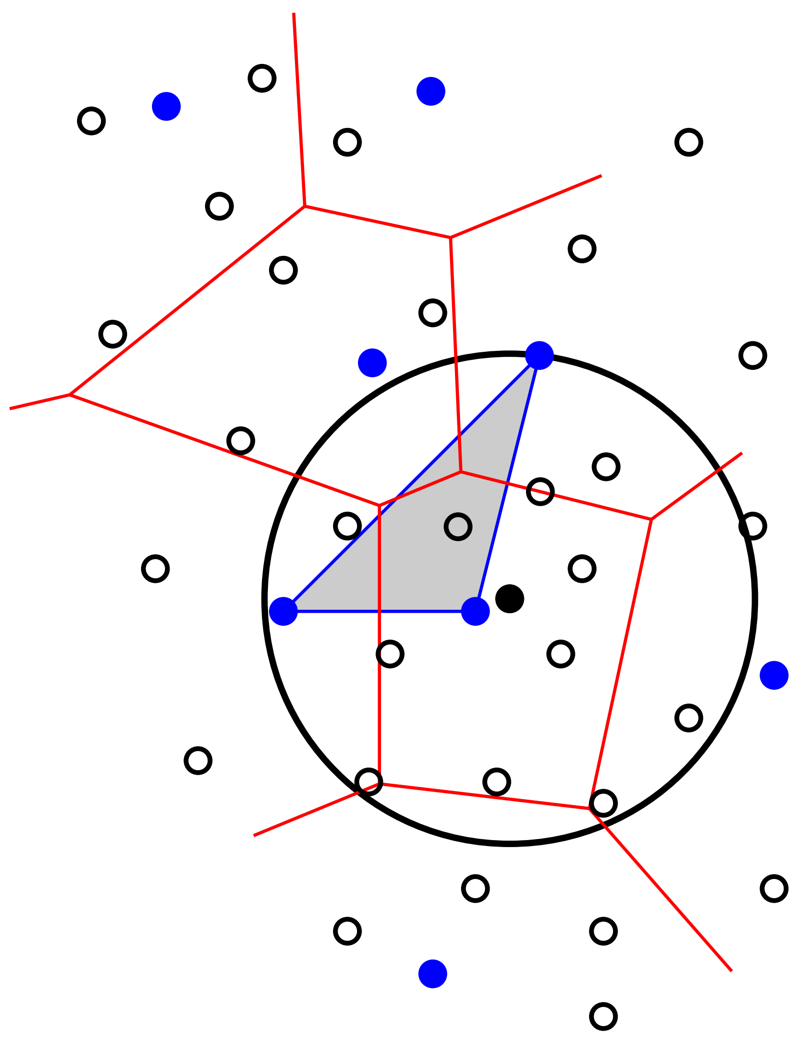

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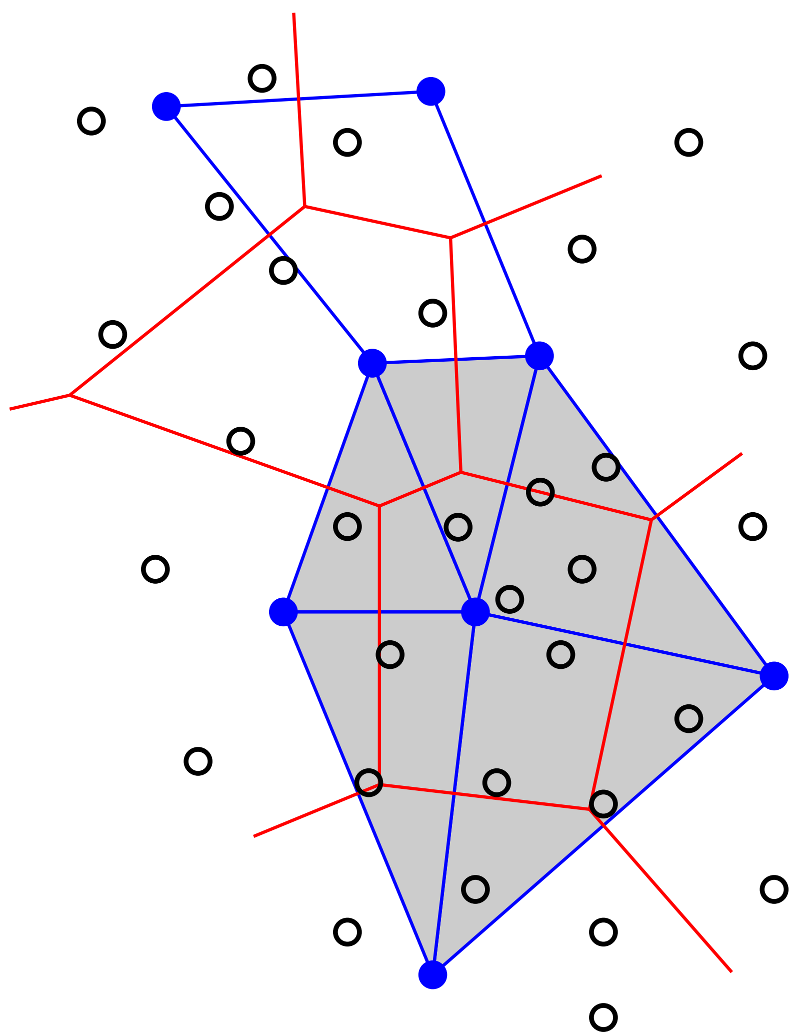
Def. $\mathcal{C}^W(L)$ is the largest abstract simplicial complex built over L, whose faces are weakly witnessed by points of W.



Witness complex
(properties)

Thm. 1 [de Silva 2003] $\forall W, L$, $\forall \sigma \in \mathcal{C}^W(L)$,
$\exists c \in \mathbb{R}^d$ that strongly witnesses σ.

$\Rightarrow \mathcal{C}^W(L)$ is a subcomplex of $\mathcal{D}(L)$

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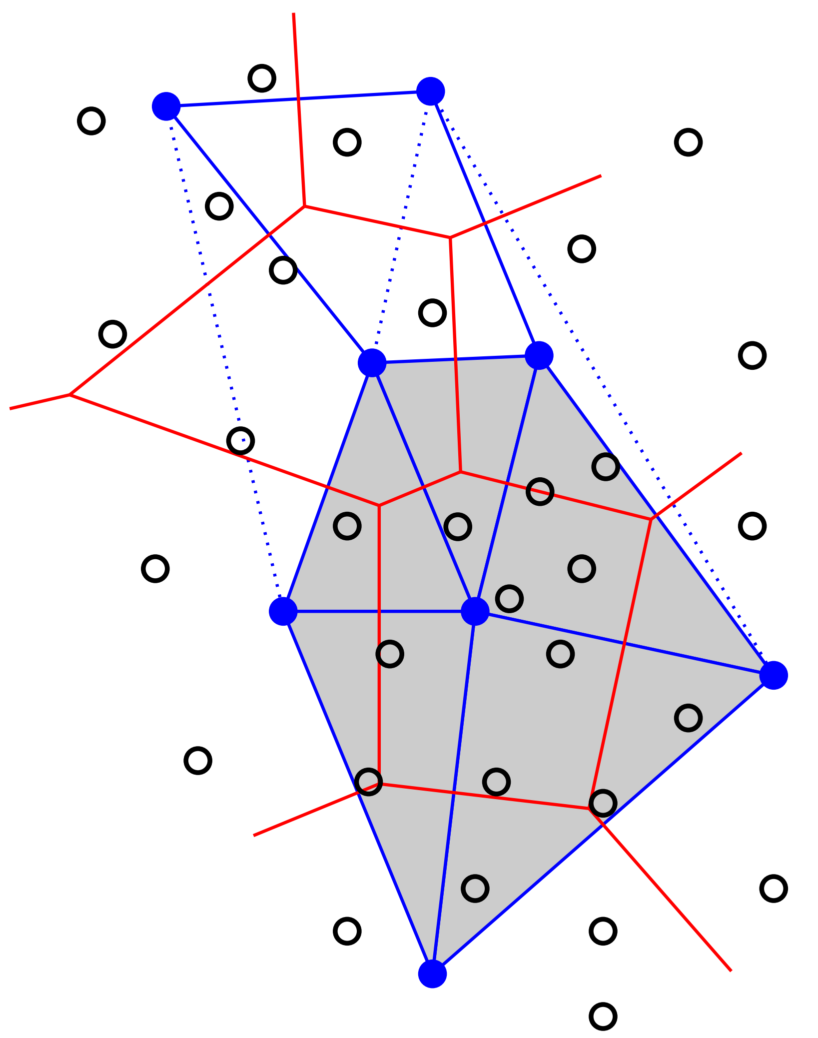

Weak witness theorem

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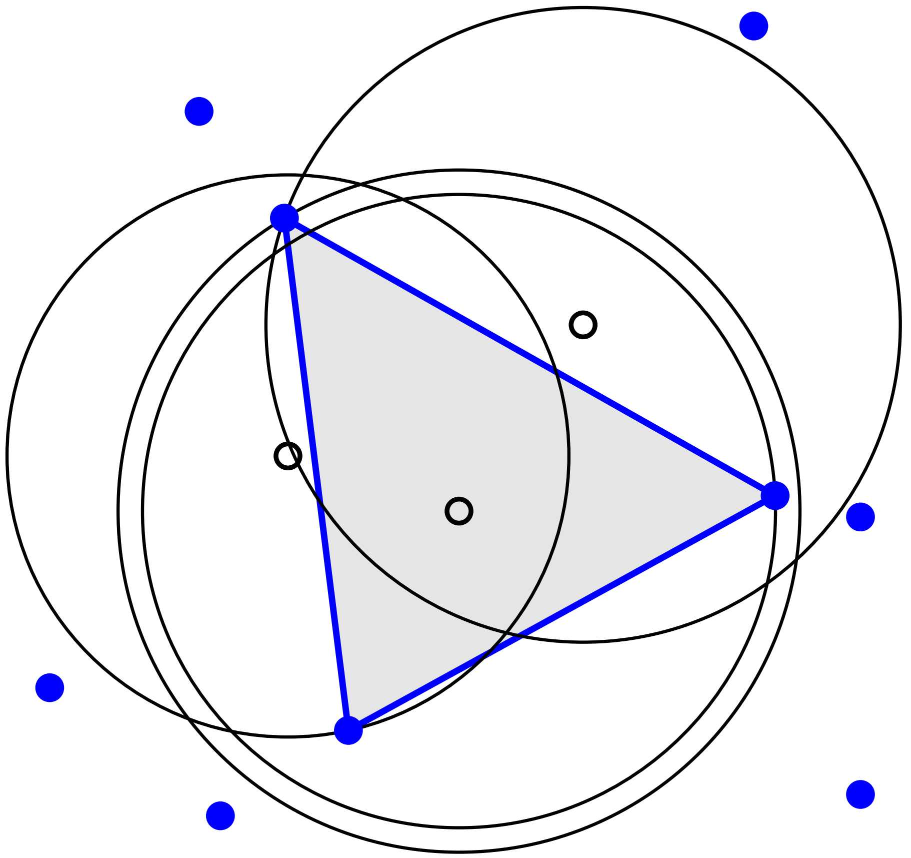

$\sigma \in \mathcal{C}^W(L)$ iff $\forall \tau \subseteq \sigma$,
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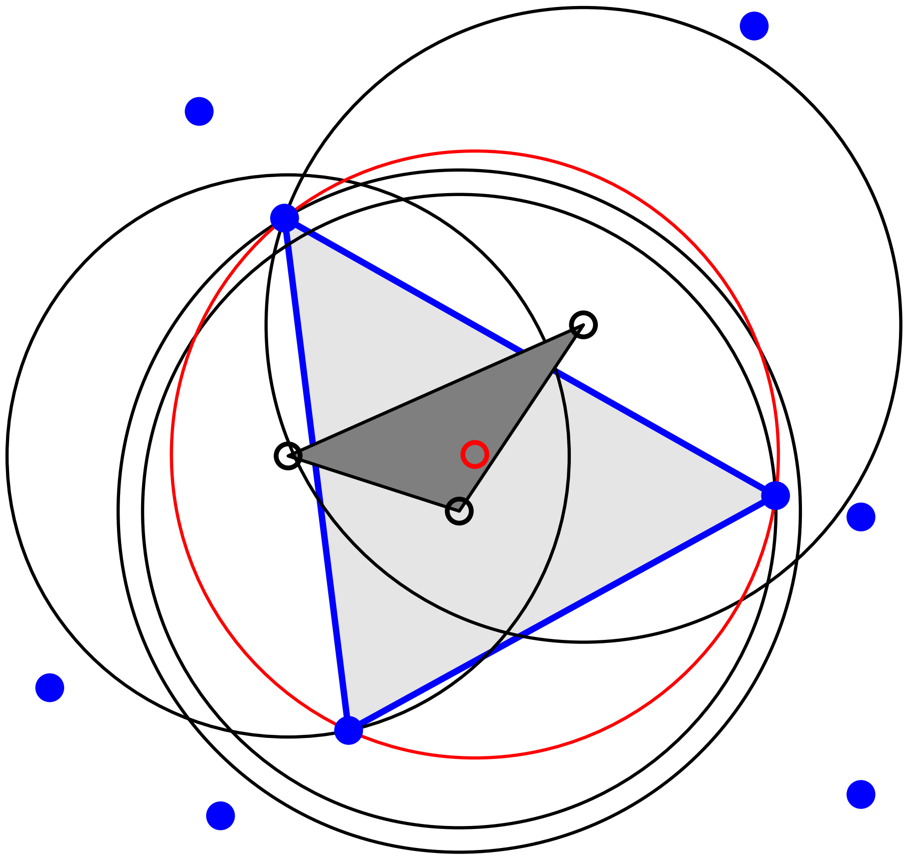

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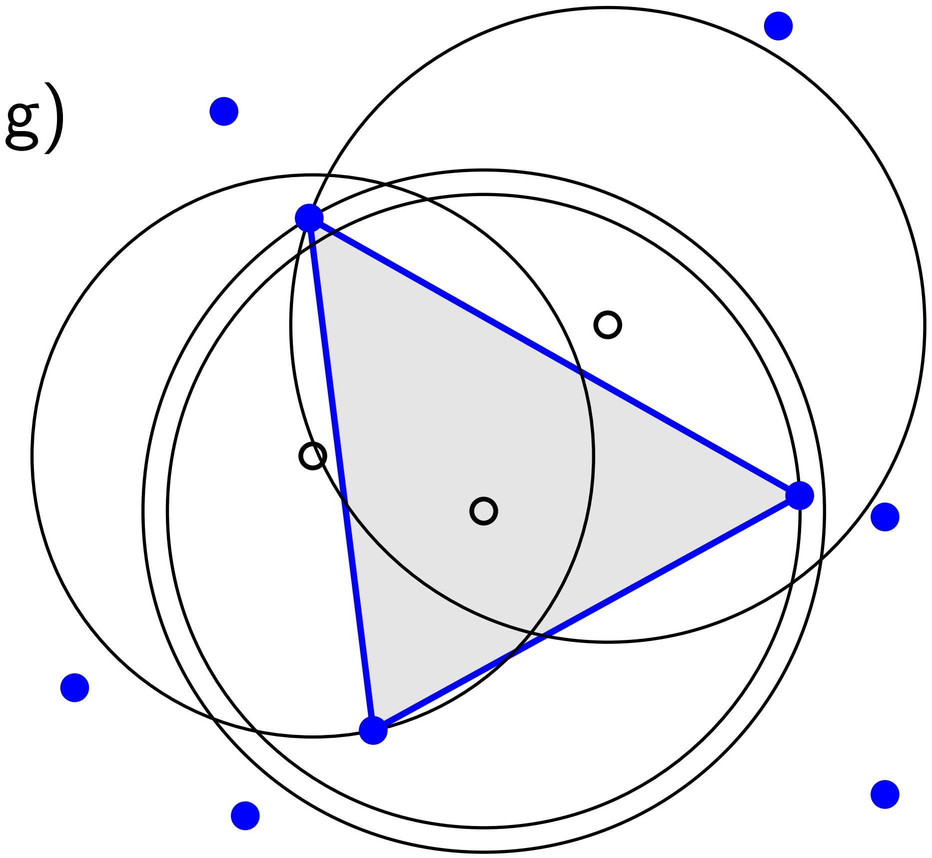
Proof. [Attali, Edelsbrunner, Mileyko 2007]

$\rightarrow$ induction on the dimension of σ:

- Case $\sigma = [v_0]$: trivial (all witnesses of v_0 are strong)



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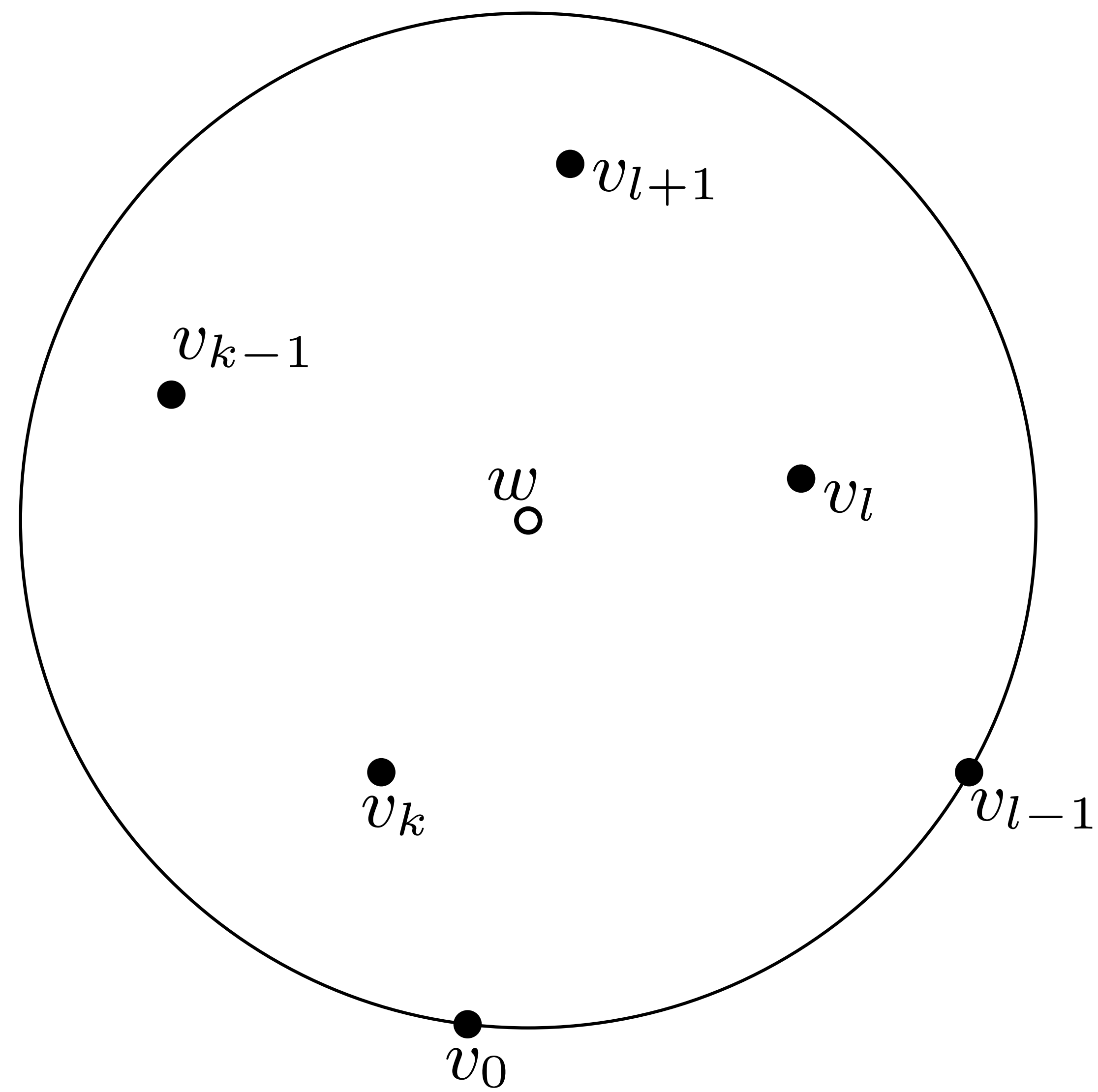
- Case $\sigma = [v_0, \cdots, v_k]$ $(k > 0)$:

assume that $\|w - v_0\| = \cdots = \|w - v_{l-1}\|$
$\geq \|w - v_i\| \; \forall i \geq l$



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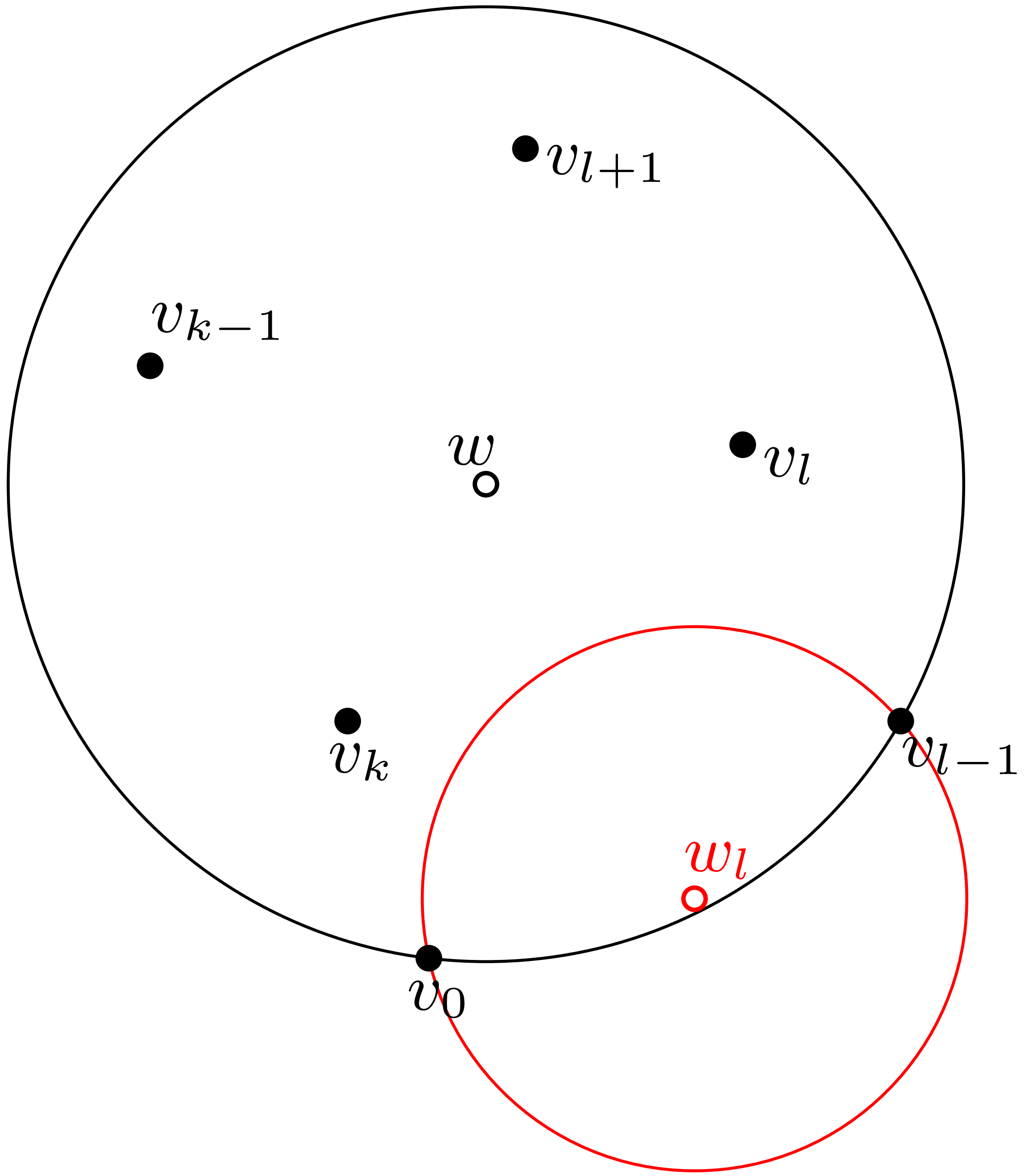

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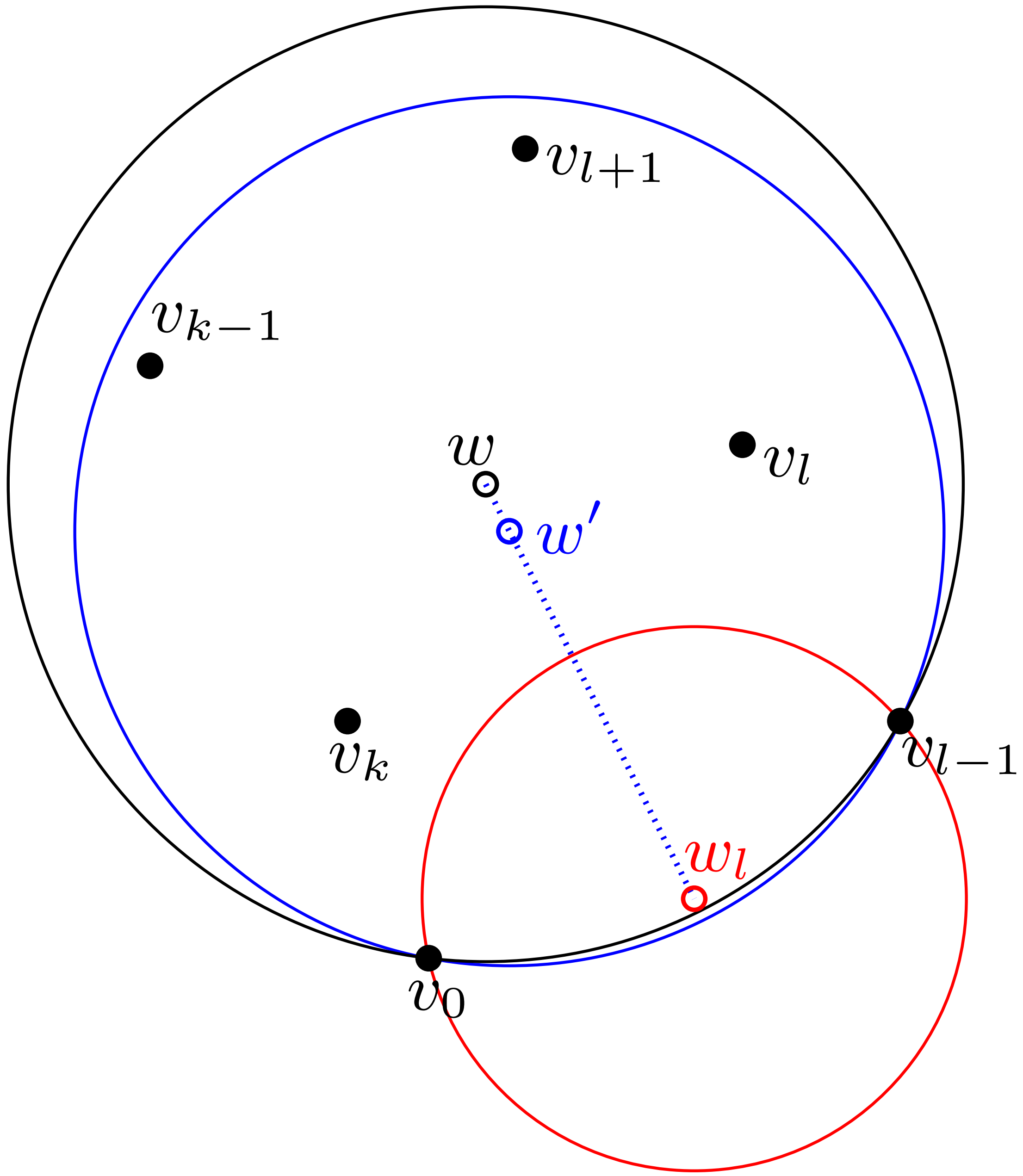

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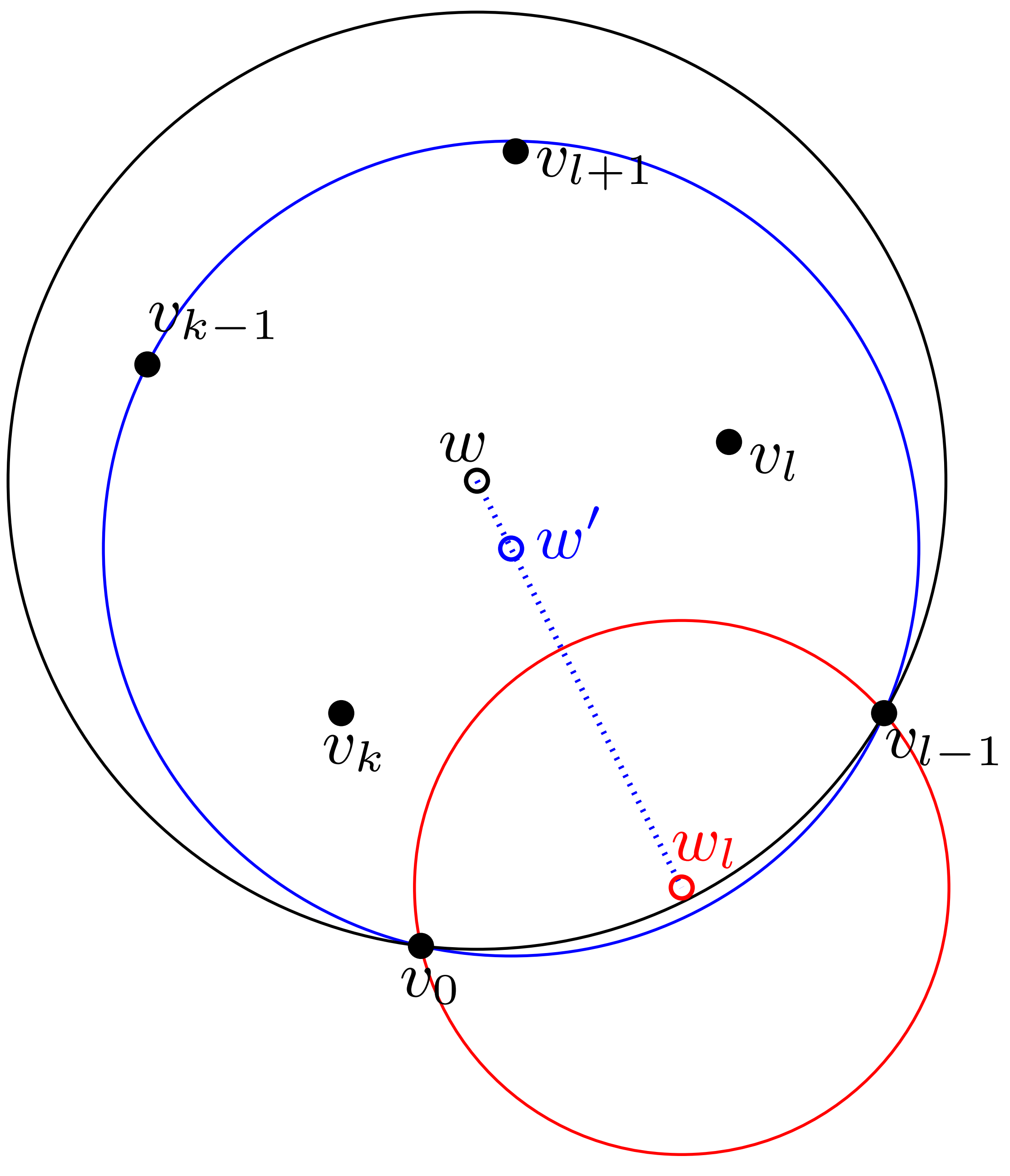
move w to w' as shown opposite

$\rightarrow B_{w'} \cap L = \{v_0, \cdots, v_k\}$

$\rightarrow |\partial B_{w'} \cap L| \geq l + 1$



Witness complex

(properties)

Thm. 1 [de Silva 2003] $\forall W, L$, $\forall \sigma \in \mathcal{C}^W(L)$,
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$\Rightarrow \mathcal{C}^W(L)$ is a subcomplex of $\mathcal{D}(L)$

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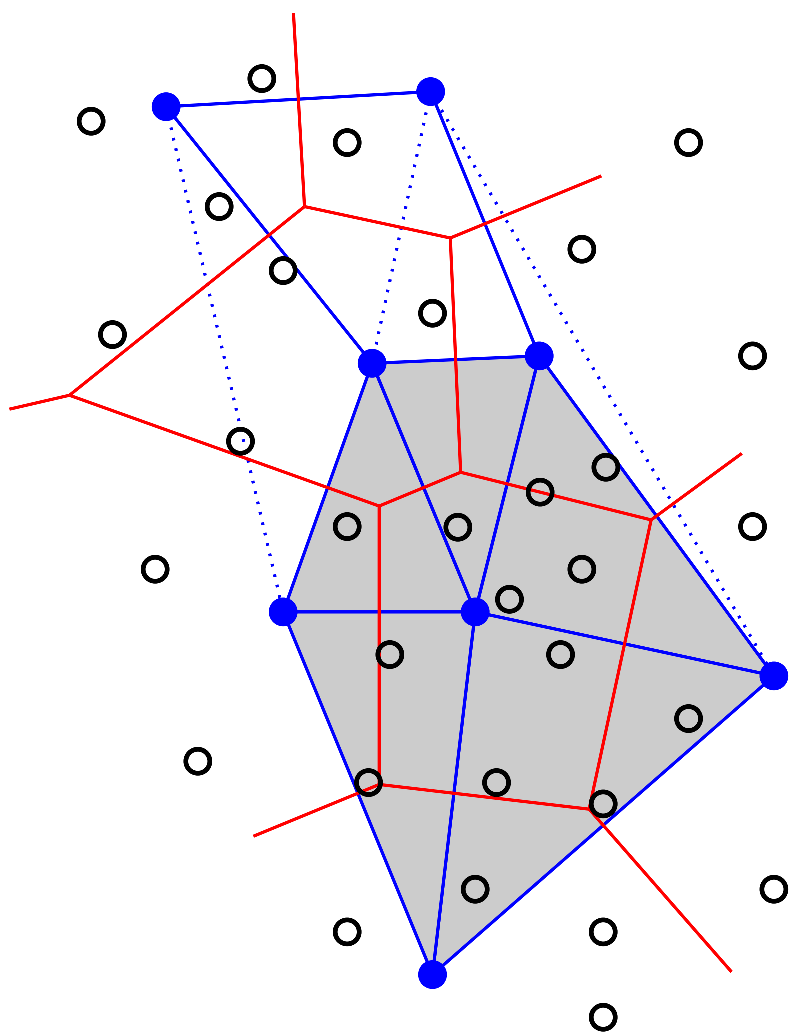

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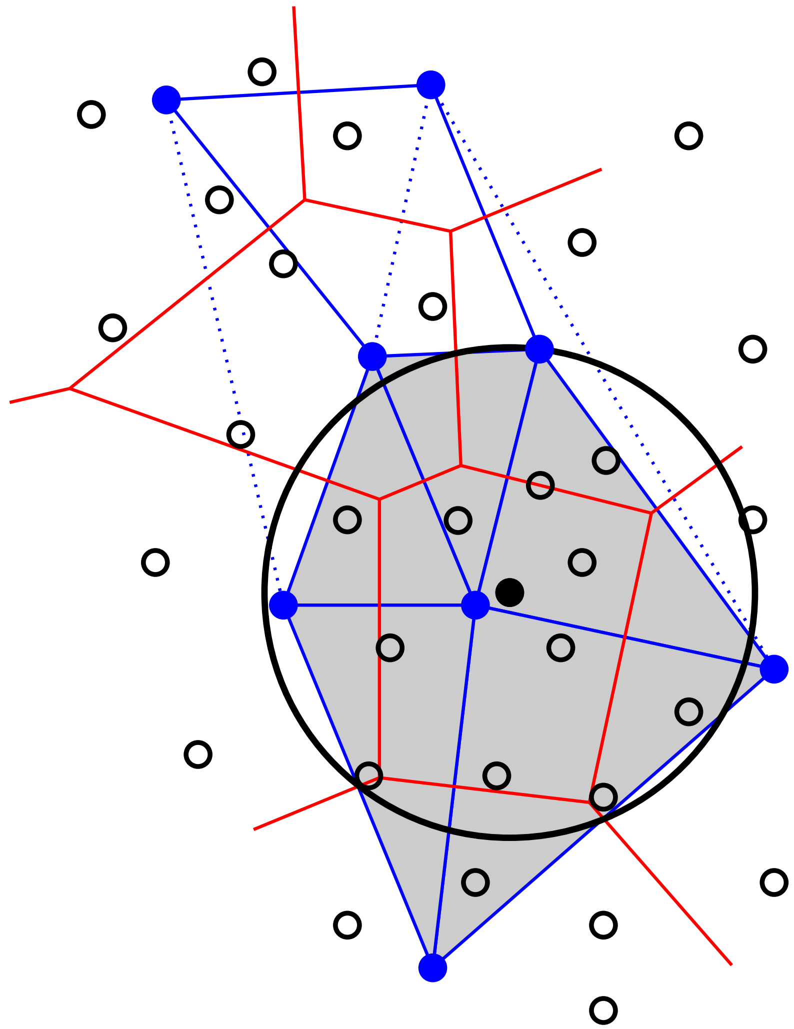

Thm. 2 [de Silva, Carlsson 2004]
- The size of $\mathcal{C}^W(L)$ is $O(d|W|)$
- The time to compute is $\mathrm{Poly}(d, |W|, |L|)$

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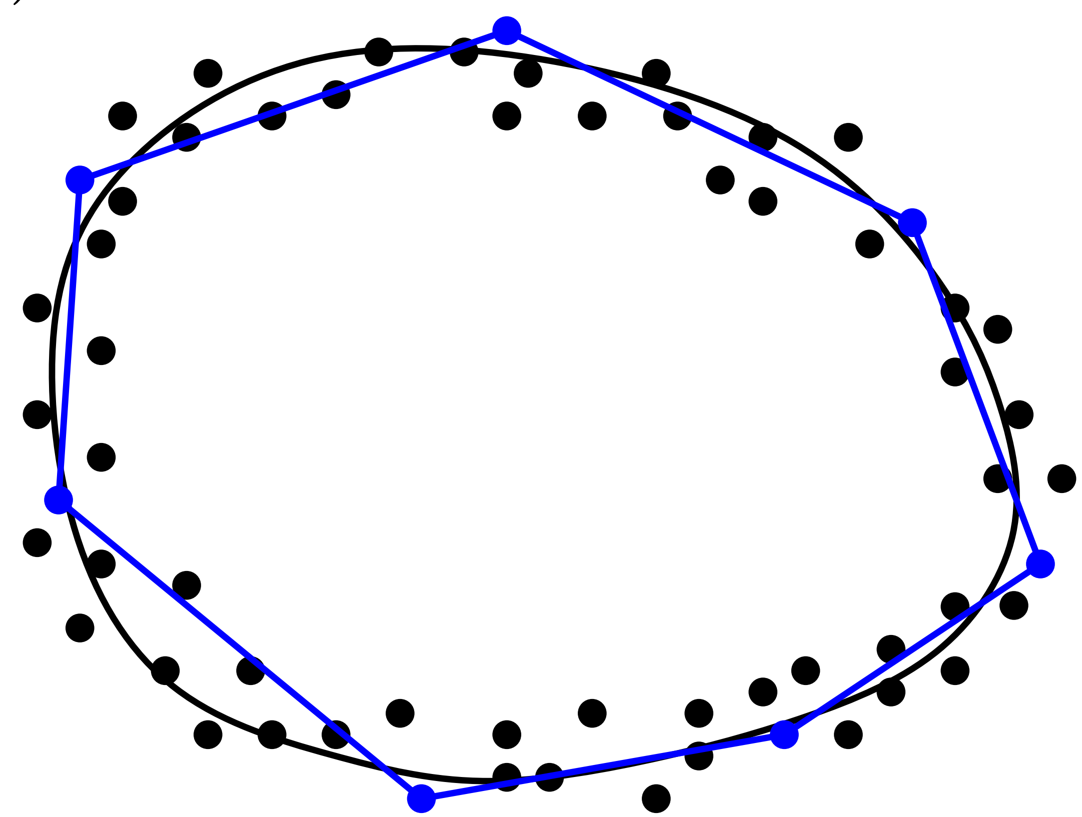
$\Rightarrow \mathcal{C}^W(L)$ is embedded in $\mathbb{R}^d$
(if L lies in general position)



Thm. 2 [de Silva, Carlsson 2004]
- The size of $\mathcal{C}^W(L)$ is $O(d|W|)$
- The time to compute is $\mathrm{Poly}(d, |W|, |L|)$

$\rightarrow$ What if W, L lie on or near a submanifold M?

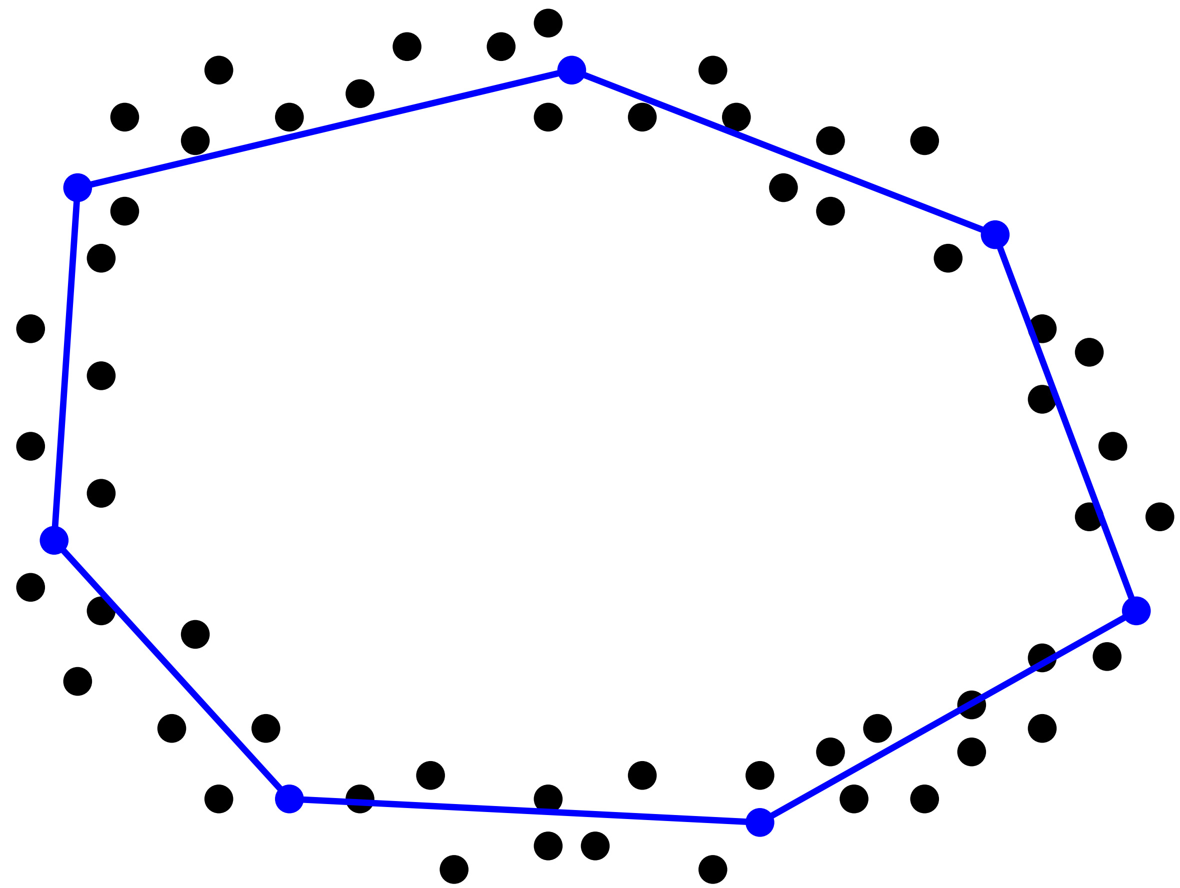
Thm. 3 [Guibas, Oudot 2007]
[Attali, Edelsbrunner, Mileyko 2007]
Under *some conditions*, $\mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M$

Witness complex

(connection to reconstruction)

- $W \subset \mathbb{R}^d$ is given as input
- $L \subseteq W$ is generated
- underlying manifold M unknown
- only distance comparisons

$\Rightarrow$ algorithm is applicable in any metric space

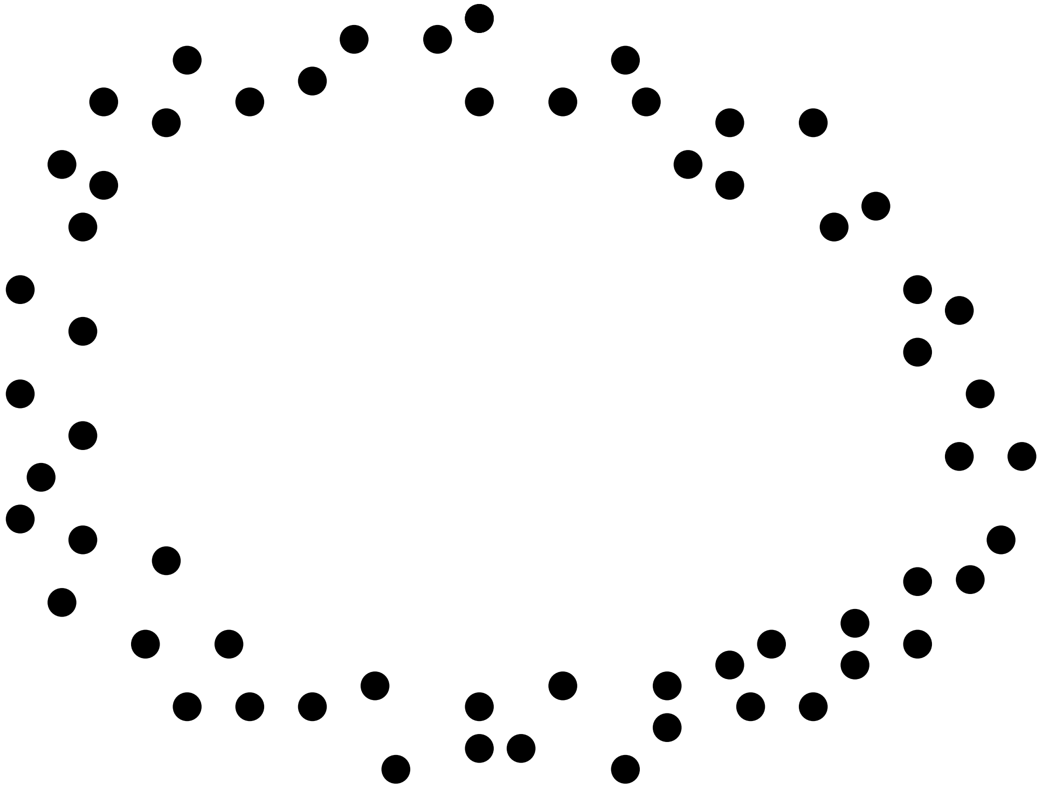


- In $\mathbb{R}^d$, $\mathcal{C}^W(L)$ can be maintained by updating, for each witness w, the list of $d+1$ nearest landmarks of w.

$$\Rightarrow \begin{array}{lcl} \text{space} & \leq & O\left(d|W|\right) \\ \text{time} & \leq & O\left(d|W|^2\right) \end{array}$$

The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.

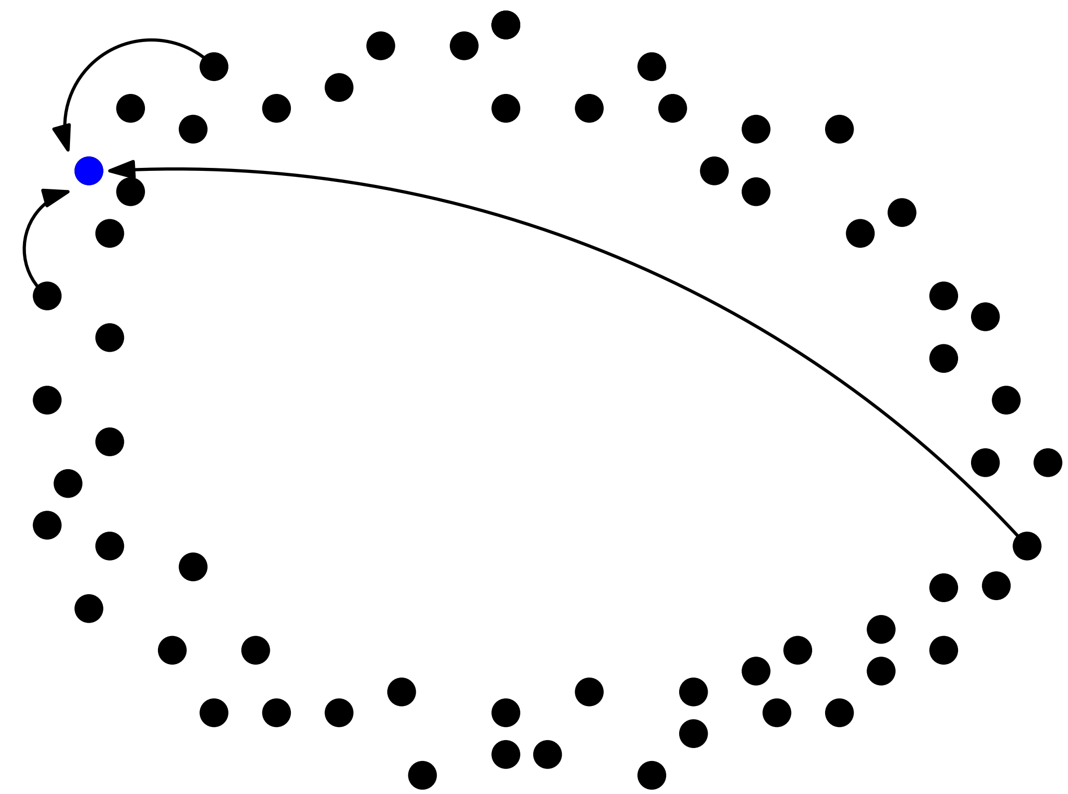


The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.

Init: $L := \{p\}$; construct lists of nearest landmarks; $\mathcal{C}^W(L) = \{[p]\}$;

Invariant: $\forall w \in W$, the list of $d+1$ nearest landmarks of w is maintained throughout the process.



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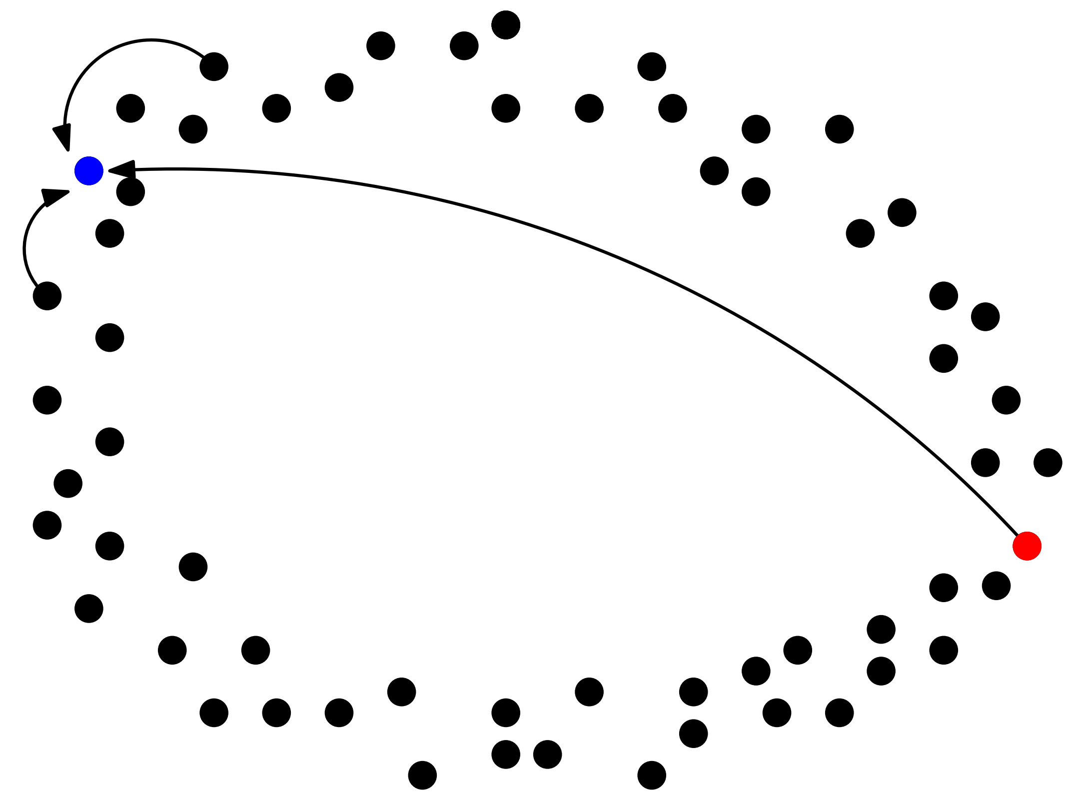

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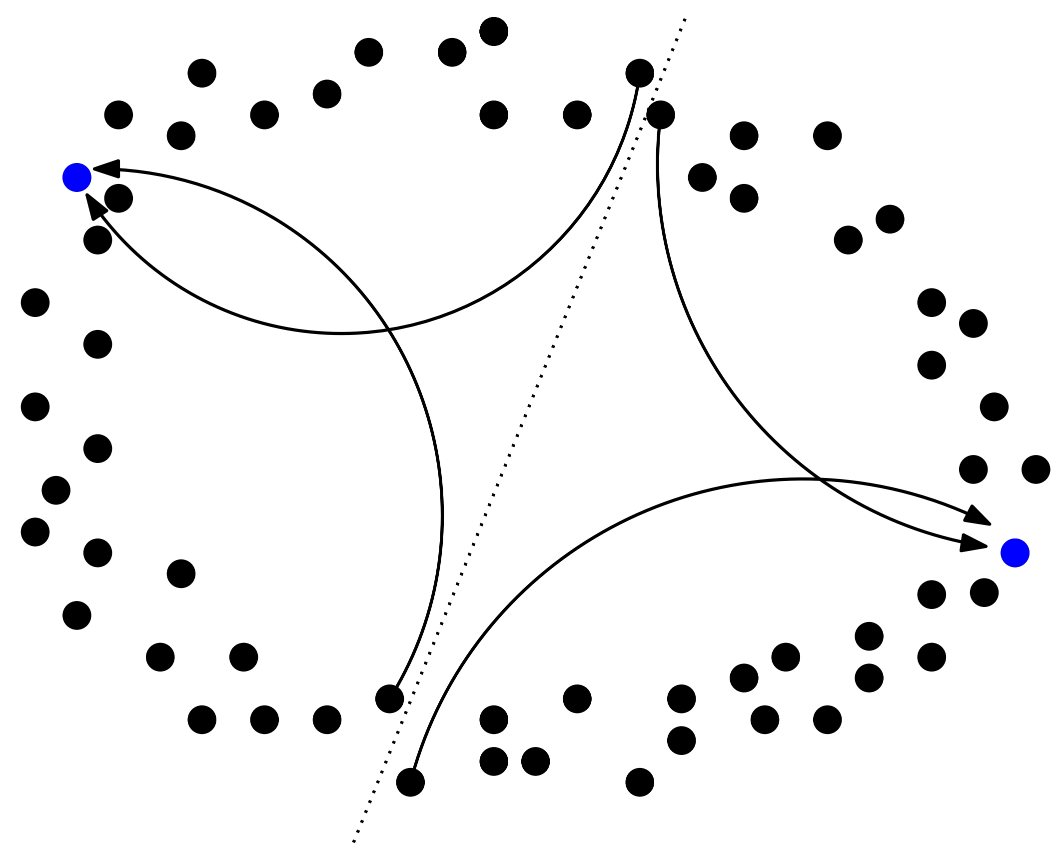
$\textsc{while}\ L \subsetneq W$

insert $\operatorname{argmax}_{w \in W} \mathsf{d}(w, L)$ in L;

update the lists of nearest neighbors;



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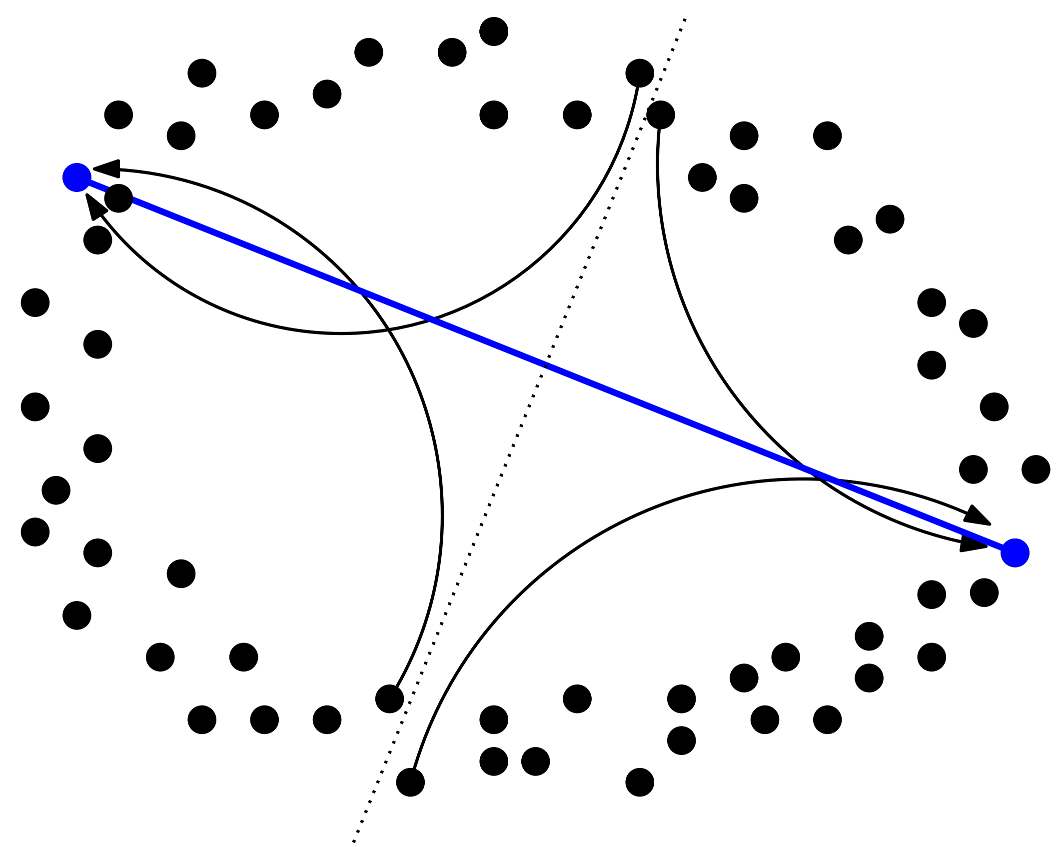
$\textsc{while}\ L \subsetneq W$

  insert $\operatorname{argmax}_{w \in W} \mathsf{d}(w, L)$ in L;

  update the lists of nearest neighbors;

  update $\mathcal{C}^W(L)$;

$\textsc{end\_while}$



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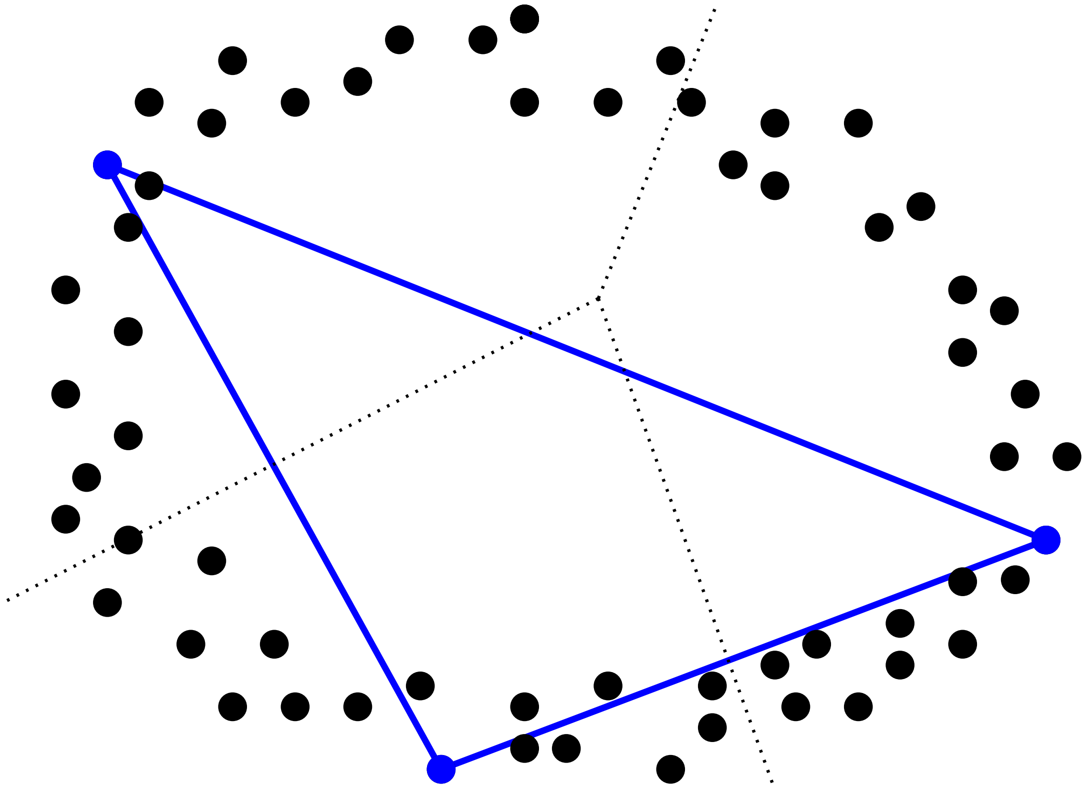
update $\mathcal{C}^W(L)$;

END_WHILE



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$\text{WHILE } L \subsetneq W$

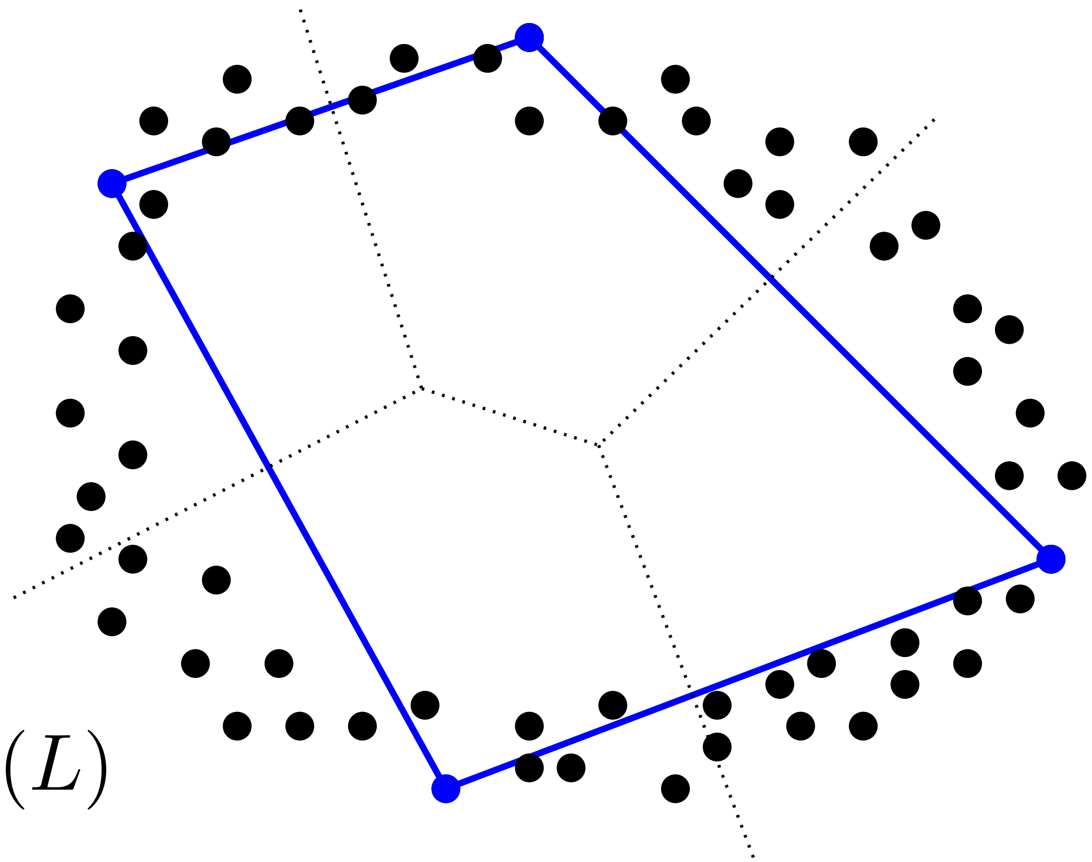
insert $\mathrm{argmax}_{w \in W} \mathsf{d}(w, L)$ in L;

update the lists of nearest neighbors;

update $\mathcal{C}^W(L)$;

END_WHILE

Output: the sequence of complexes $\mathcal{C}^W(L)$



Relation with the restricted Delaunay

If M is a closed k-manifold smoothly embedded in $\mathbb{R}^d$, then, under sufficient sampling conditions, $\mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M$

Relation with the restricted Delaunay

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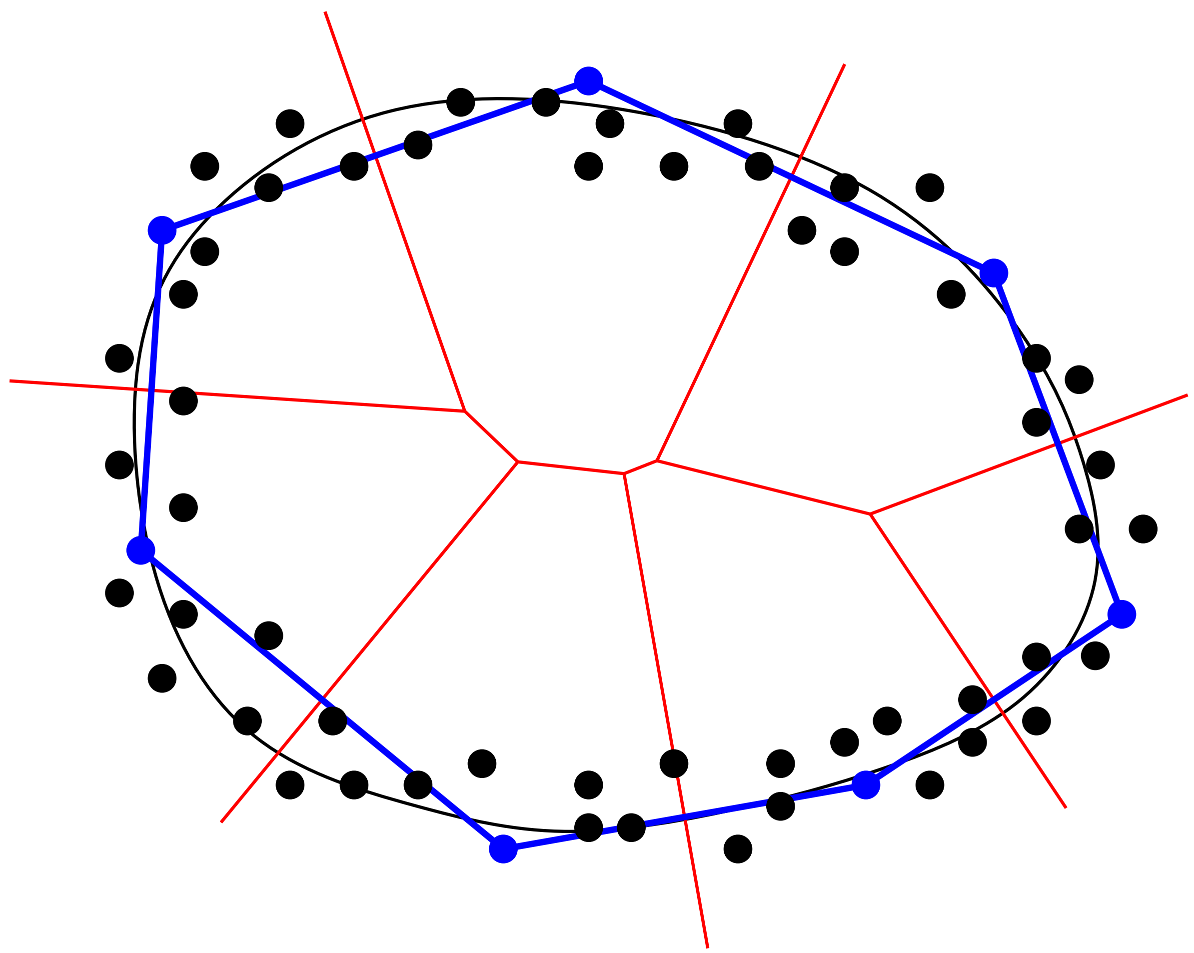
- Case $k = 1$:
  - $\mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M$

[Guibas, O. 07]

[Attali, Edelsbrunner, Mileyko 07]



Relation with the restricted Delaunay

$\boxed{\text{If } M \text{ is a closed } k\text{-manifold smoothly embedded in } \mathbb{R}^d\text{, then, under sufficient sampling conditions, } \mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M}$

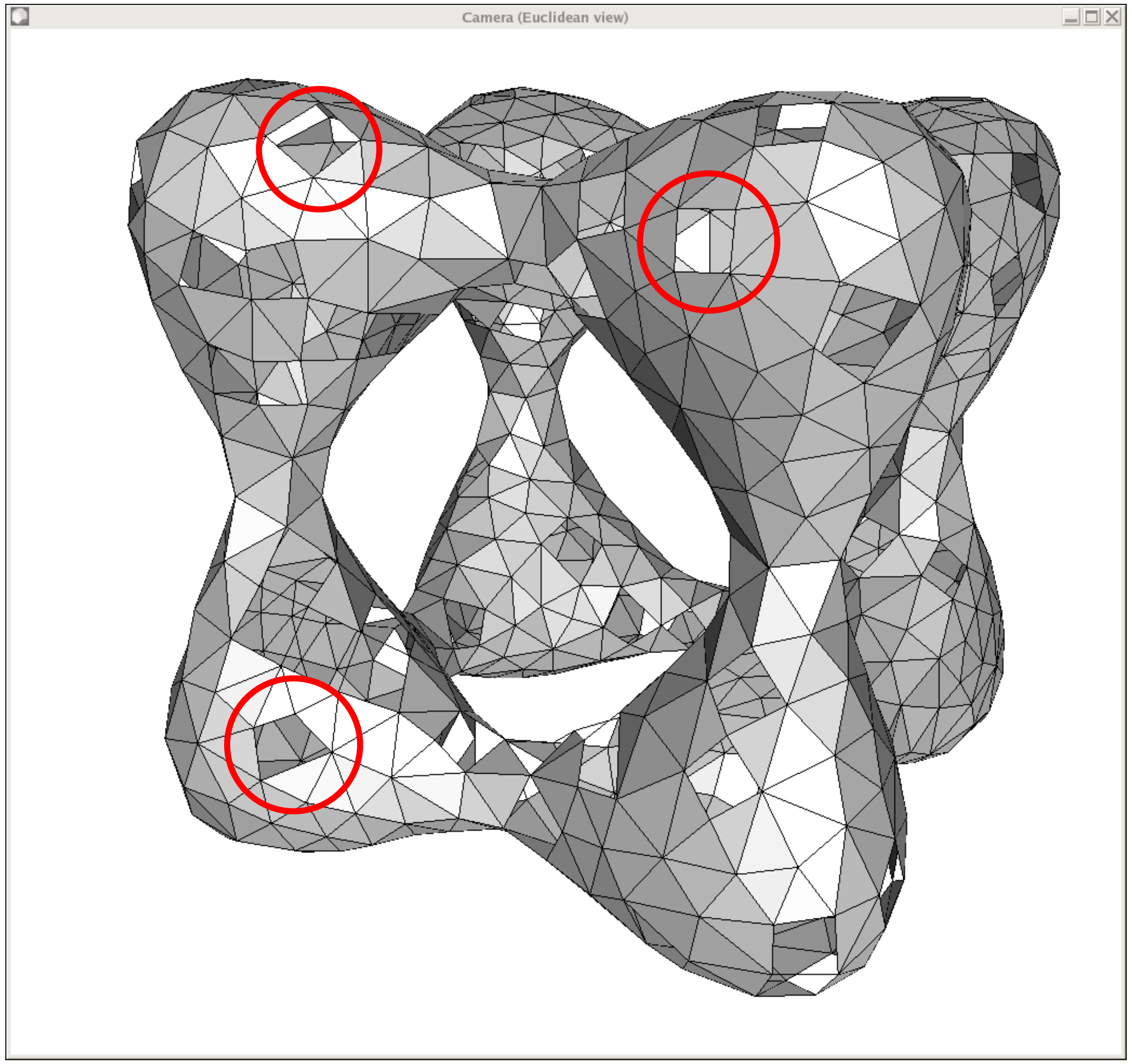
- Case $k = 1$:
  - $\mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M$
- Case $k = 2$:
  - $\mathcal{C}^W(L) \subseteq \mathcal{D}^M(L) \simeq M$
  - $\mathcal{C}^W(L) \not\supseteq \mathcal{D}^M(L)$

[Amenta, Bern 98]

[Attali, Edelsbrunner, Mileyko 07]

[de Silva, Carlsson 04]

[Guibas, O. 07]



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- Case $k \geq 3$:
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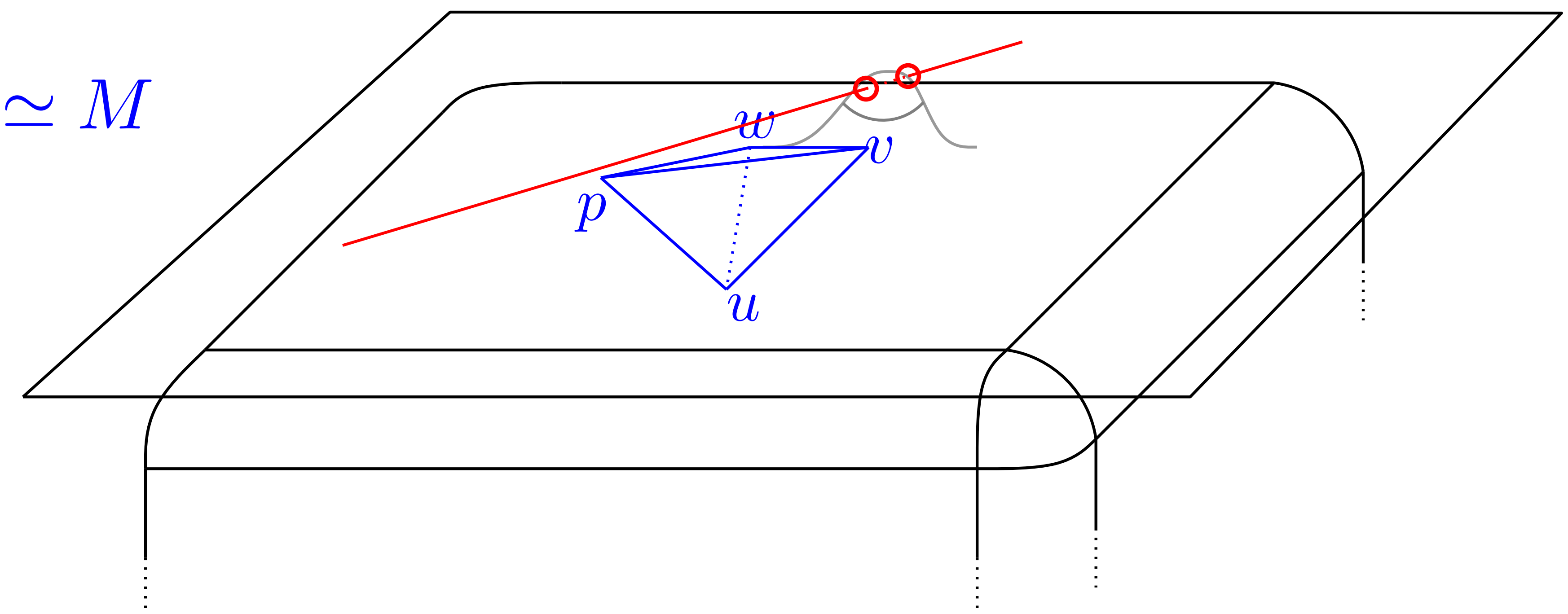
  - $\mathcal{D}^M(L) \not\simeq M$

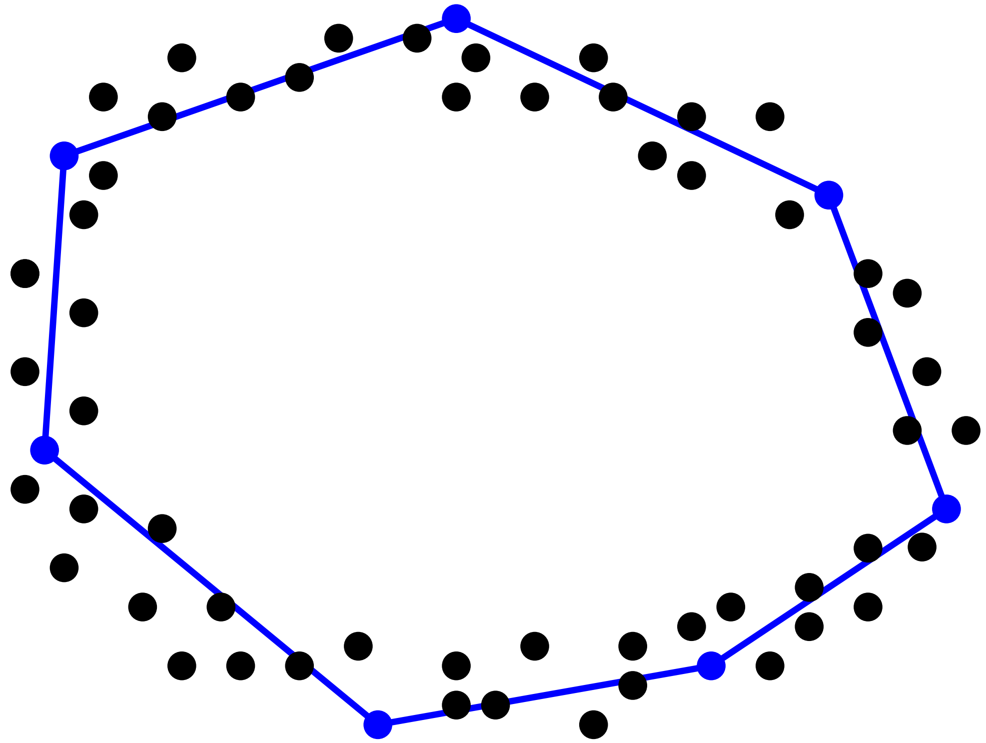


[Cheng, Dey, Ramos 05]
[O. 07]

Relation with the restricted Delaunay

(case of curves)

Conjecture [Carlsson, de Silva 2004]
$\mathcal{C}^W(L)$ coincides with $\mathcal{D}^M(L)$...



Relation with the restricted Delaunay

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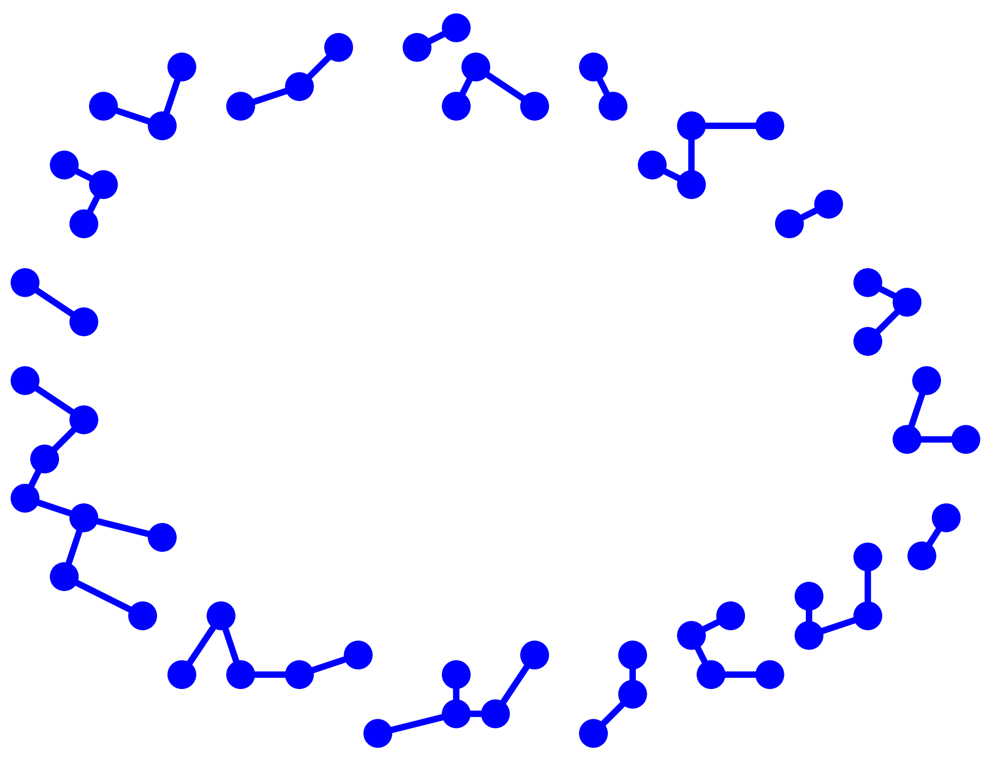
Conjecture [Carlsson, de Silva 2004]
$\mathcal{C}^W(L)$ coincides with $\mathcal{D}^M(L)$...

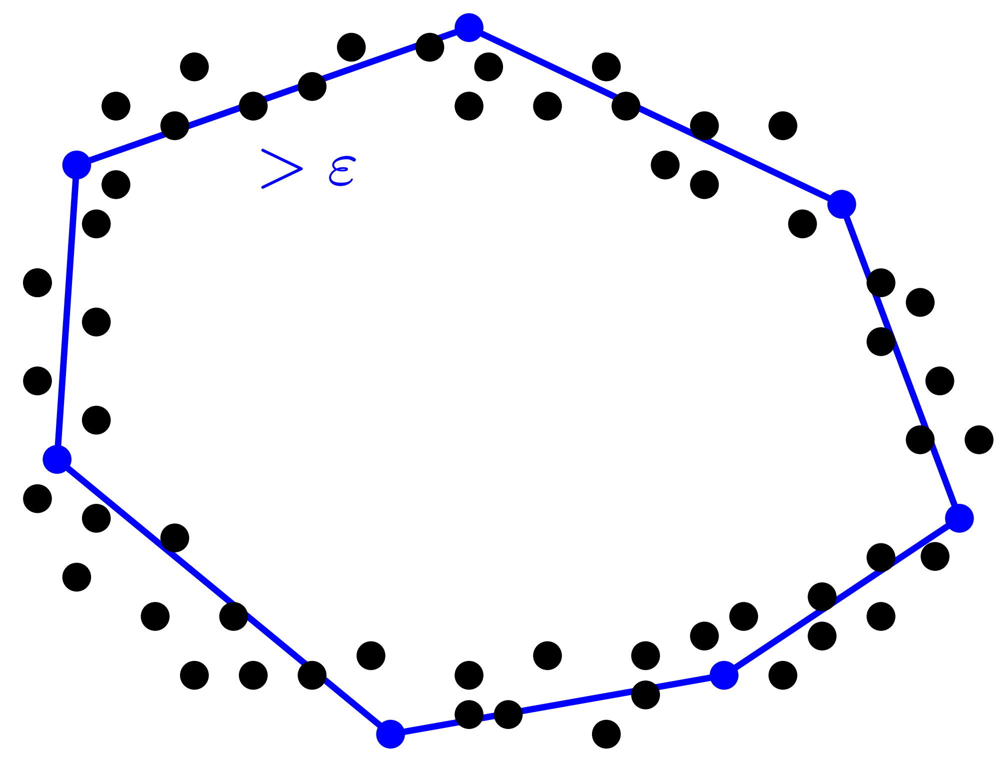
... under some conditions on W and L



Relation with the restricted Delaunay

(case of curves)

Thm: If M is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $\mathrm{d_H}(W, M) \leq \delta$, $L \subseteq W$ ε-sparse ε-sample of W with $\delta \ll \varepsilon \ll \varrho_M$, then $\mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M$.



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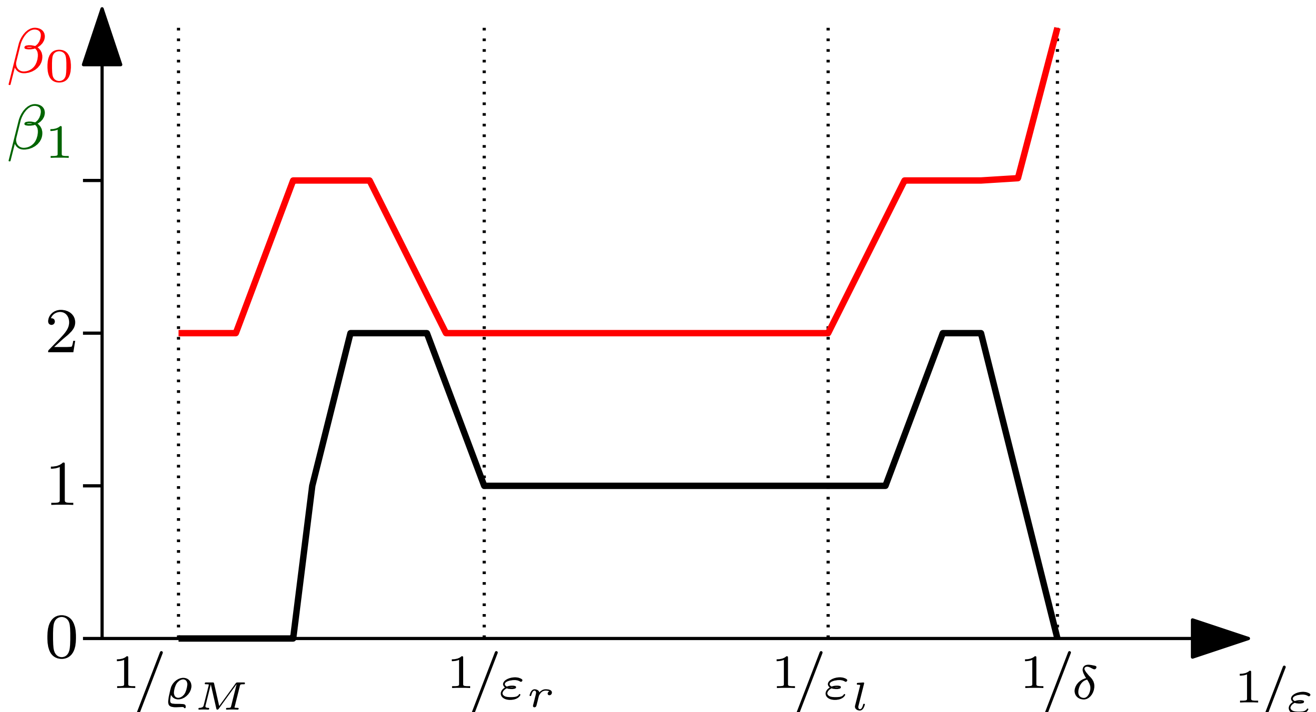

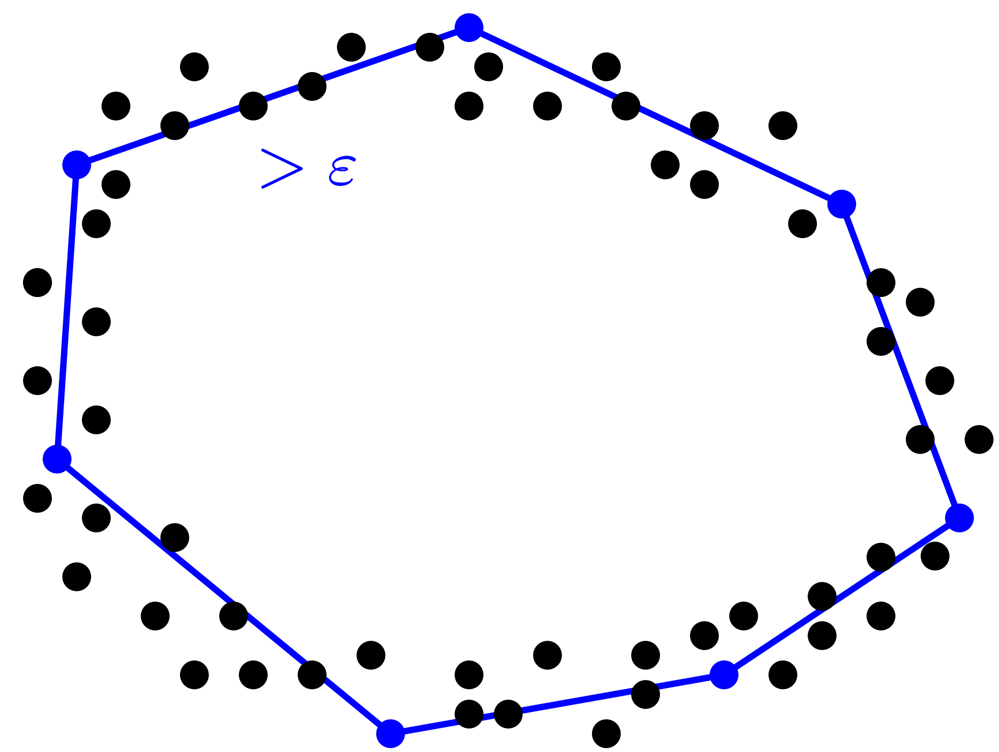

$\rightarrow$ There is a plateau in the diagram of Betti numbers of $\mathcal{C}^W(L)$.

Relation with the restricted Delaunay

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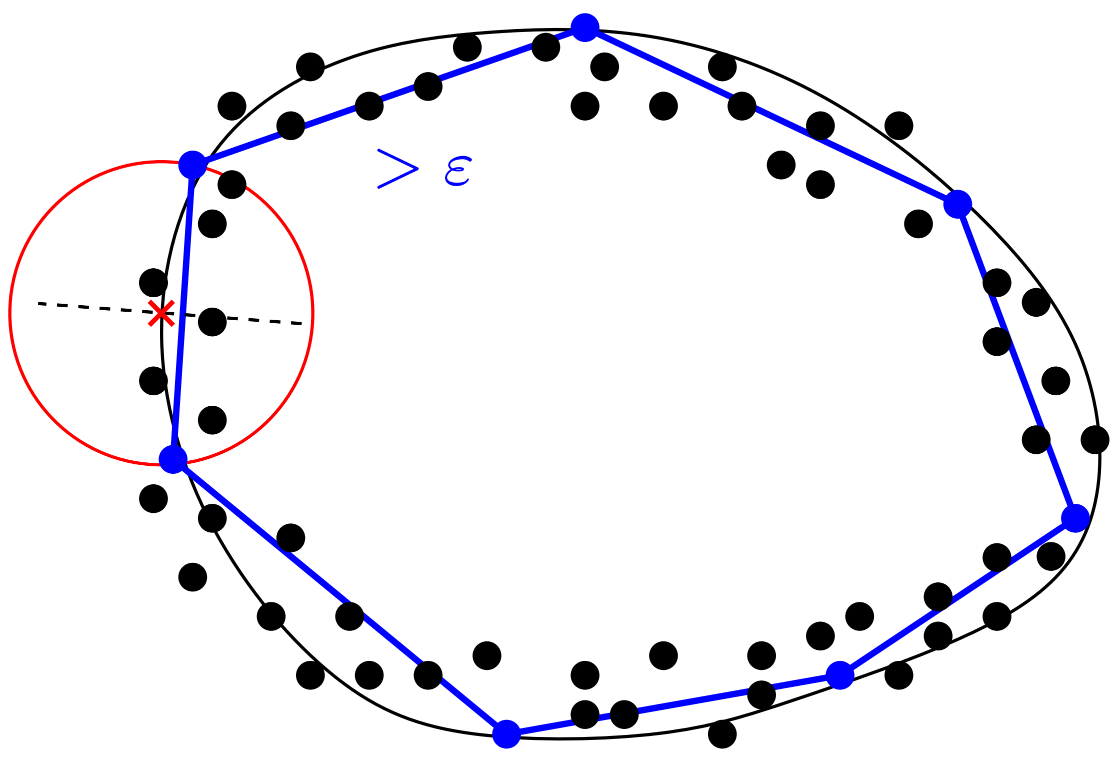
- $\mathcal{D}^M(L) \subseteq \mathcal{C}^W(L)$



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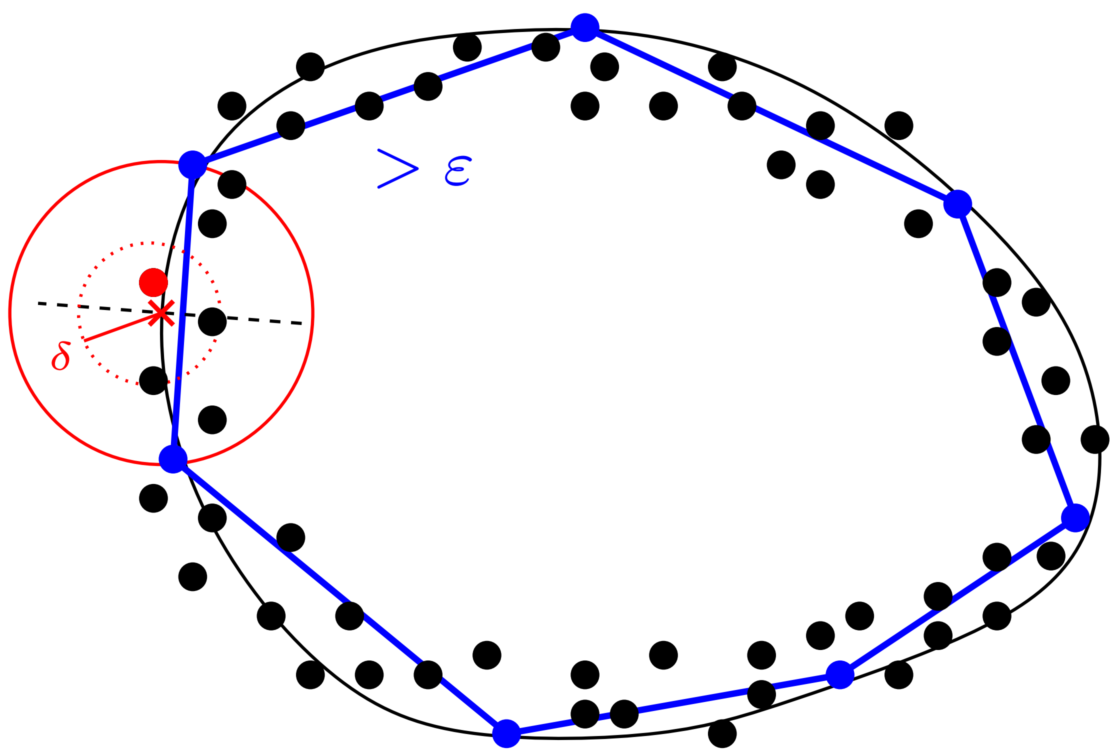

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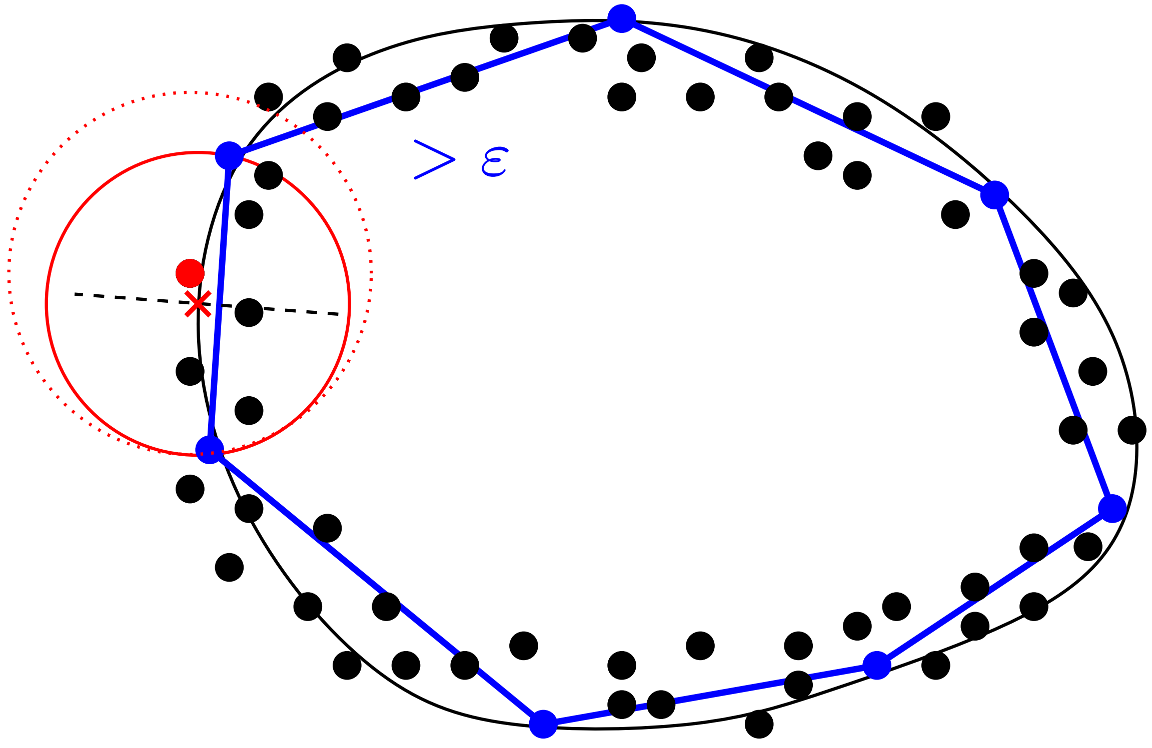

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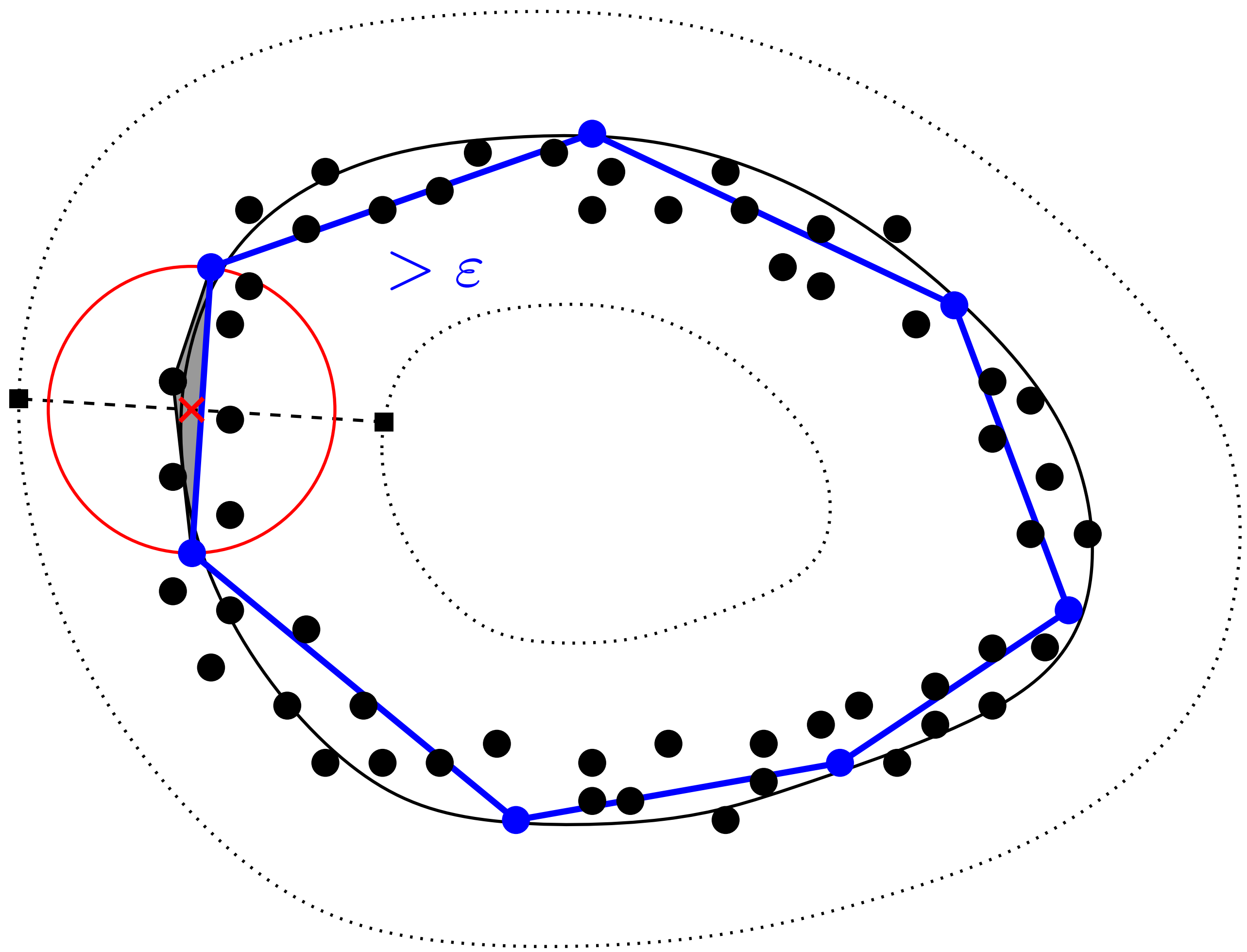

Relation with the restricted Delaunay

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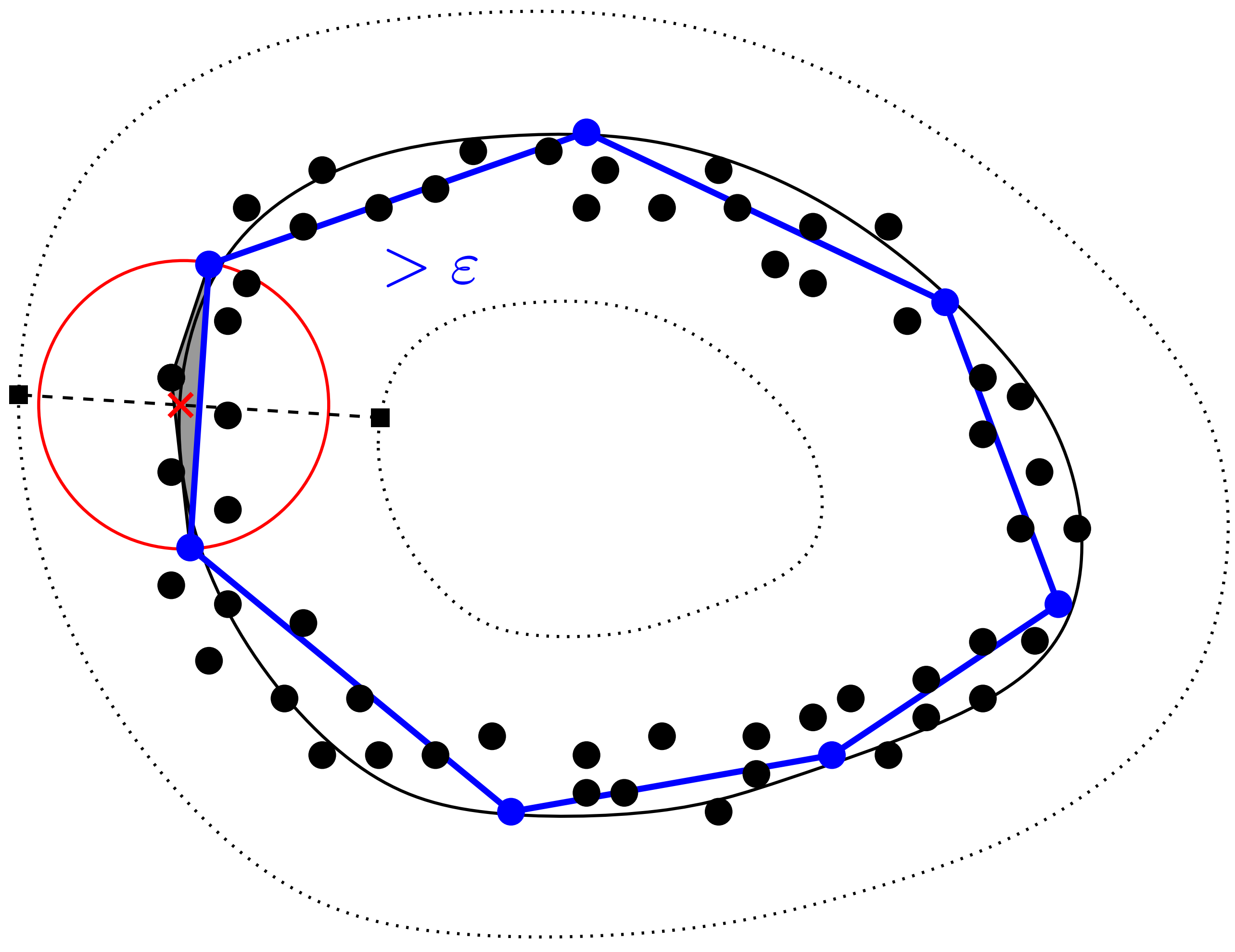
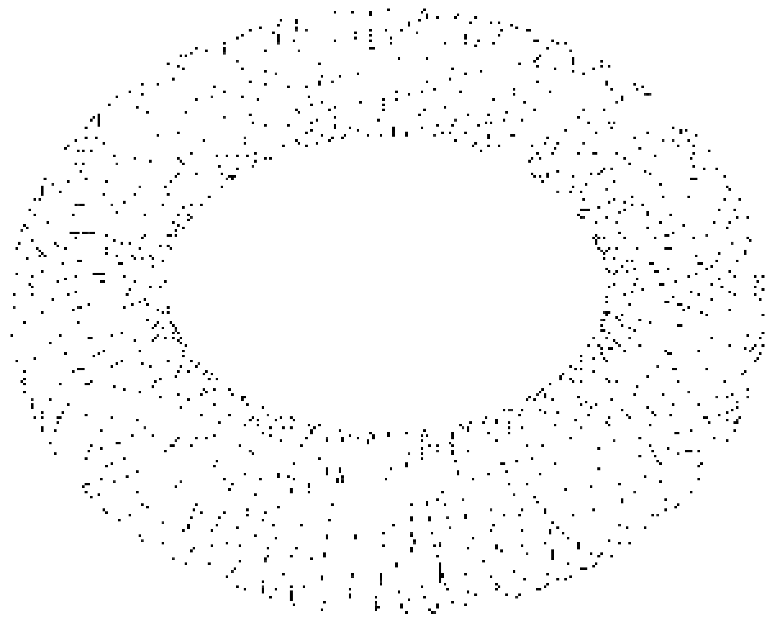
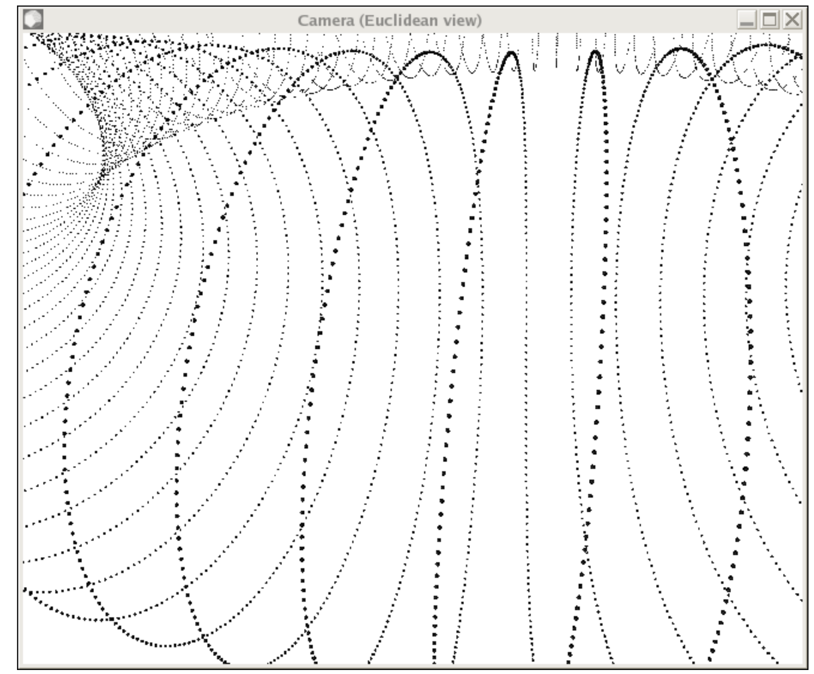


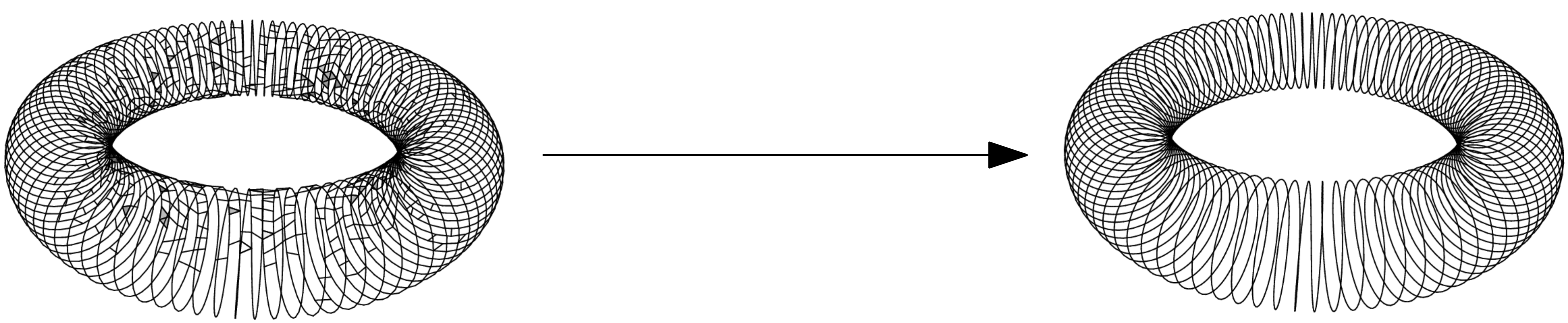
Some results

Input:





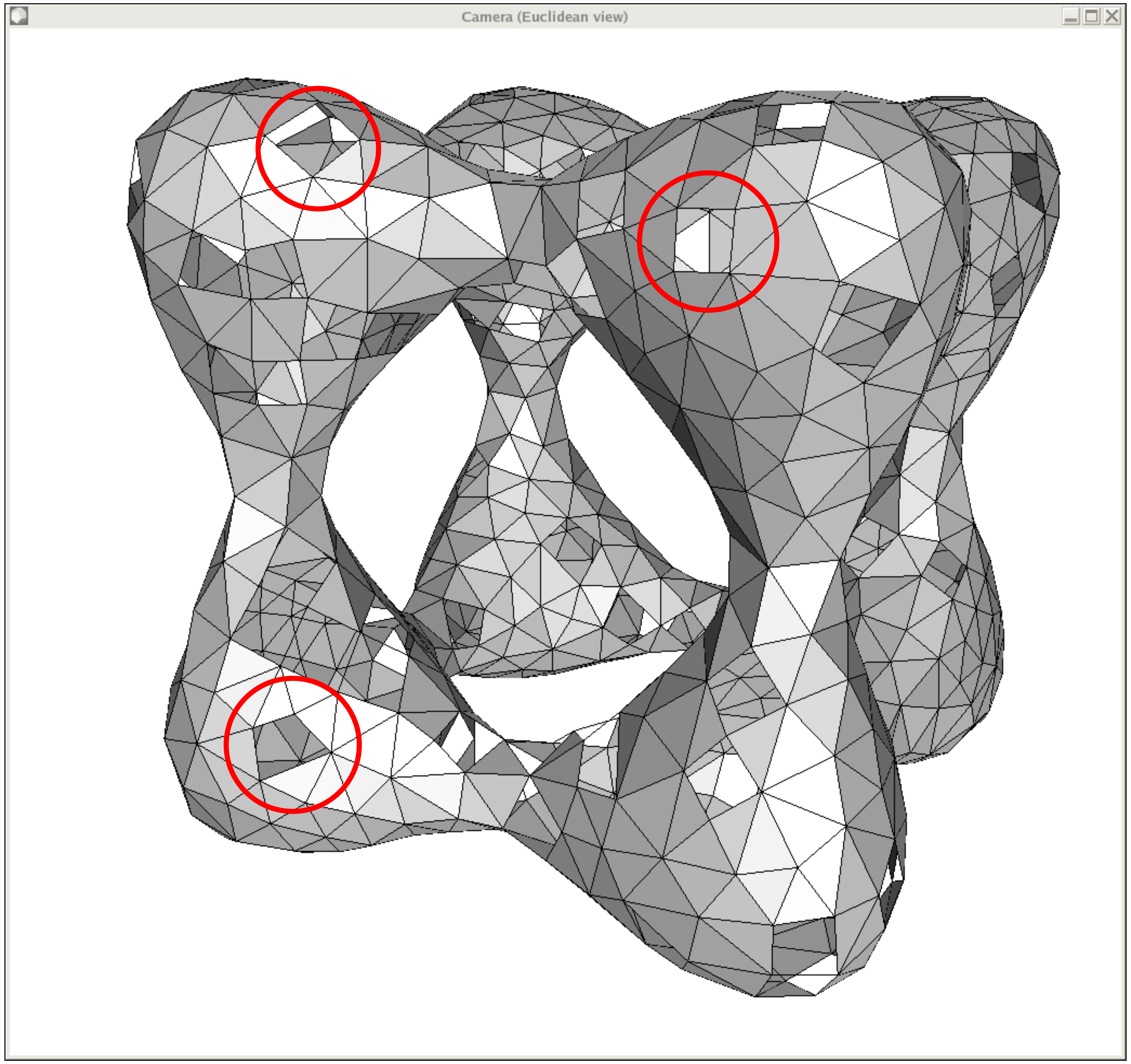
Output:



Relation with the restricted Delaunay

(case of surfaces)

$\mathcal{D}^M(L) \not\subseteq \mathcal{C}^W(L)$ if $W \subsetneq M$



$\varepsilon = 0.2$, $\mathrm{rch}(M) \approx 0.25$

Relation with the restricted Delaunay

(case of surfaces)

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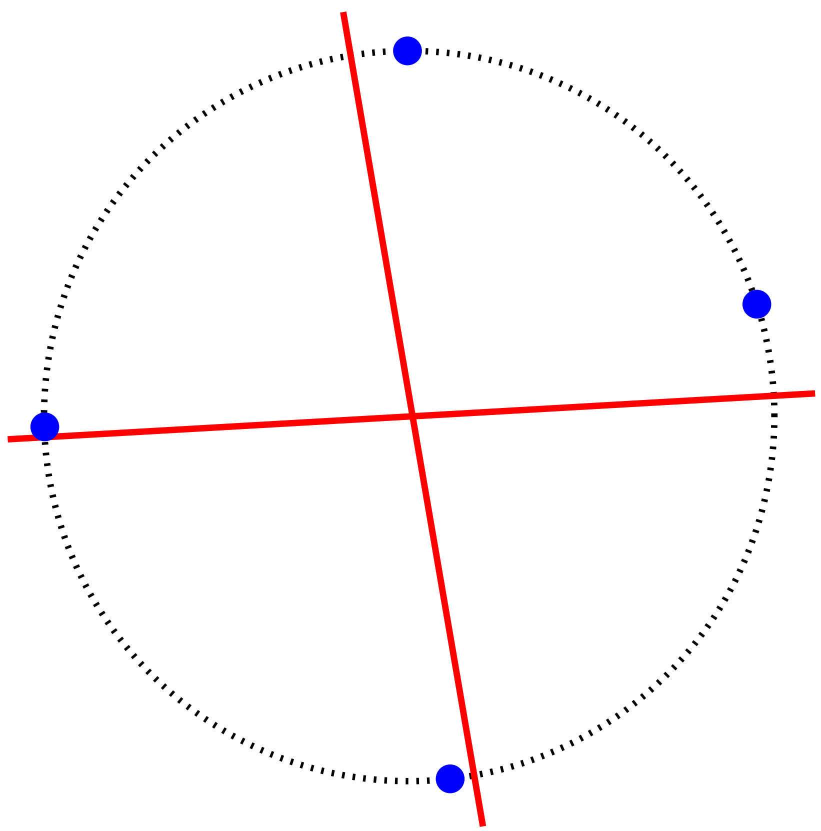
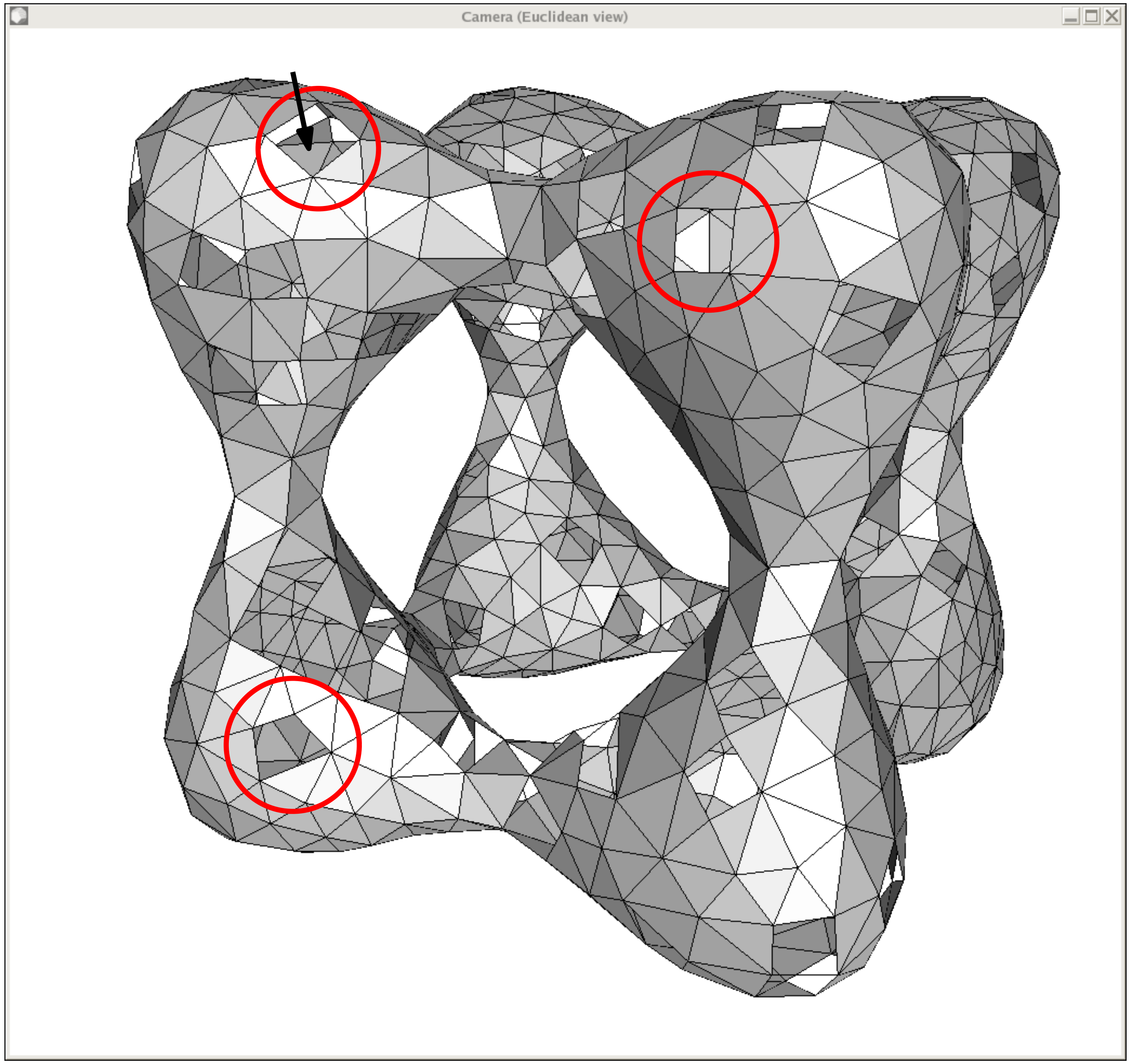


order-2 Voronoi diagram



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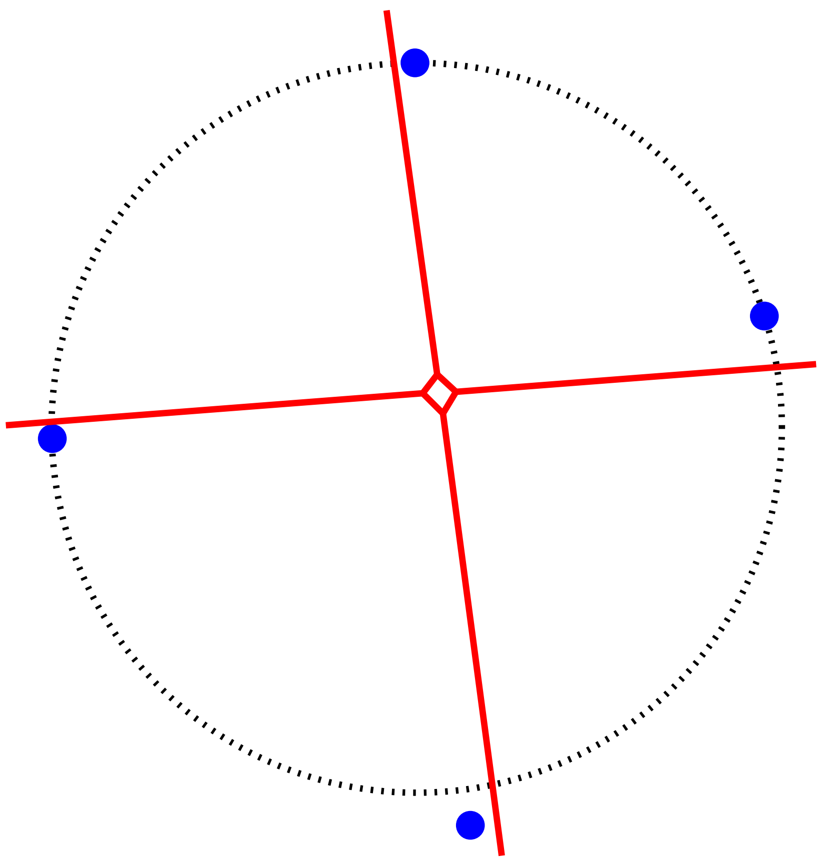

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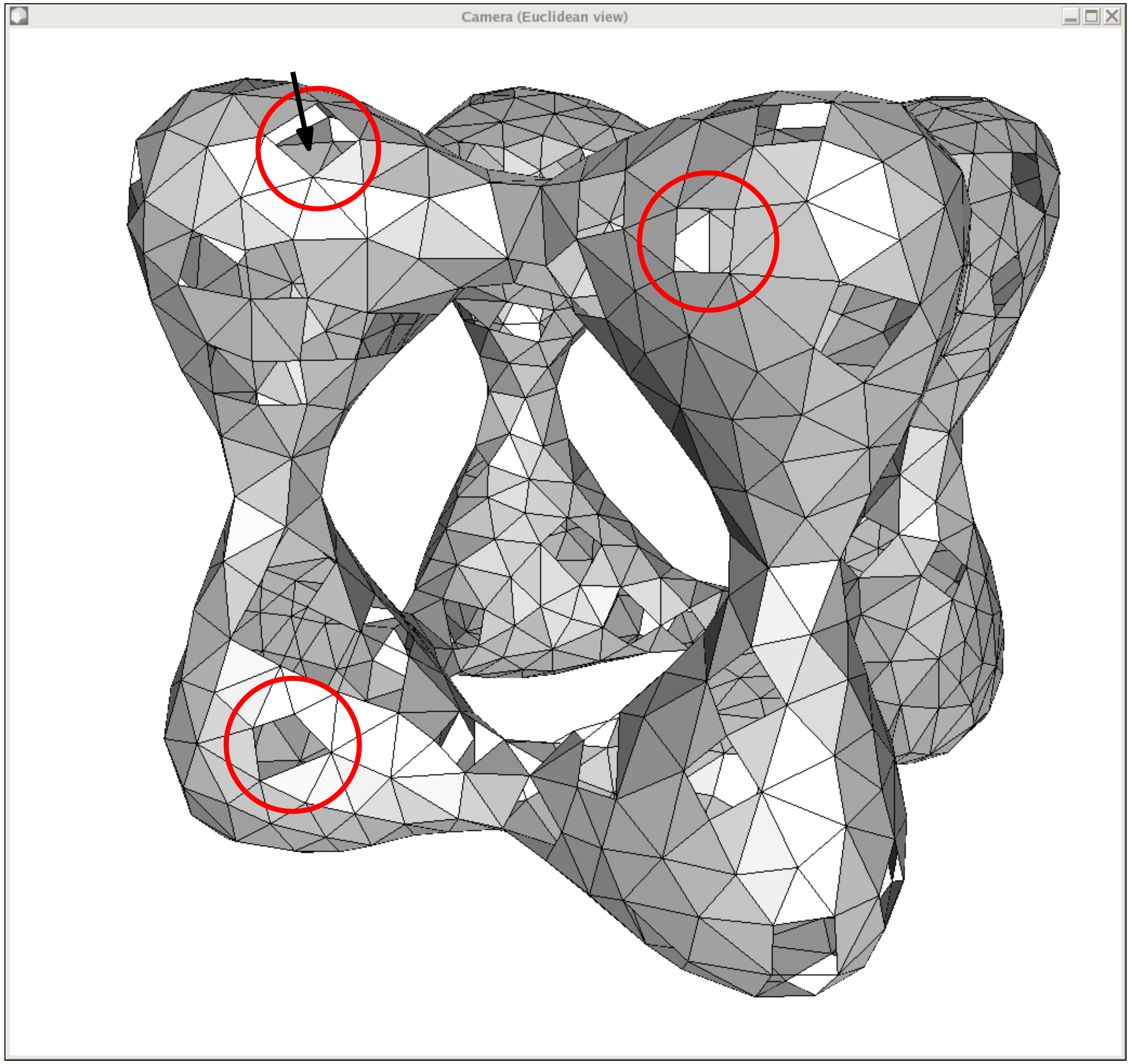


$\varepsilon = 0.2$, $\mathrm{rch}(M) \approx 0.25$

Relation with the restricted Delaunay

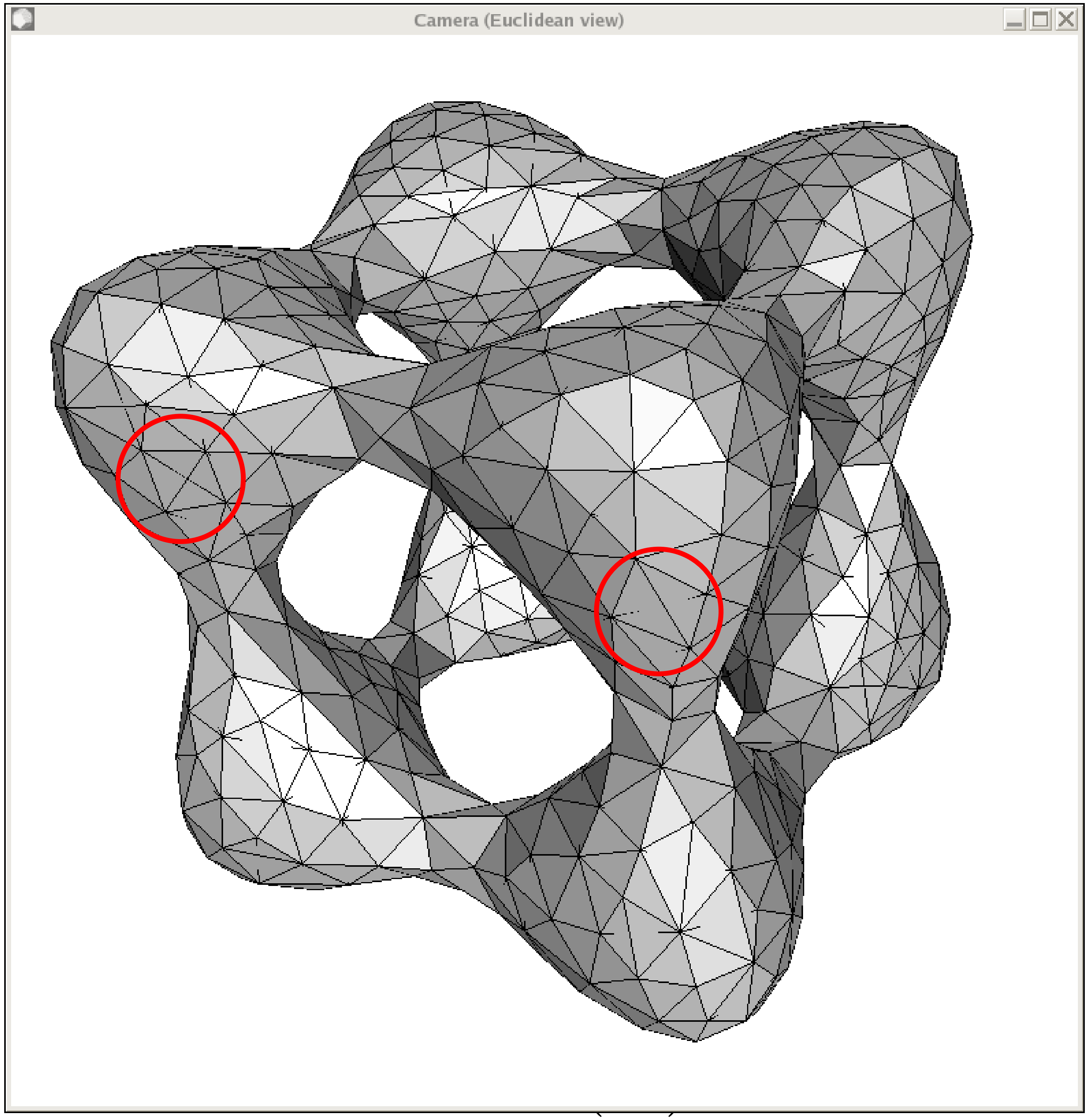
(case of surfaces)

$$\mathcal{D}^M(L) \not\subseteq \mathcal{C}^W(L) \text{ if } W \subsetneq M$$

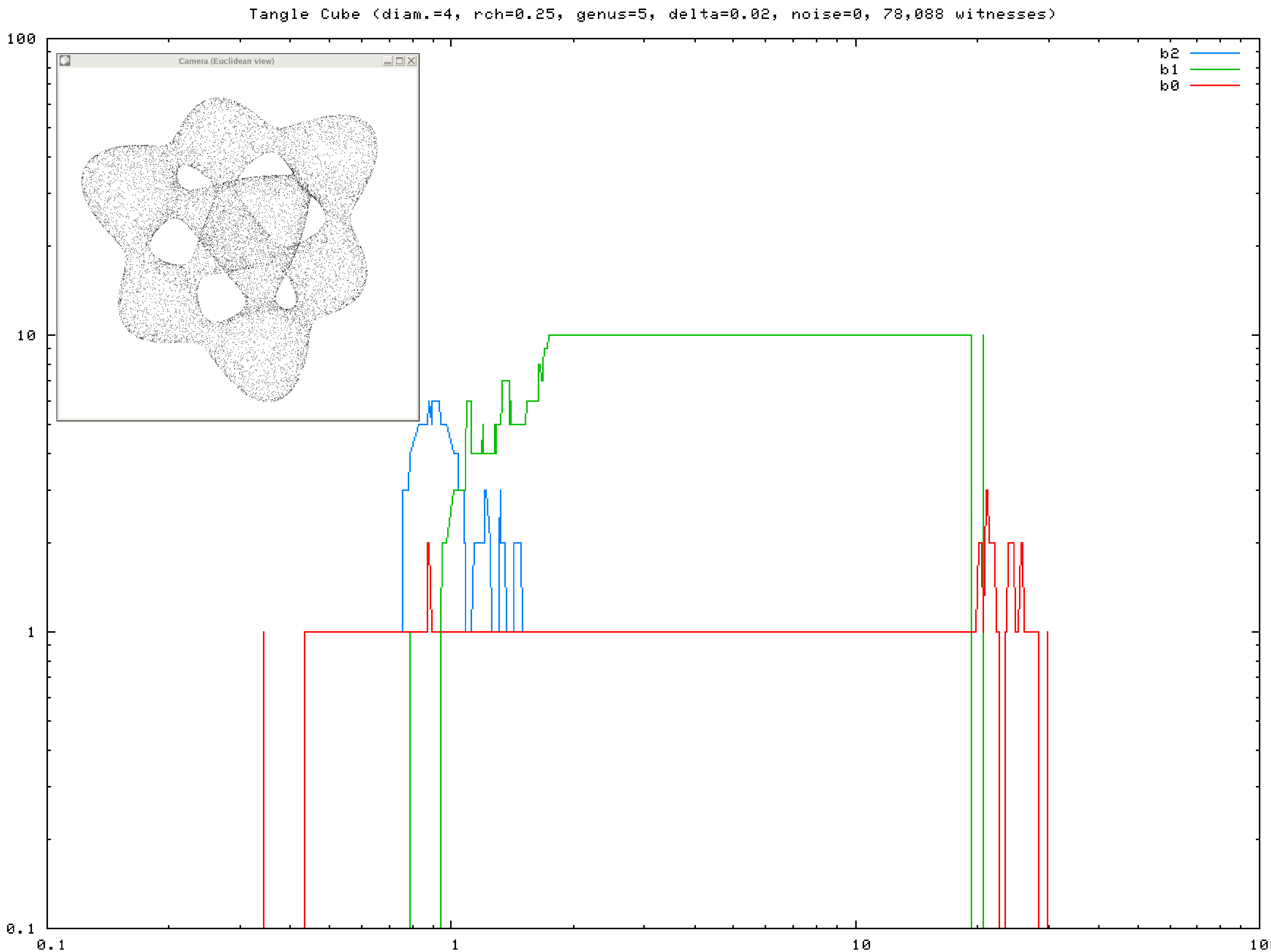
Solution relax witness test.

$\Rightarrow \mathcal{C}_\nu^W(L) = \mathcal{D}^M(L) +$ slivers
$\Rightarrow \mathcal{C}_\nu^W(L) \not\subseteq \mathcal{D}(L)$
$\Rightarrow \mathcal{C}_\nu^W(L)$ not embedded.

Post-process extract manifold M from $\mathcal{C}_\nu^W(L) \cap \mathcal{D}(L)$
[Amenta, Choi, Dey, Leekha]



Some results



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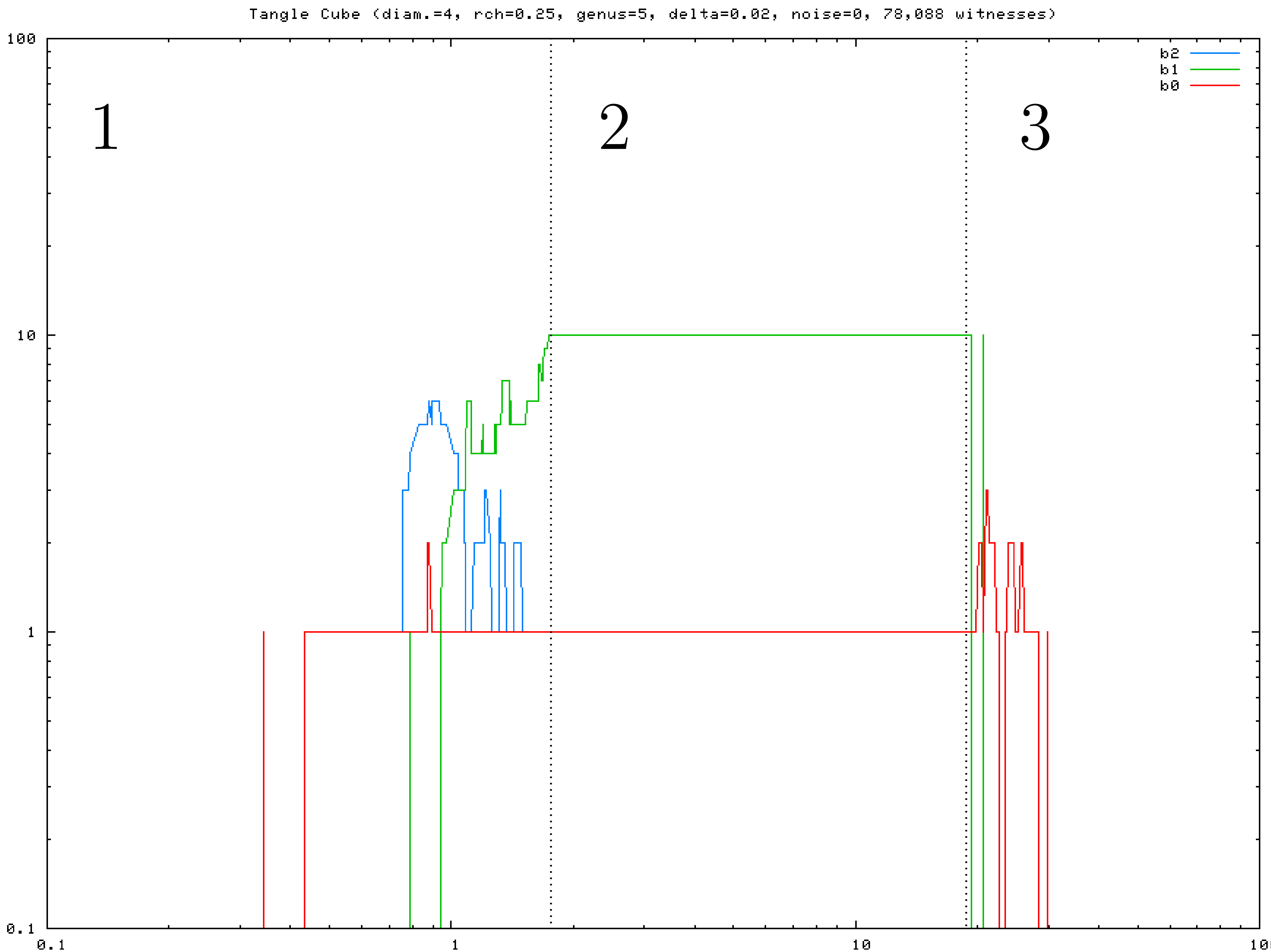
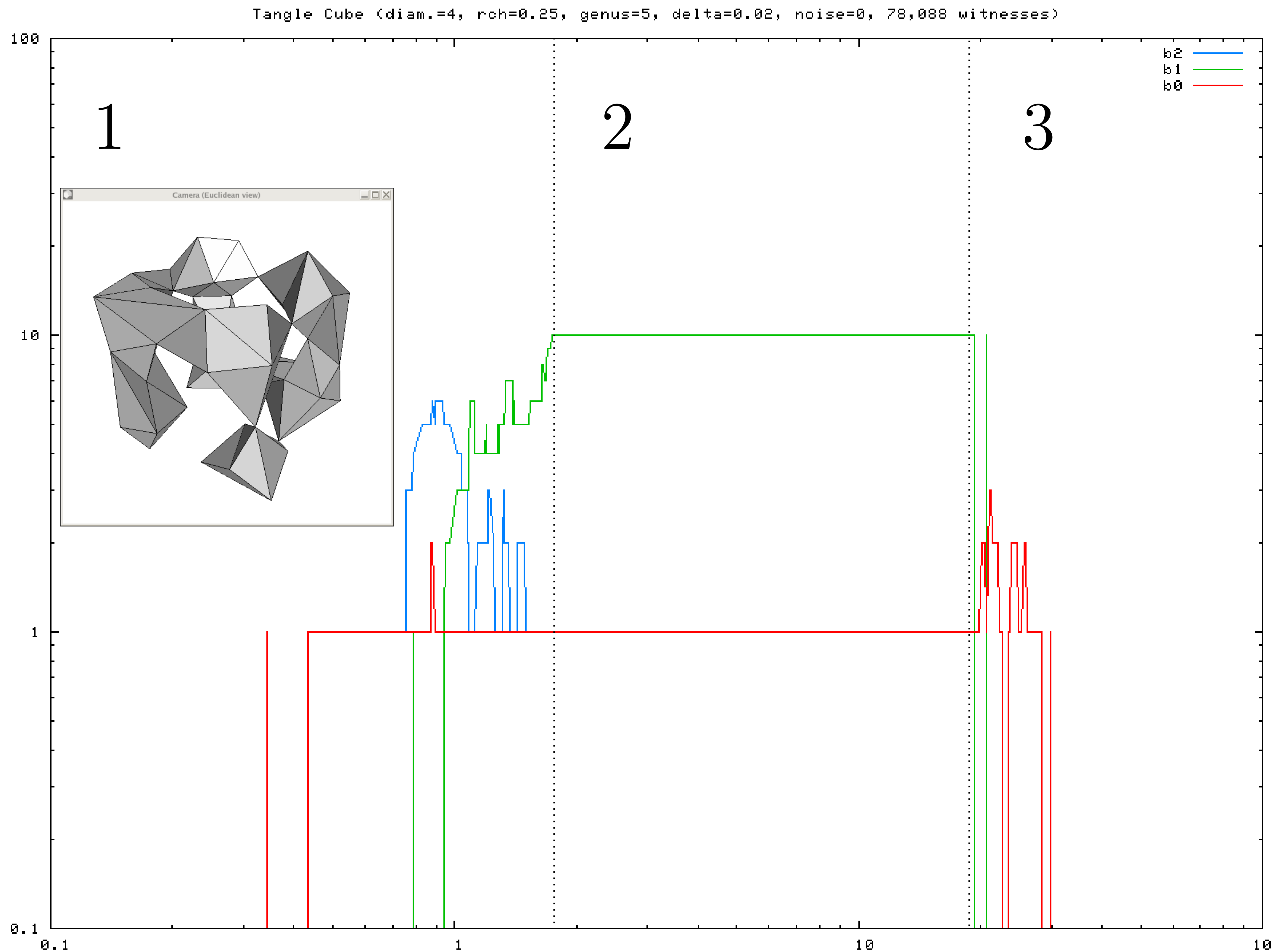


Some results



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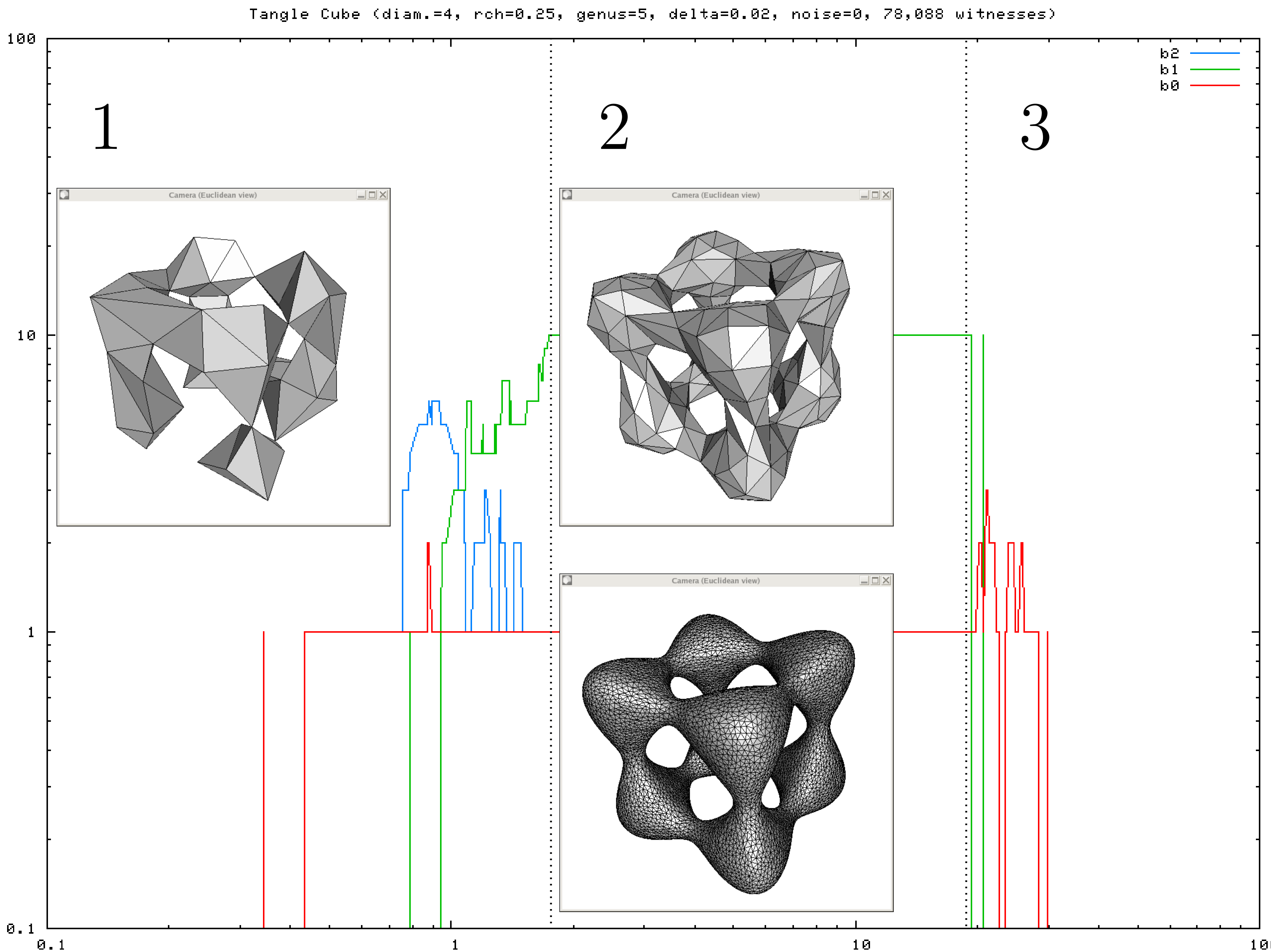

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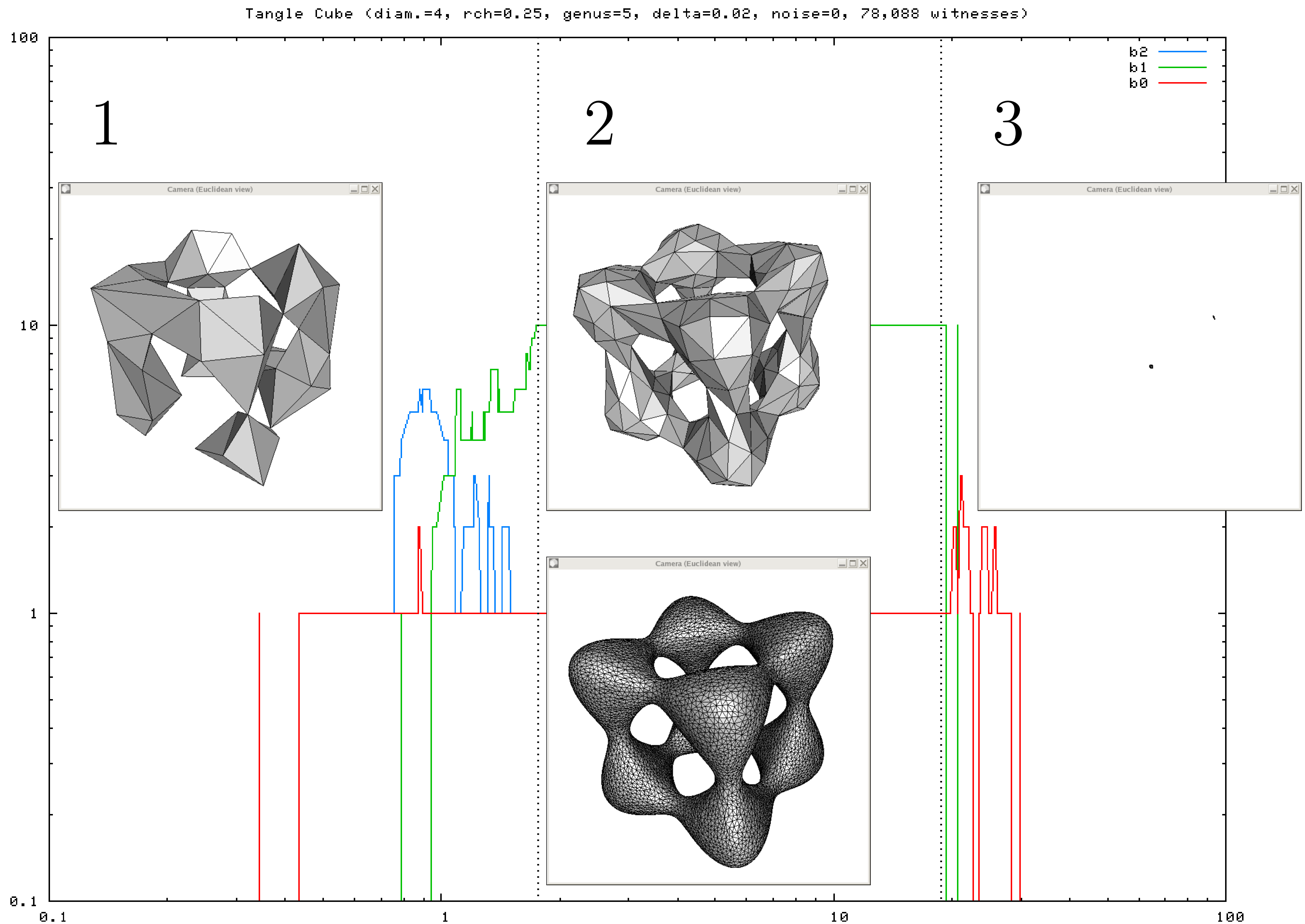

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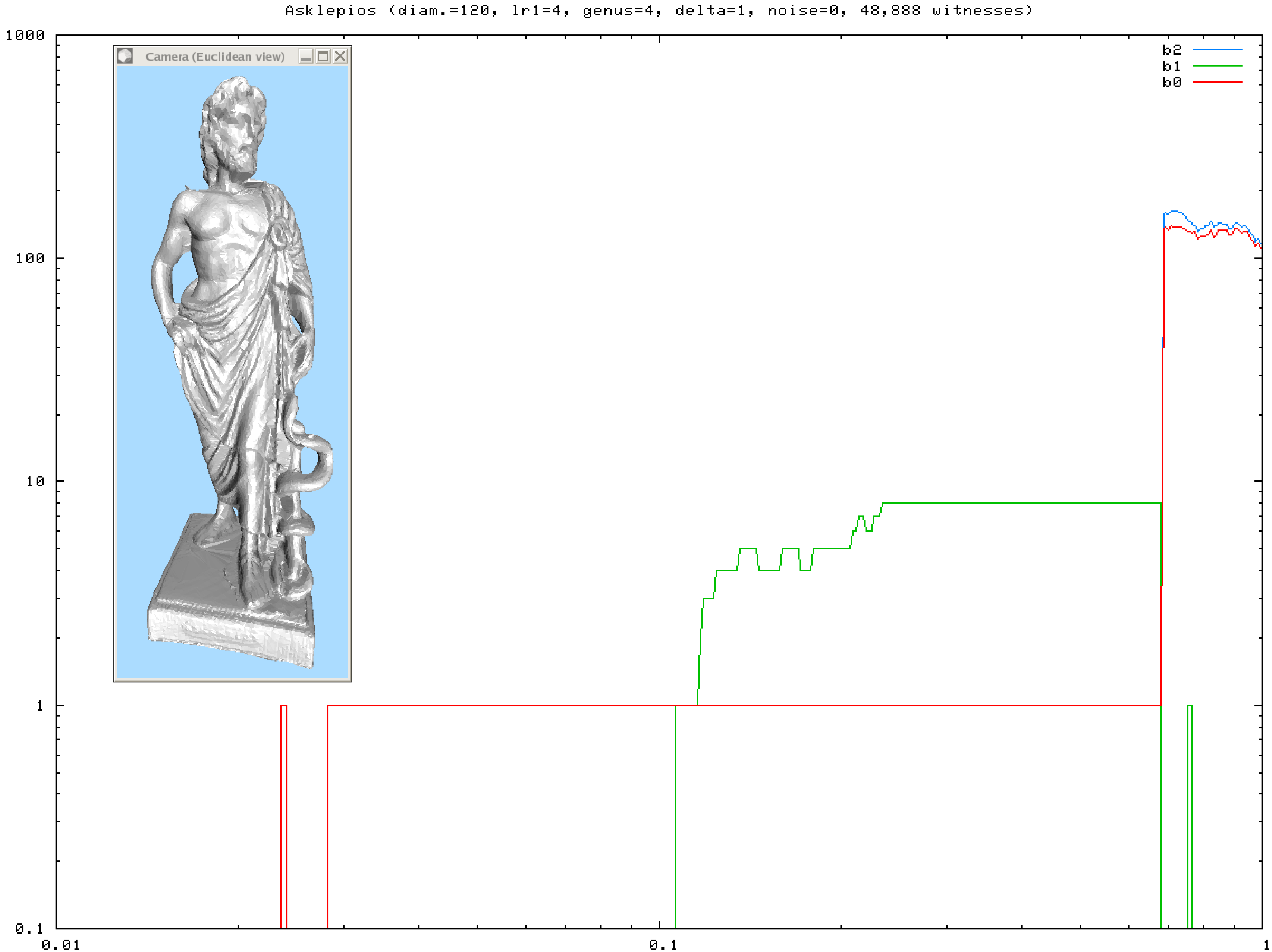
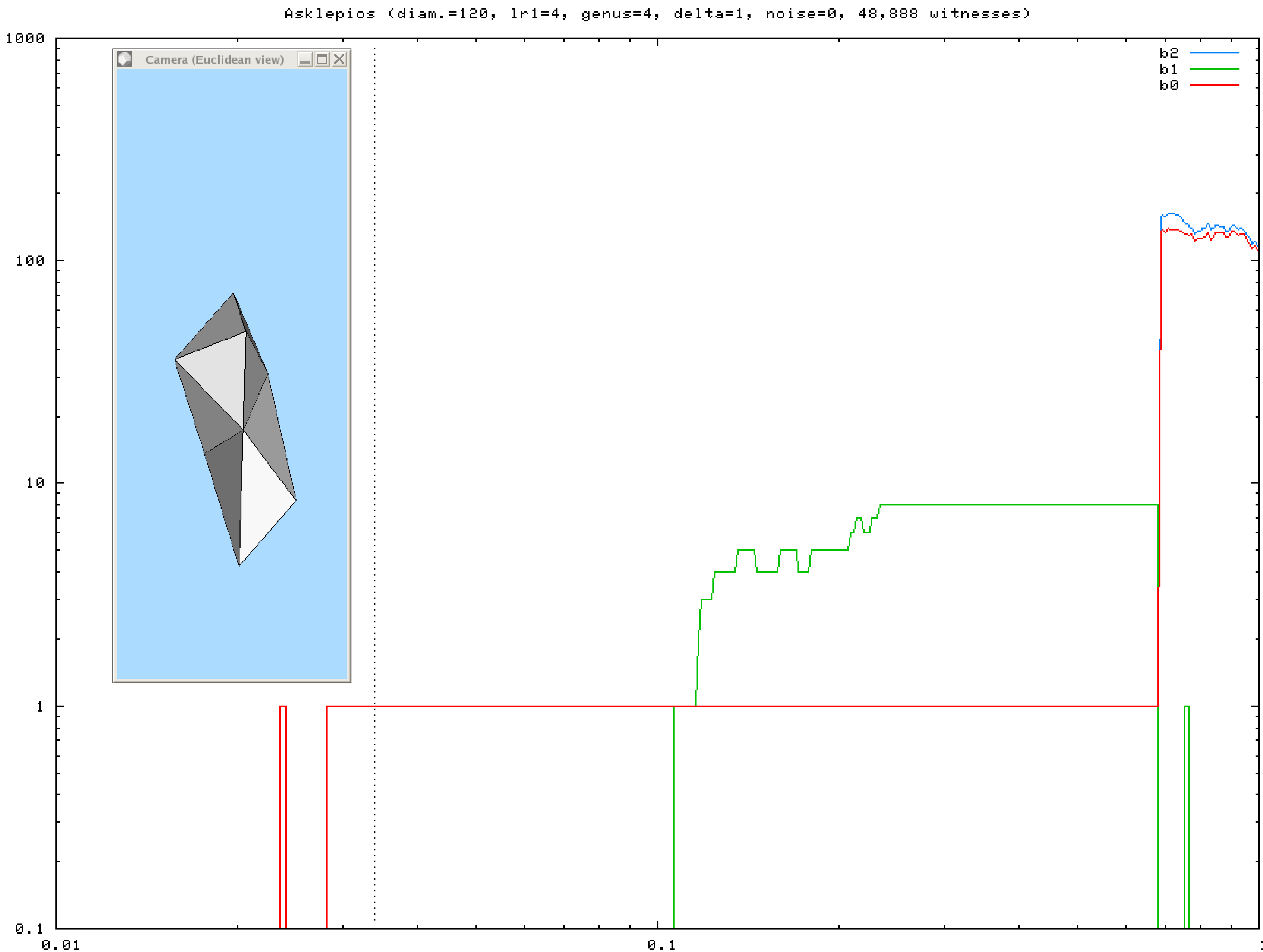


input model provided courtesy of IMATI by the Aim@Shape repository

Some results



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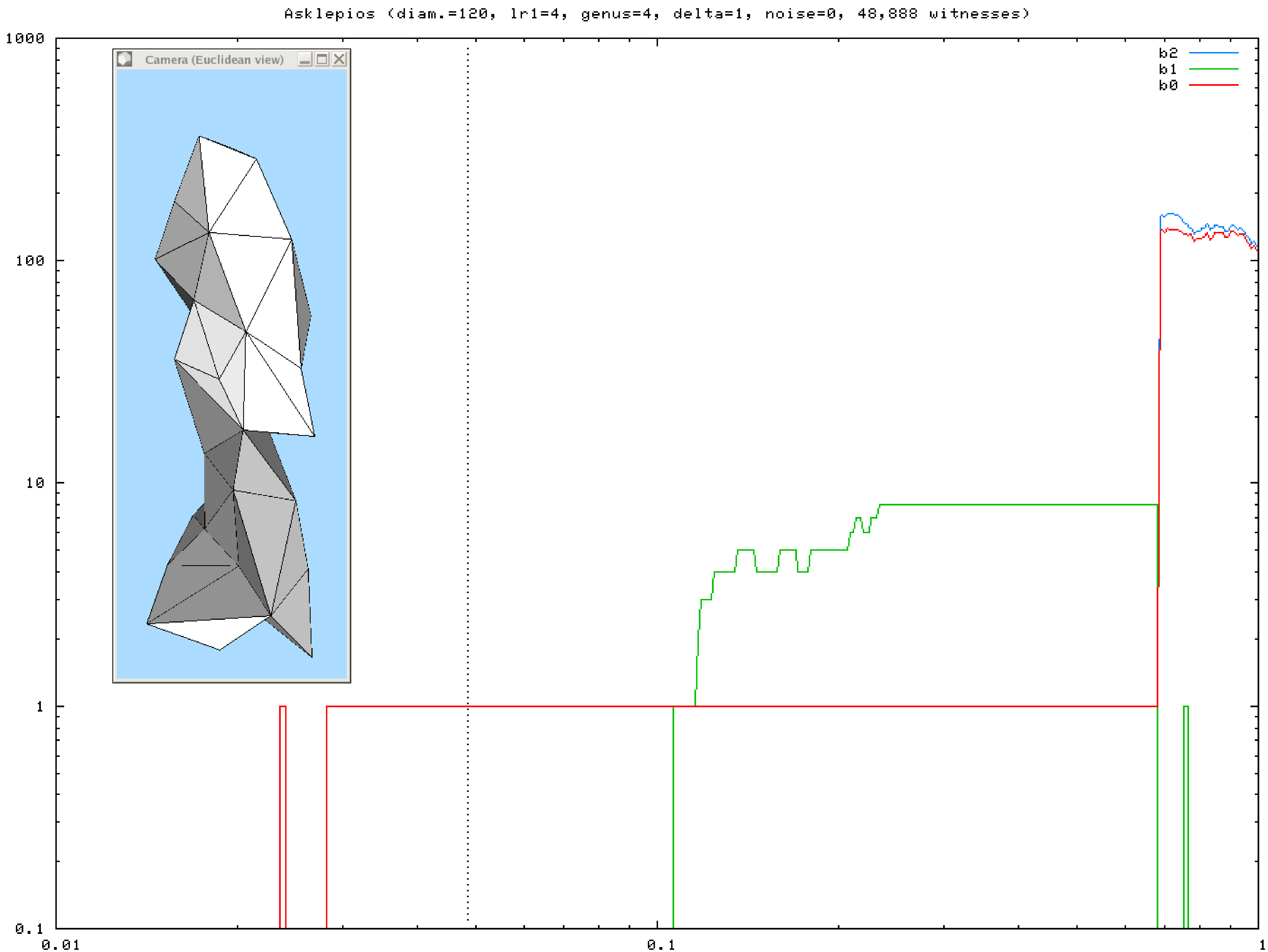

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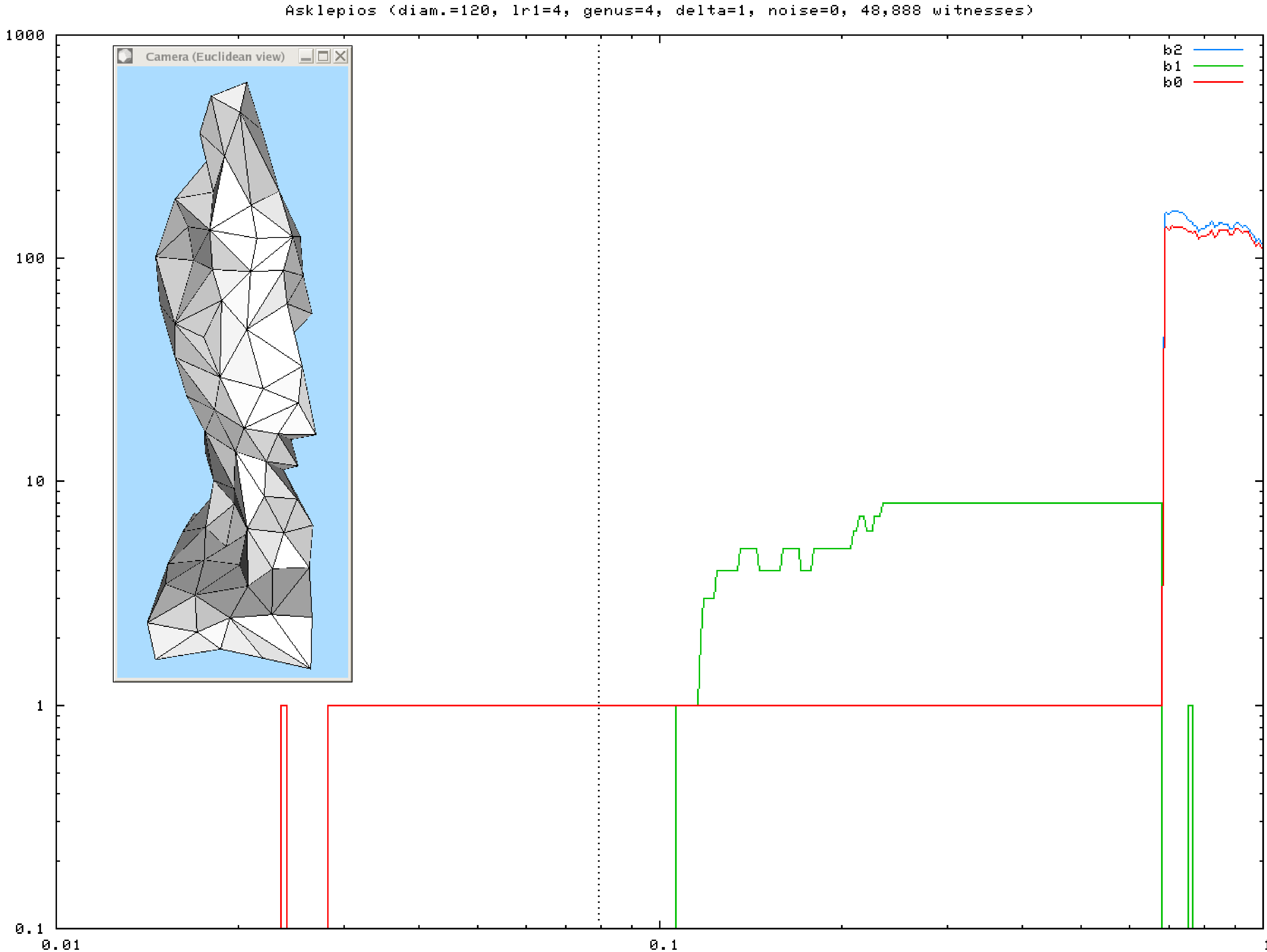

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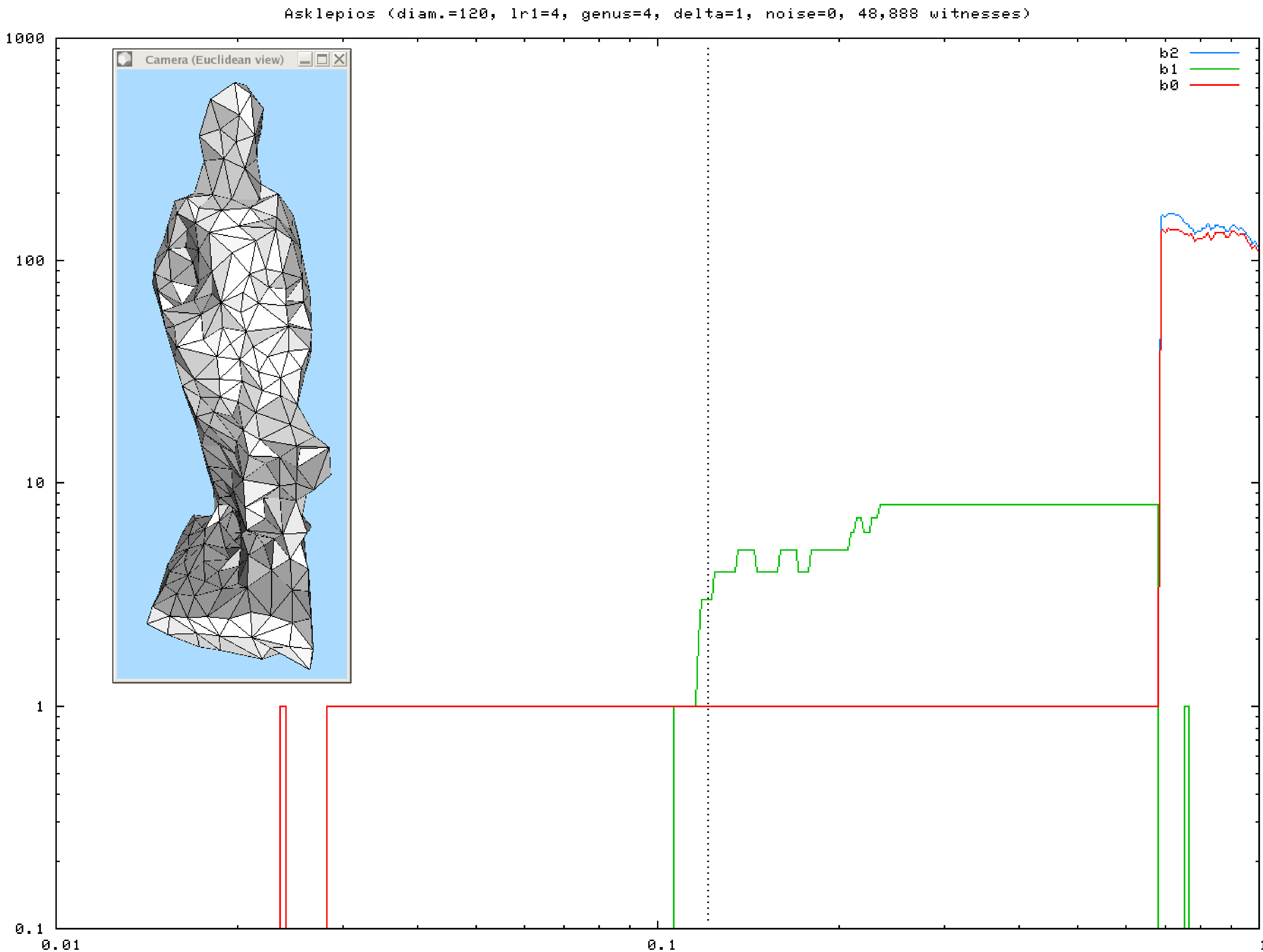

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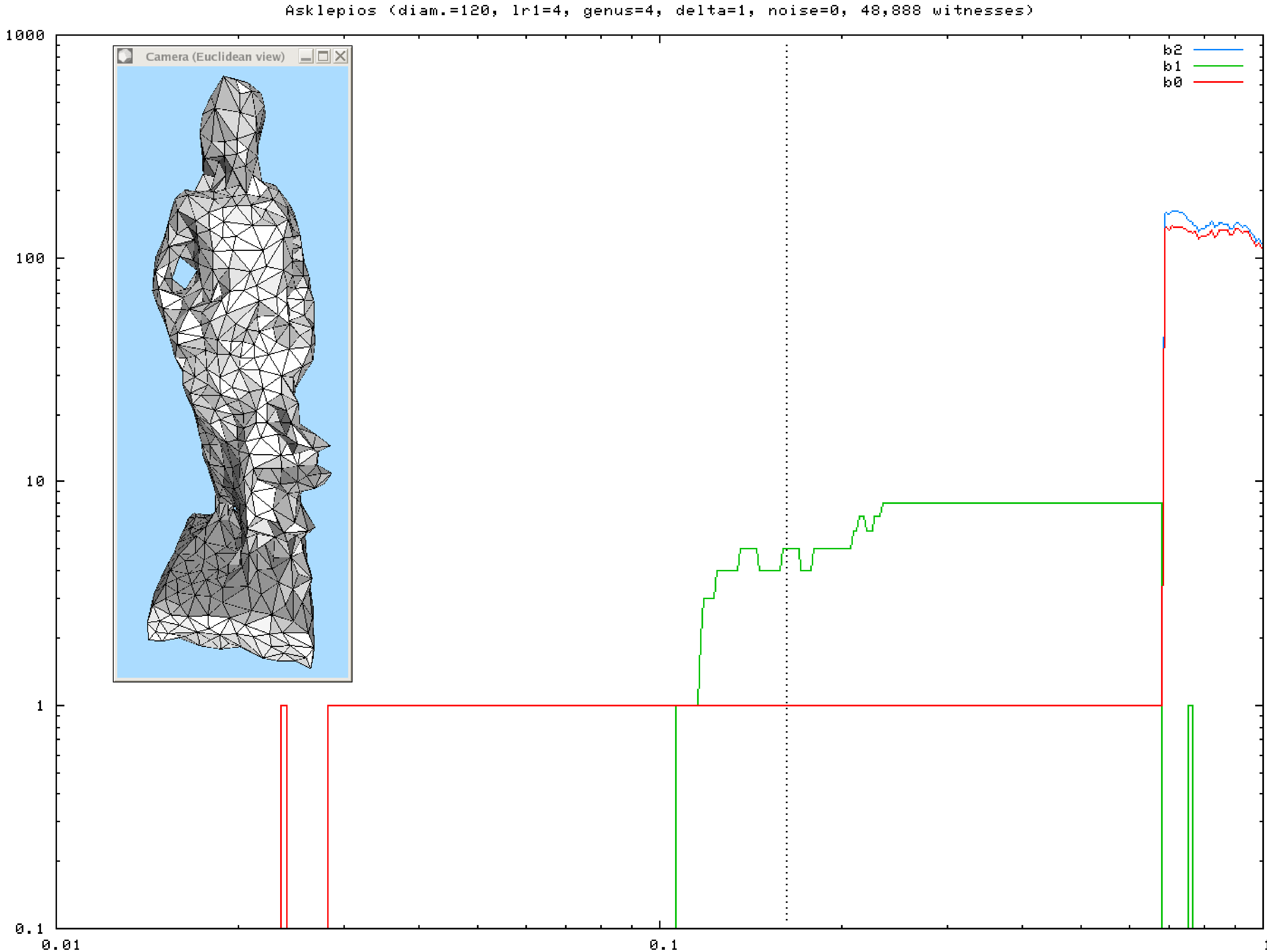

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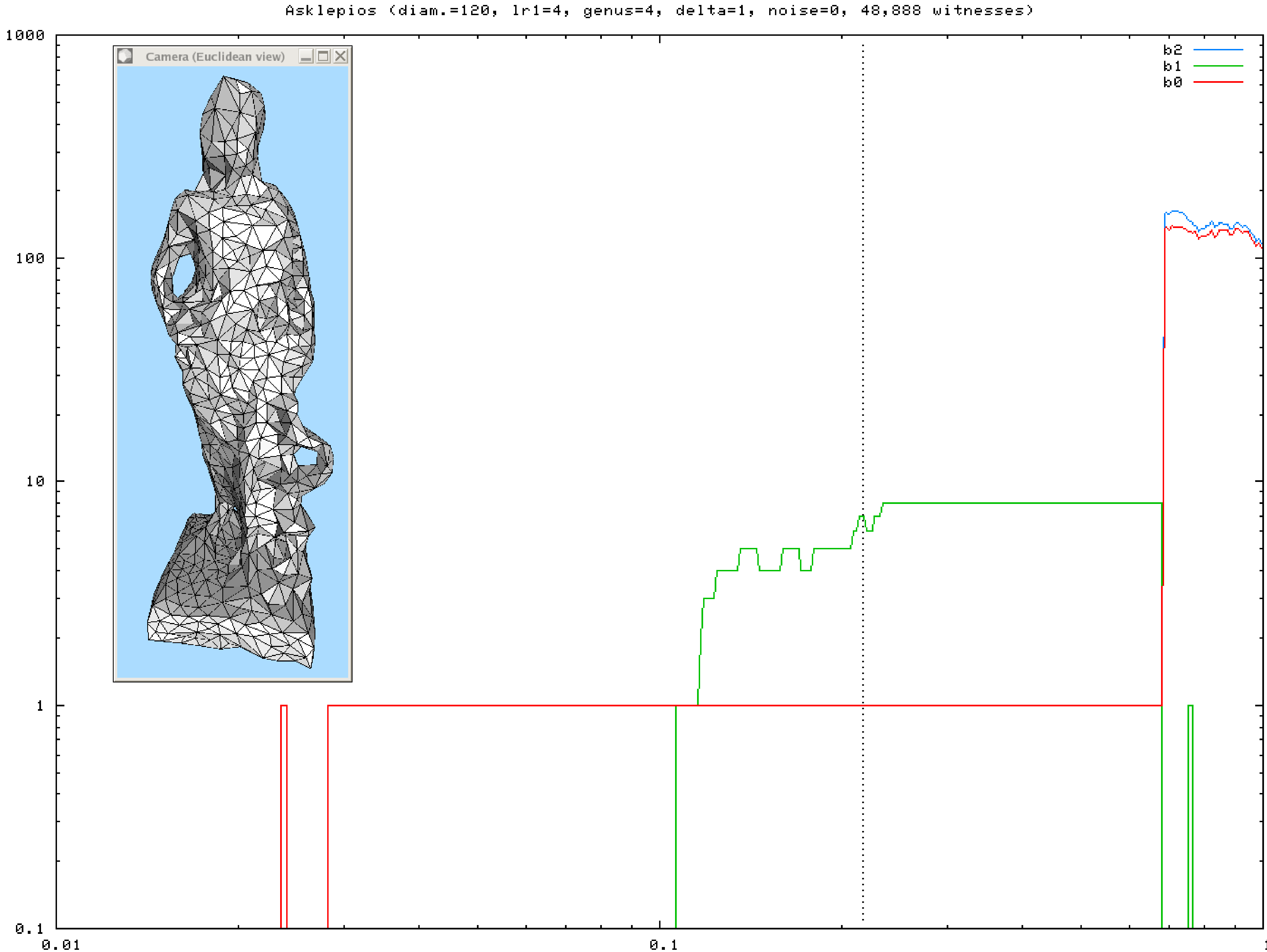

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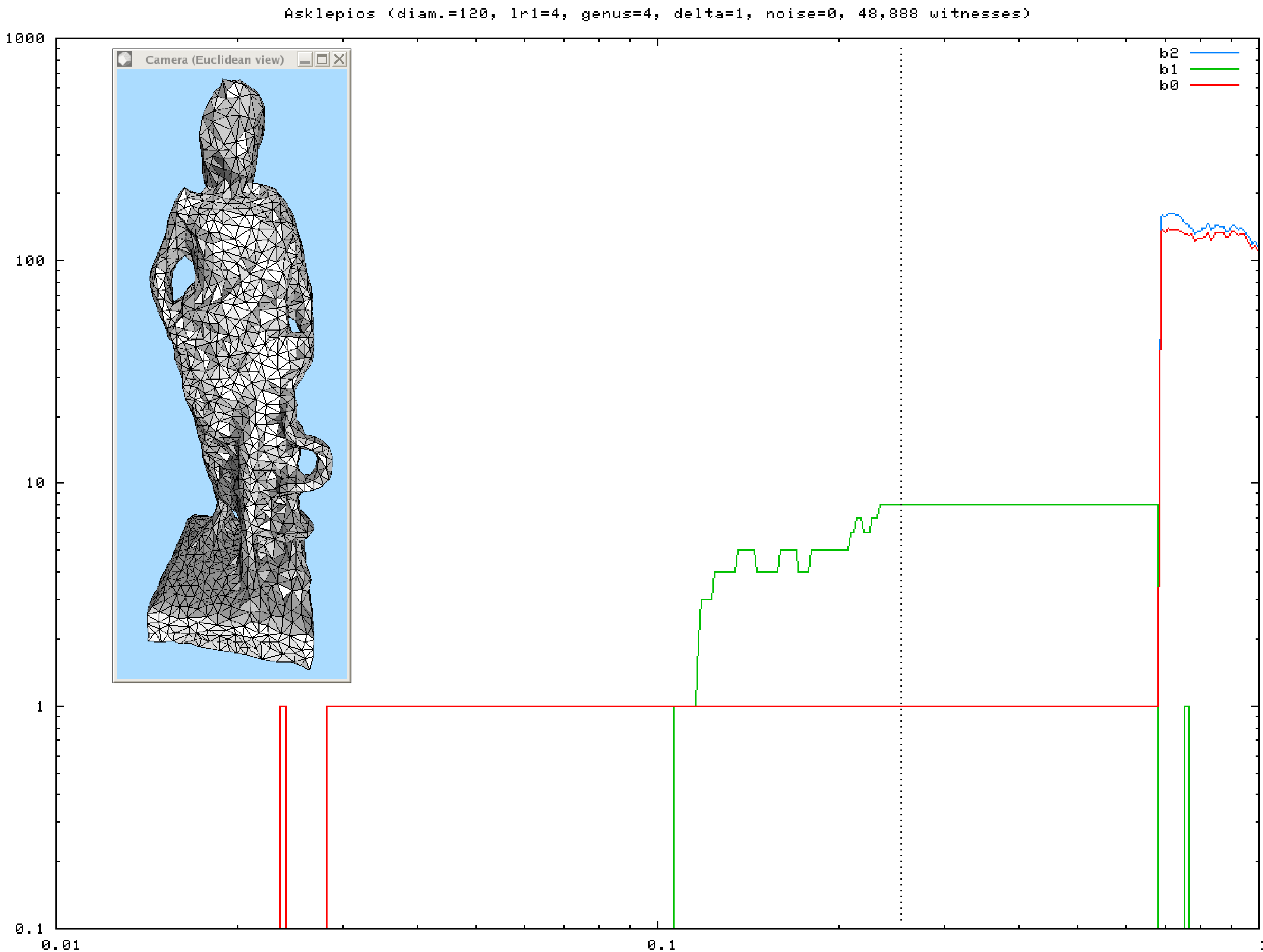
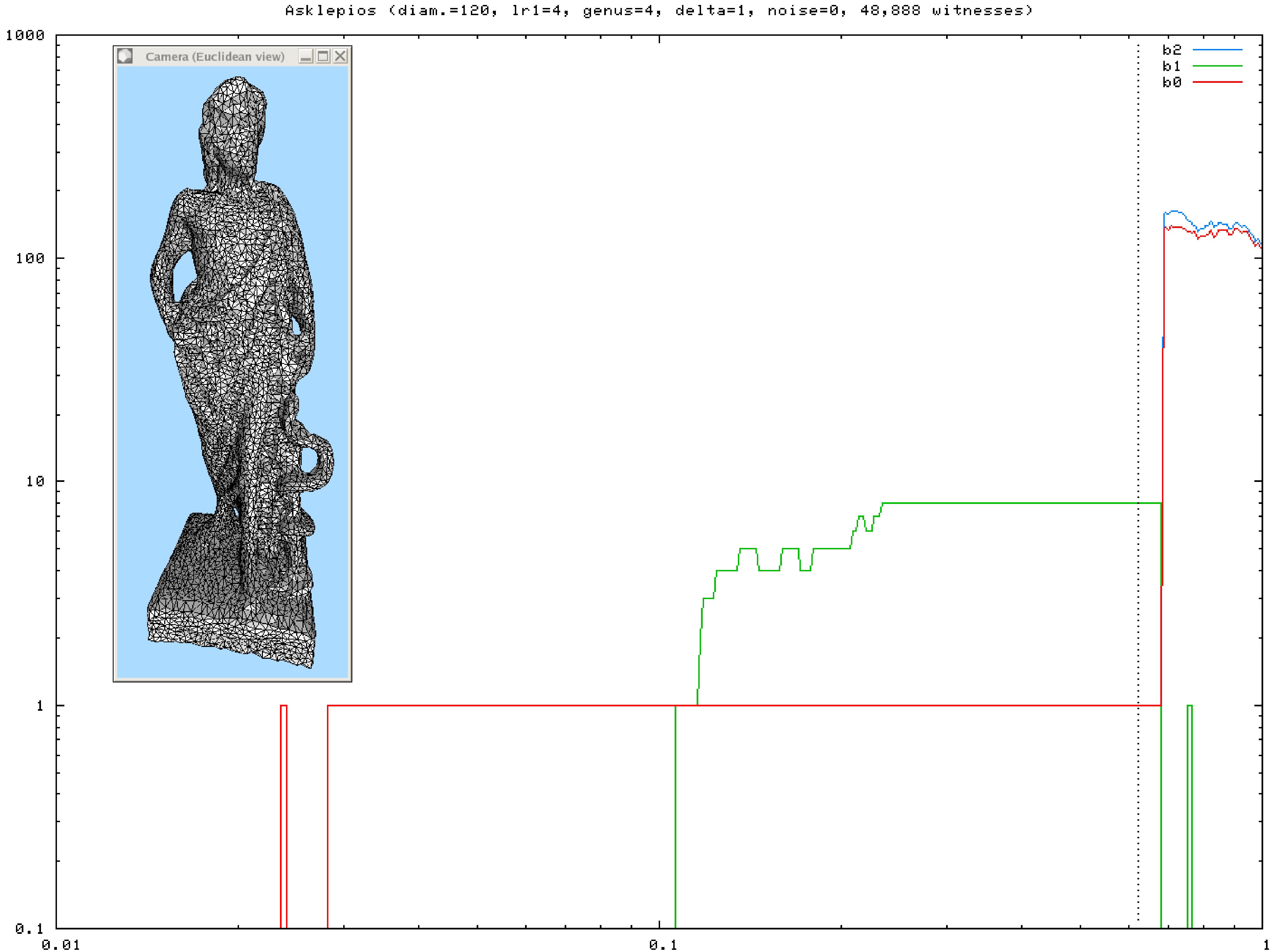


input model provided courtesy of IMATI by the Aim@Shape repository

Some results



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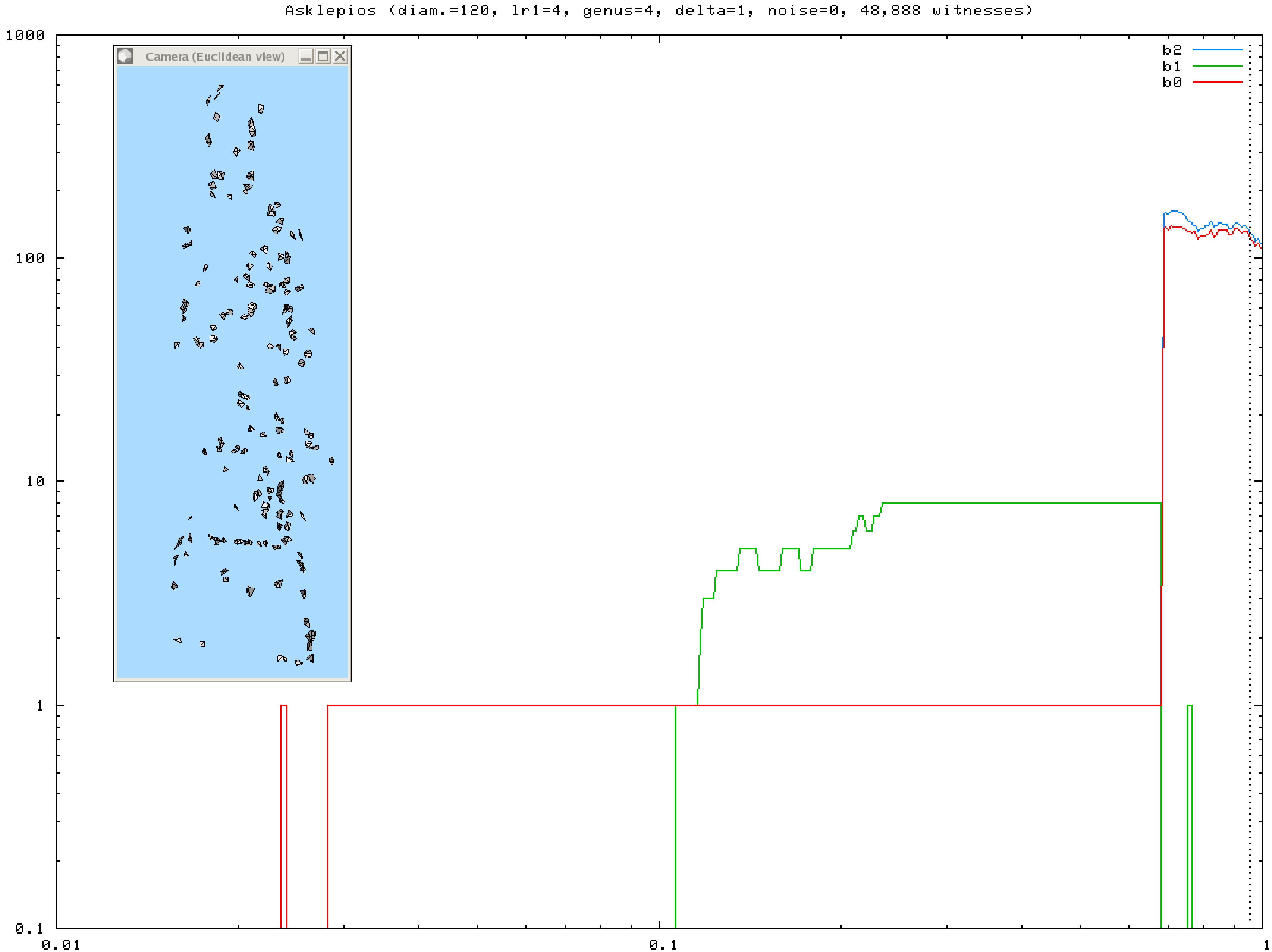

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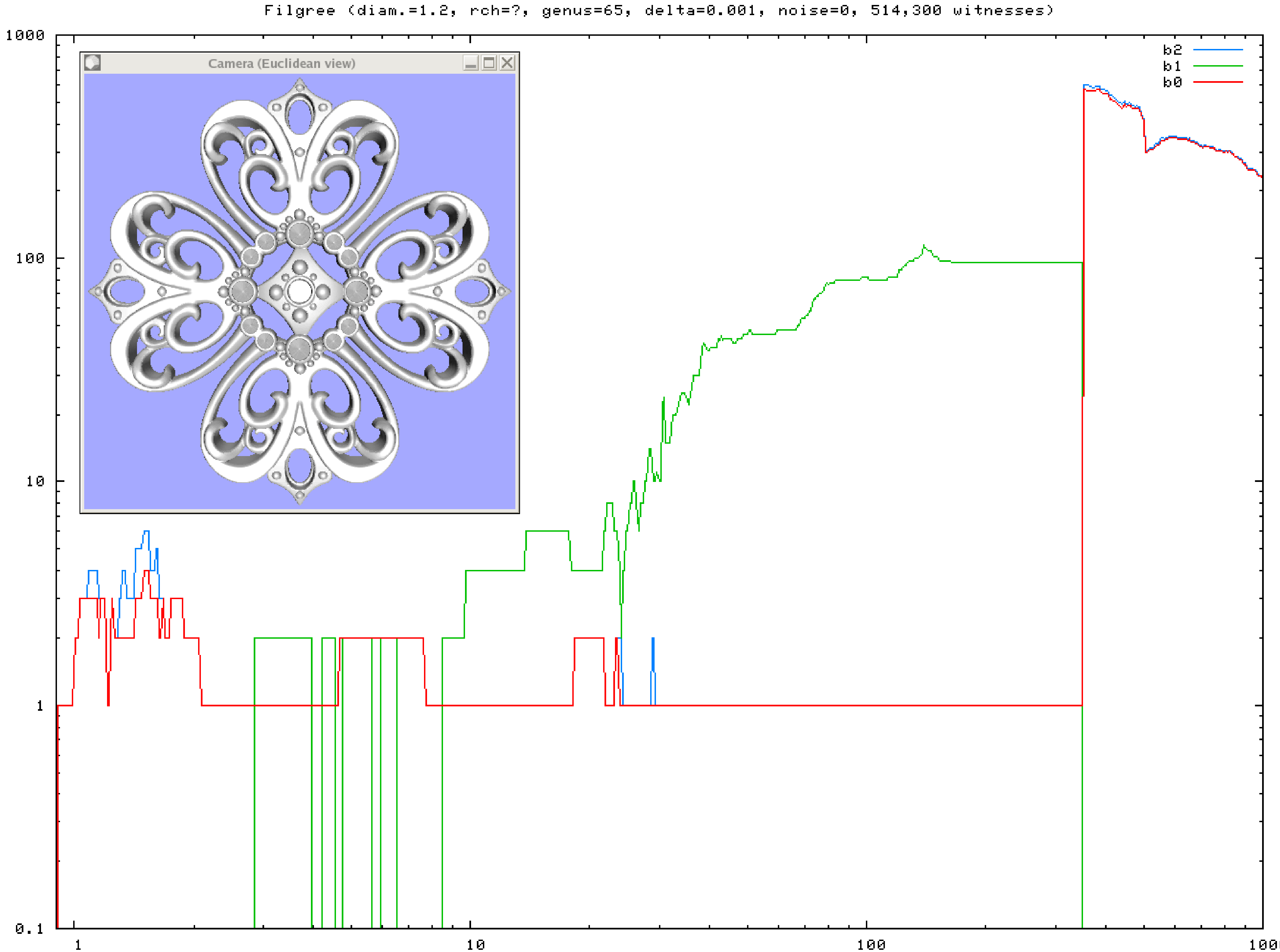

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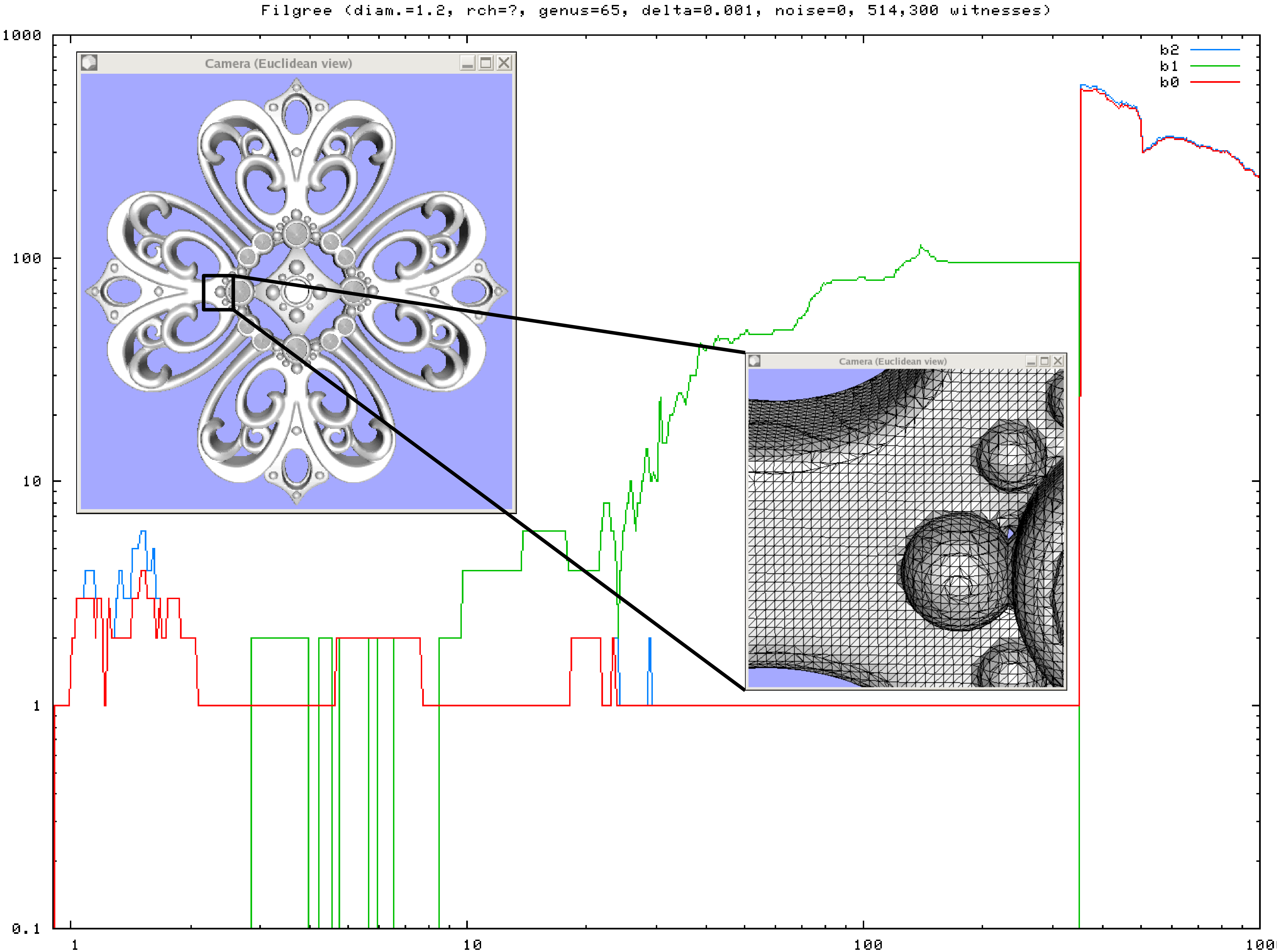

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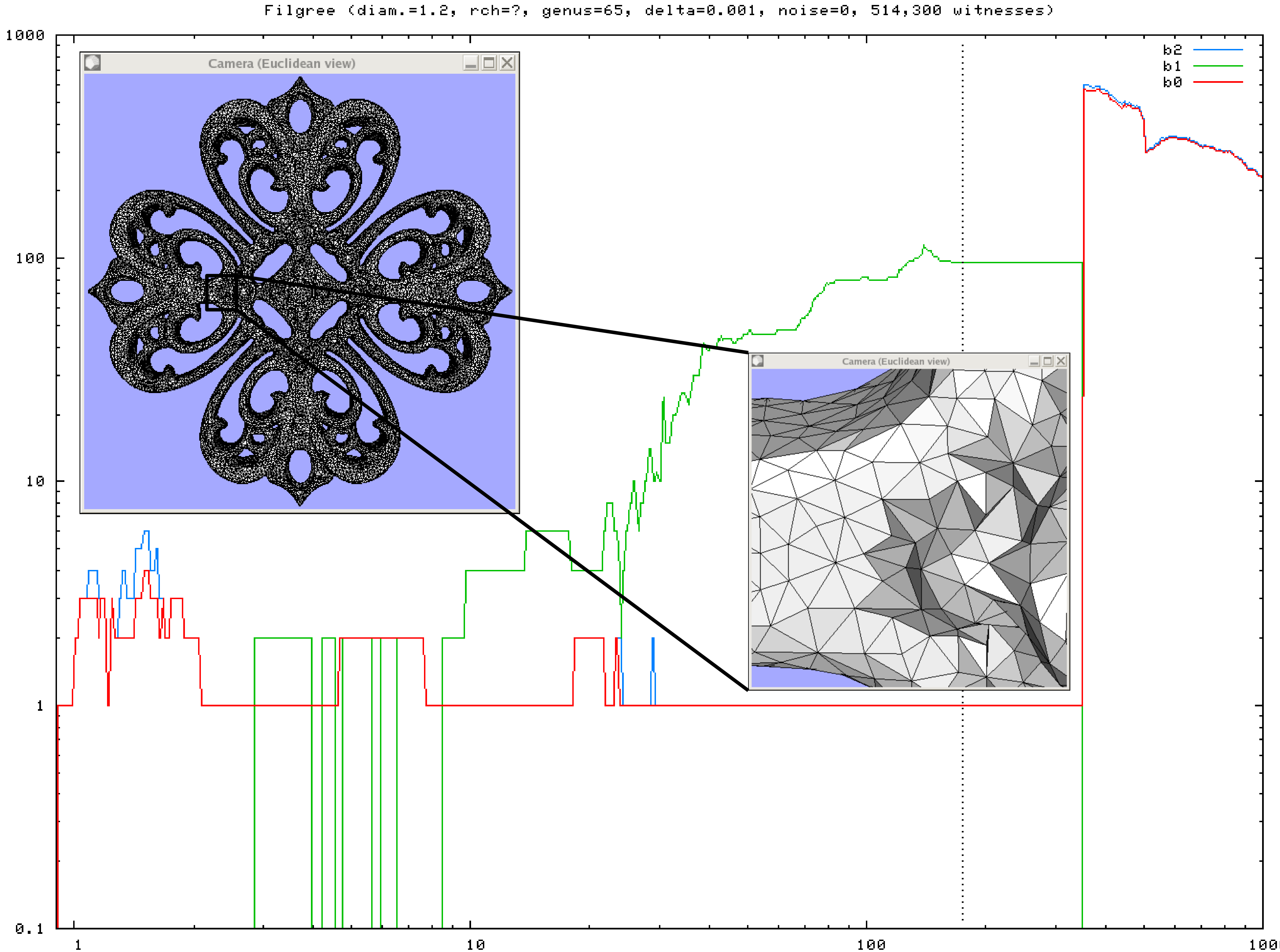

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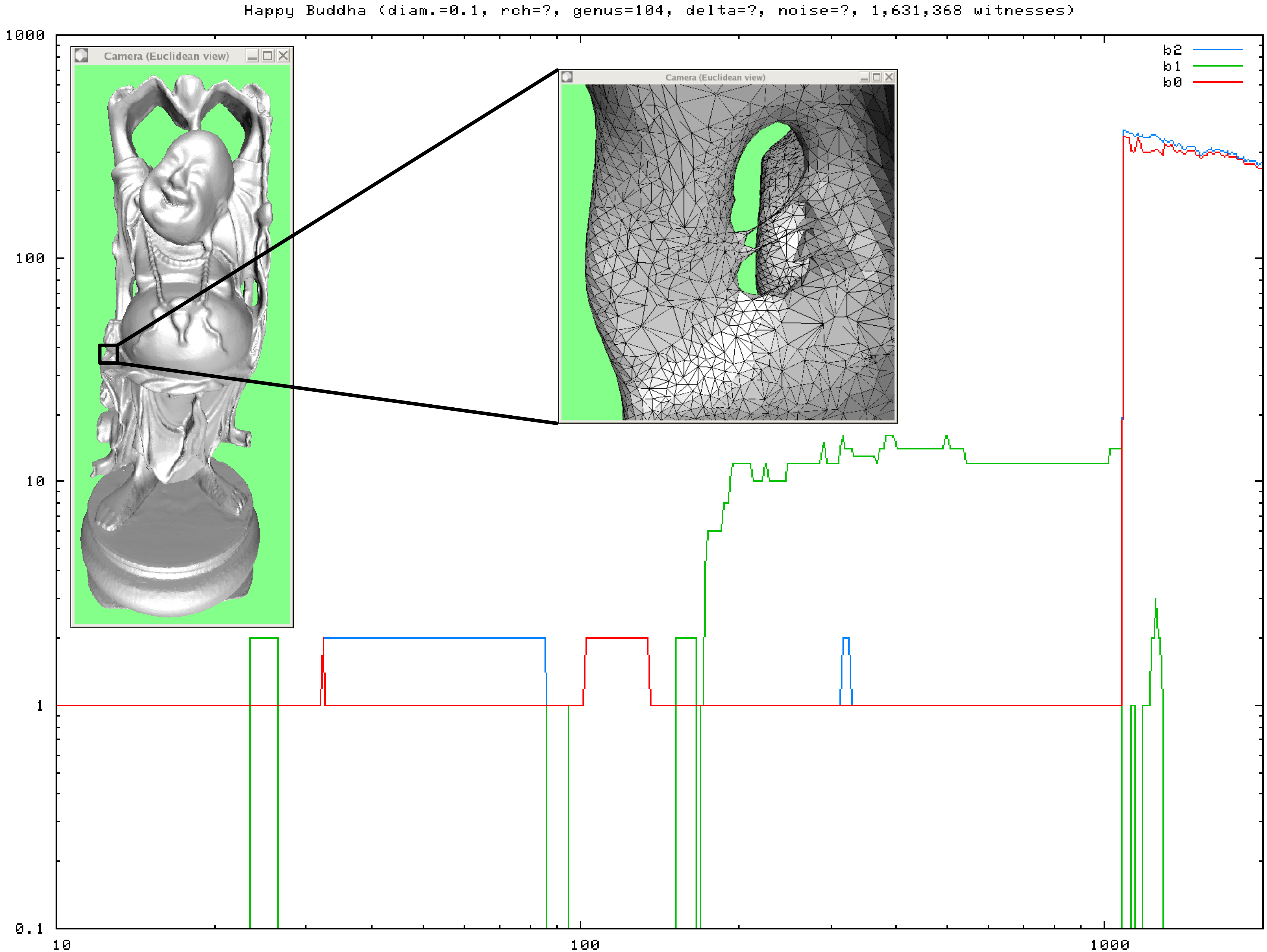

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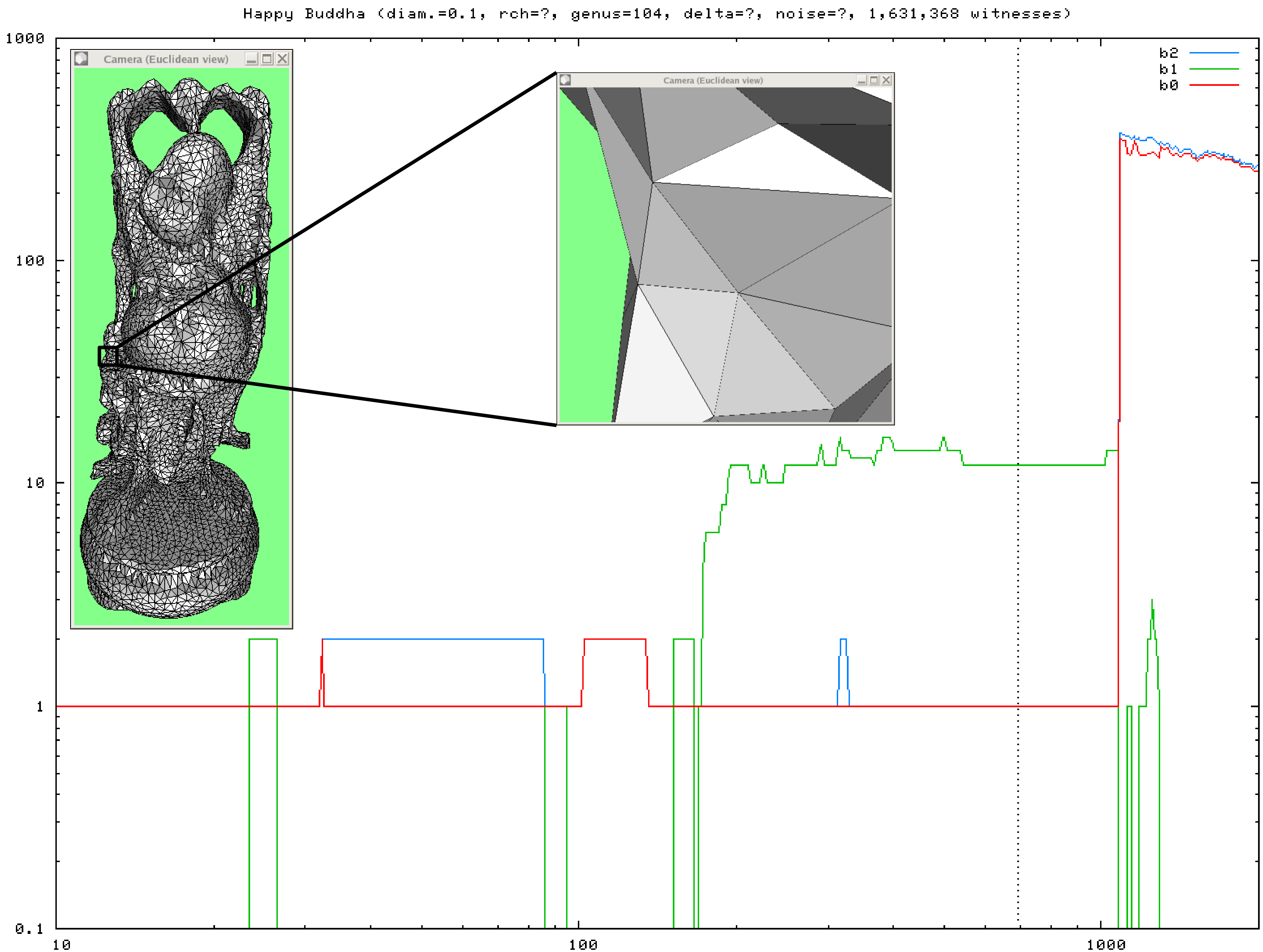

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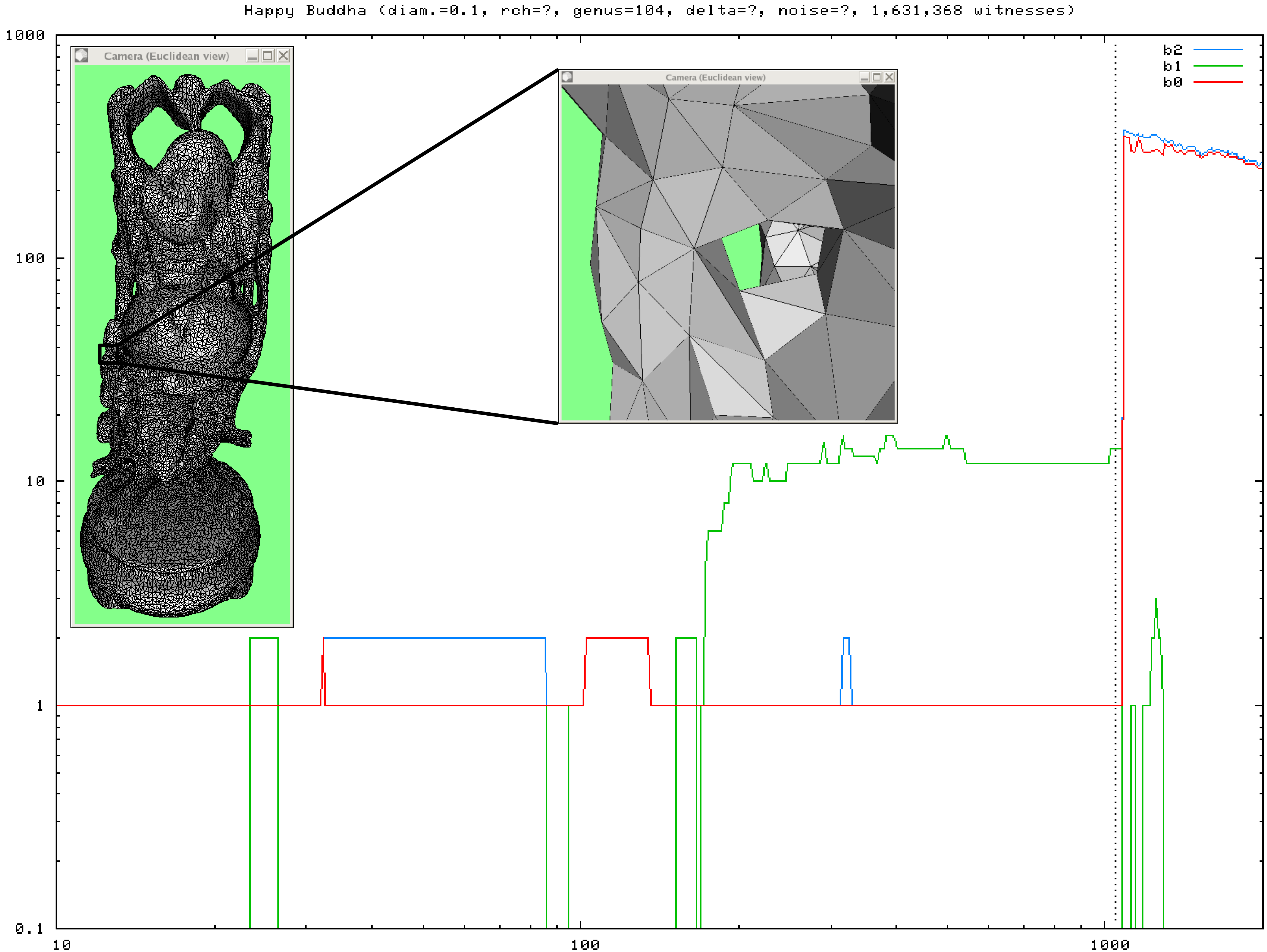

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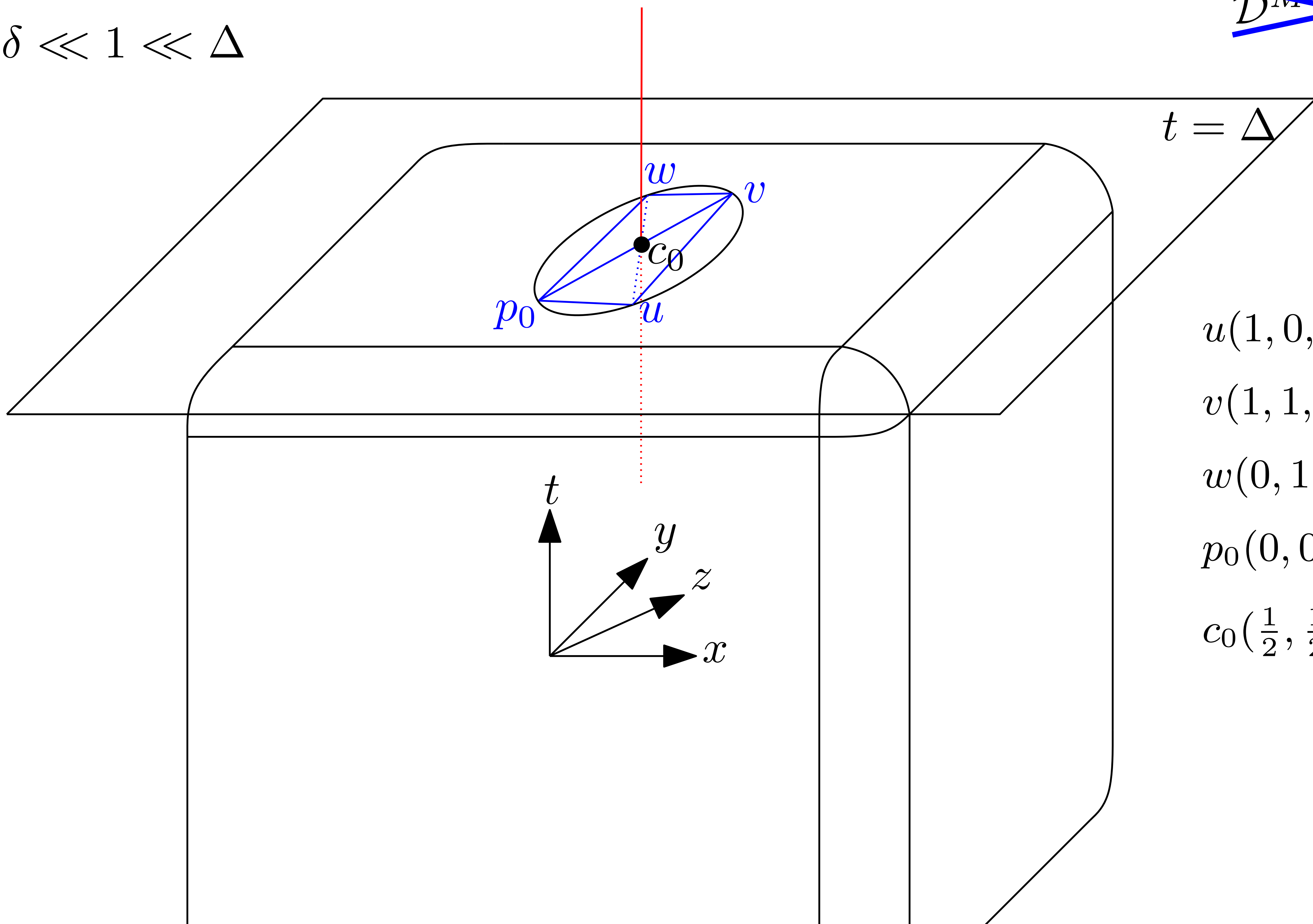
Relation with the restricted Delaunay

(intrinsic dim. ≥ 3) [O. 2007]

$\partial[-\Delta,\Delta]^4 \subset \mathbb{R}^4$

$\delta \ll 1 \ll \Delta$

$\cancel{\mathcal{D}^M(L) \not\approx M}$



$u(1,0,0,\Delta)$

$v(1,1,0,\Delta)$

$w(0,1,0,\Delta)$

$p_0(0,0,\delta,\Delta)$

$c_0(\frac{1}{2},\frac{1}{2},\frac{\delta}{2},\Delta)$

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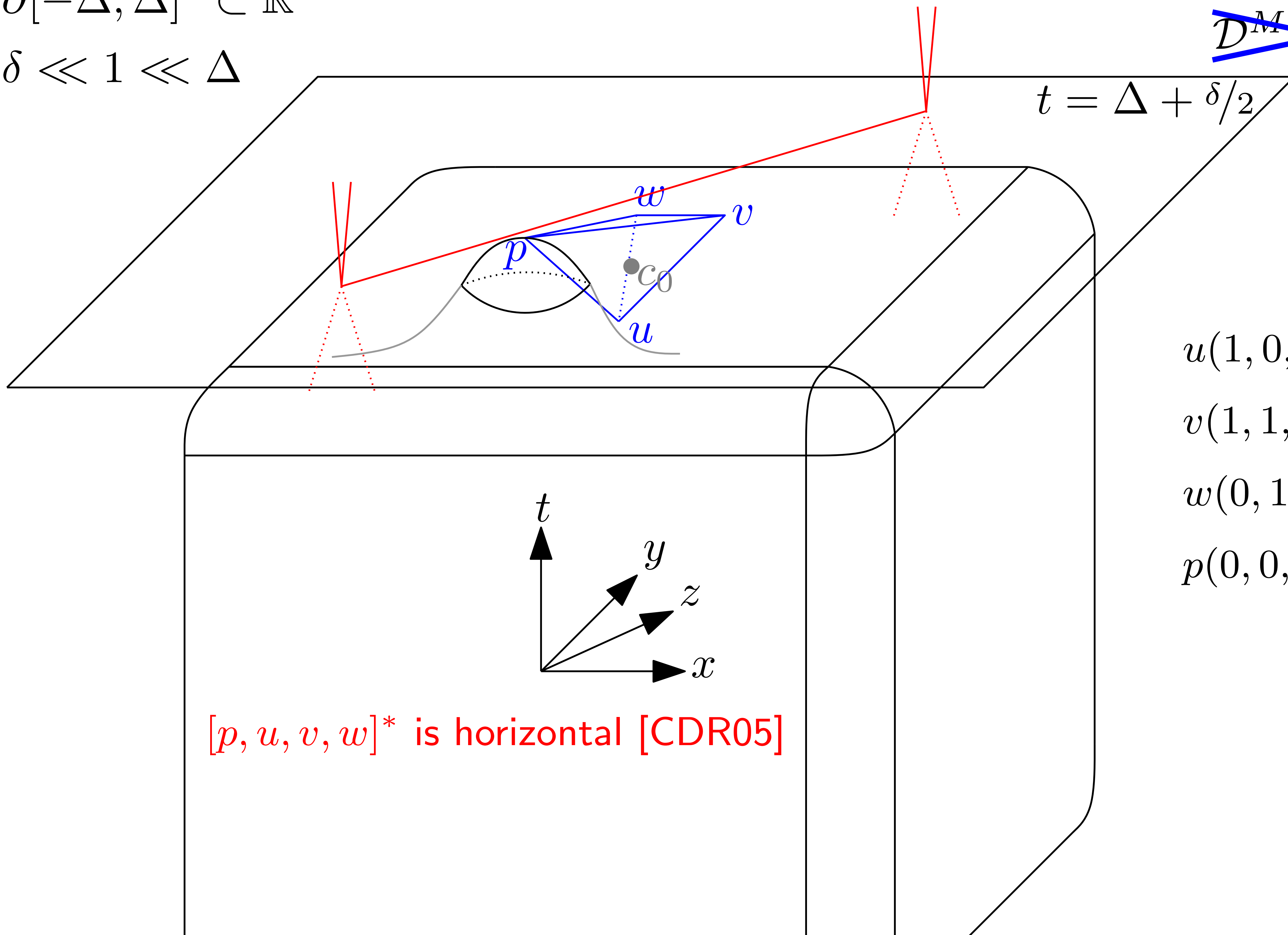

$[p, u, v, w]^*$ is horizontal [CDR05]

$u(1, 0, 0, \Delta)$

$v(1, 1, 0, \Delta)$

$w(0, 1, 0, \Delta)$

$p(0, 0, 0, \Delta + \delta)$

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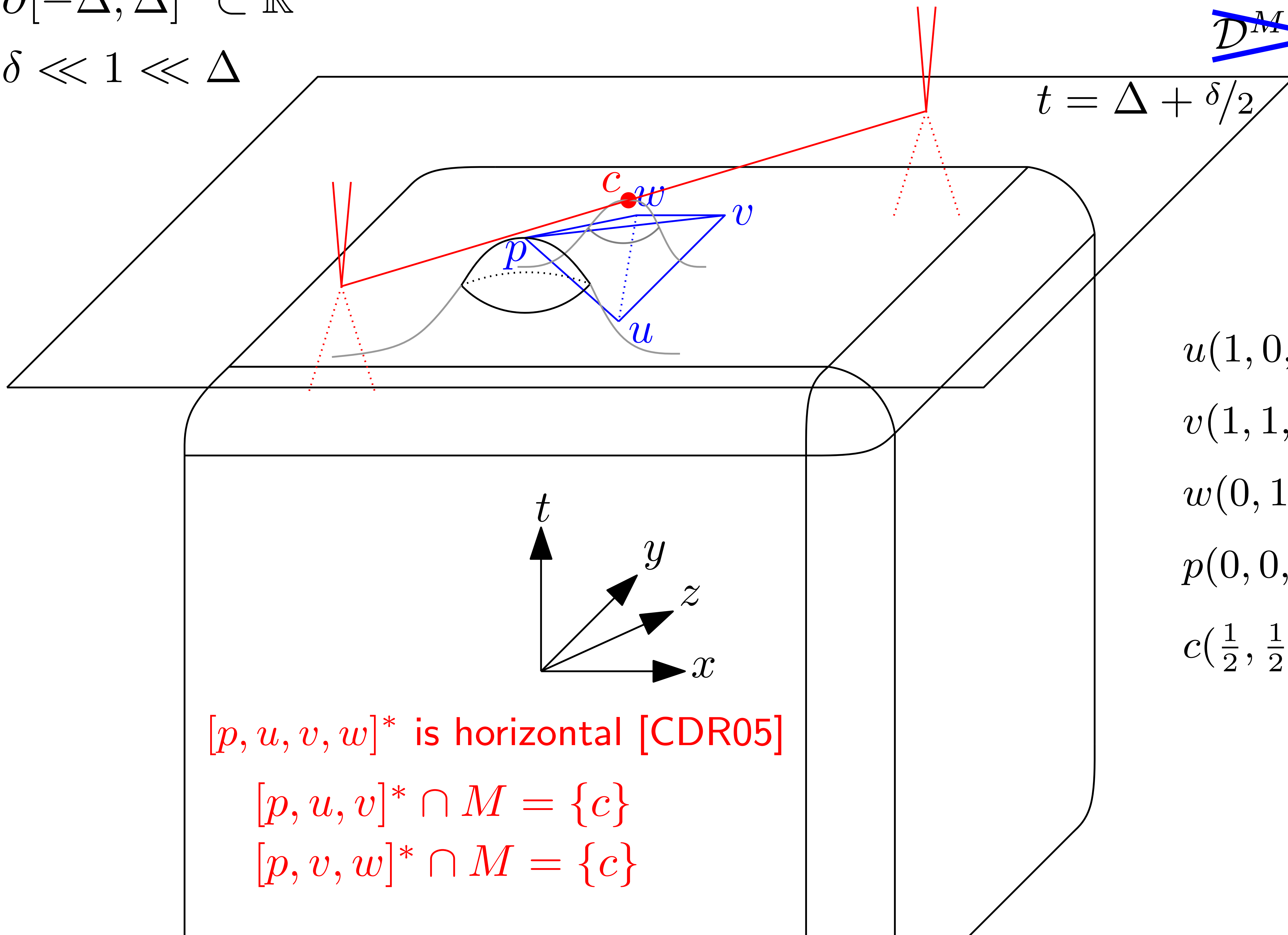

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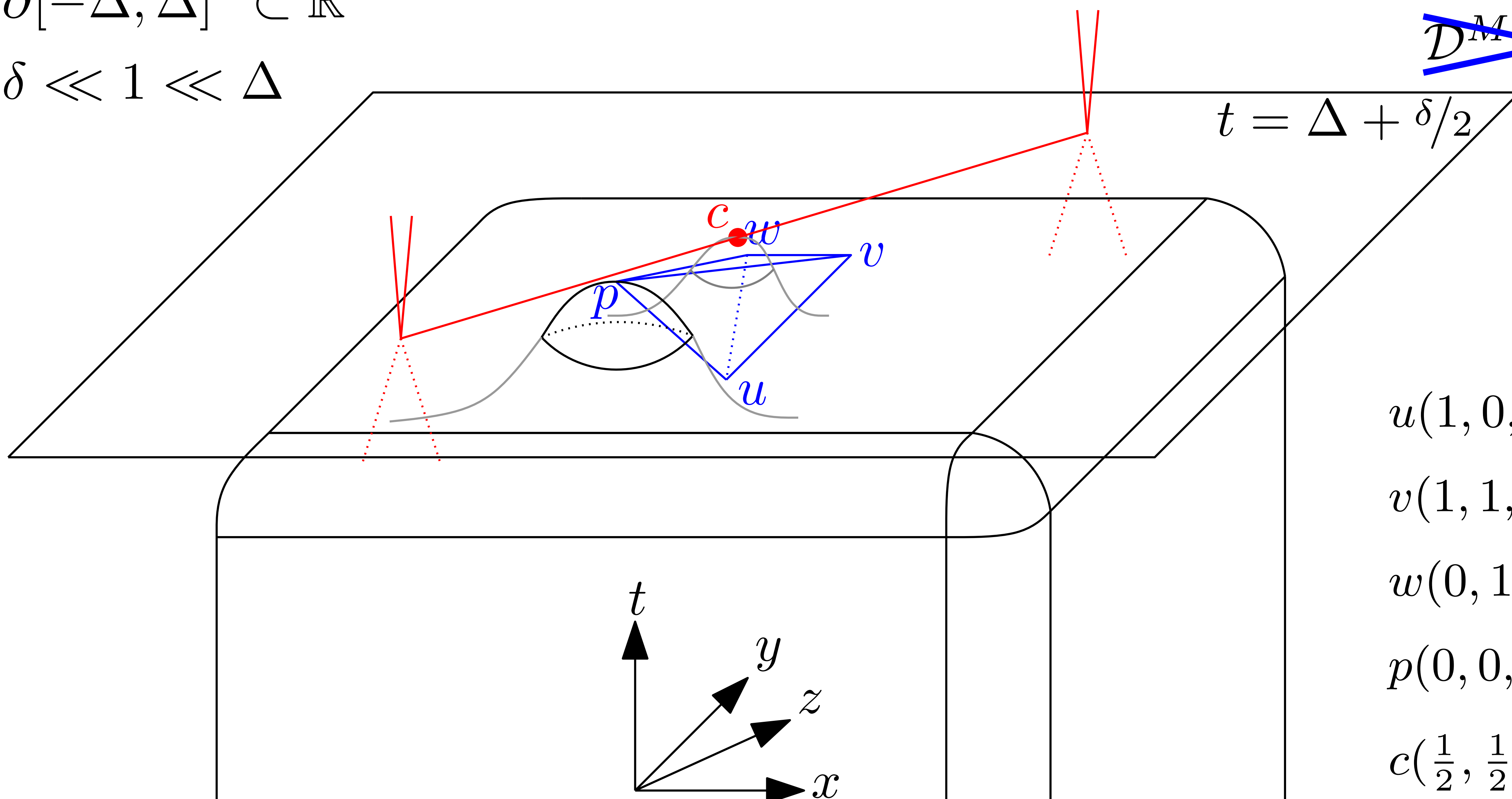

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$$\left.\begin{array}{l}[p, u, v]^* \cap M = \{c\} \\ [p, v, w]^* \cap M = \{c\}\end{array}\right\} \Rightarrow$$

$\mathcal{D}^M(L)$ is no longer a closed hypersurface if c is moved downwards slightly

Relation with the restricted Delaunay

(arbitrary dimensions)

If M is a closed k-manifold smoothly embedded in $\mathbb{R}^d$, then, under reasonable sampling conditions, $\mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M$

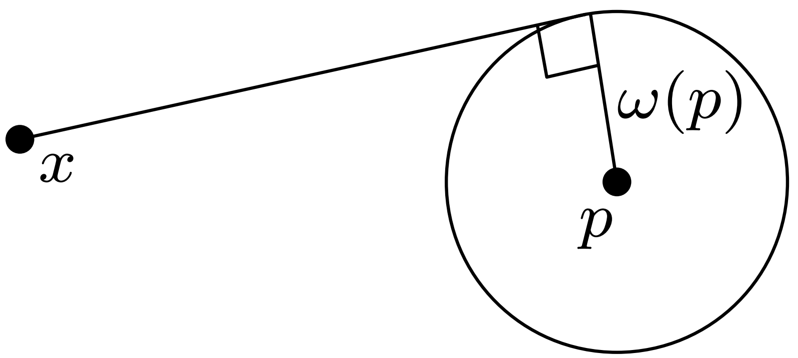
- Case $k = 1$:
  - $\mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M$
- Case $k = 2$:
  - $\mathcal{C}^W(L) \subseteq \mathcal{D}^M(L) \simeq M$
  - $\mathcal{C}^W(L) \not\supseteq \mathcal{D}^M(L)$
- Case $k \geq 3$:
  - $\mathcal{C}^W(L) \not\subseteq \mathcal{D}^M(L)$
  - $\mathcal{D}^M(L) \not\simeq M$

assign weights to the landmarks to remove all slivers from the vicinity of $\mathcal{D}^M(L)$ [Cheng *et al.* 00]

$\rightarrow$ Source of problems: slivers

Weighted Voronoi / Delaunay

Input: point cloud P, weight function $\omega : P \to \mathbb{R}_{\geq 0}$



Metric: $\mathsf{d}(x,(p,\omega(p)))^2 = \|x-p\|^2 - \omega(p)^2$

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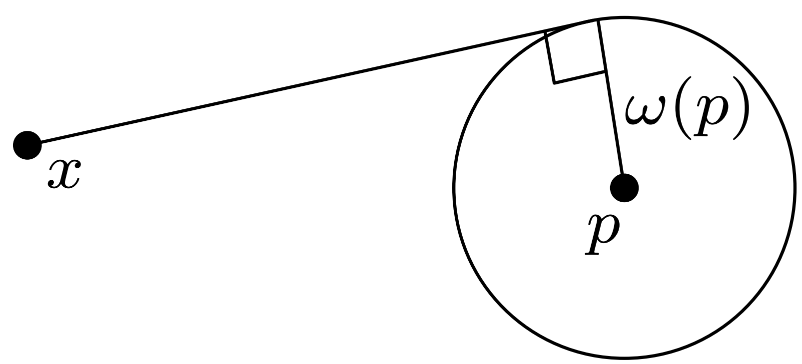

Metric: $\mathsf{d}(x, (p, \omega(p)))^2 = \|x - p\|^2 - \omega(p)^2$

Induced diagram: $\mathcal{V}(p) = \{x \in \mathbb{R}^d \mid \mathsf{d}(x, (p, \omega(p)) \leq \mathsf{d}(x, (q, \omega(q)) \; \forall q \in P\}$

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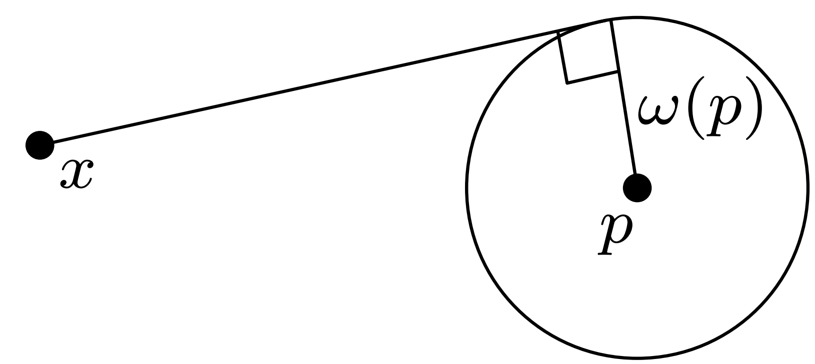

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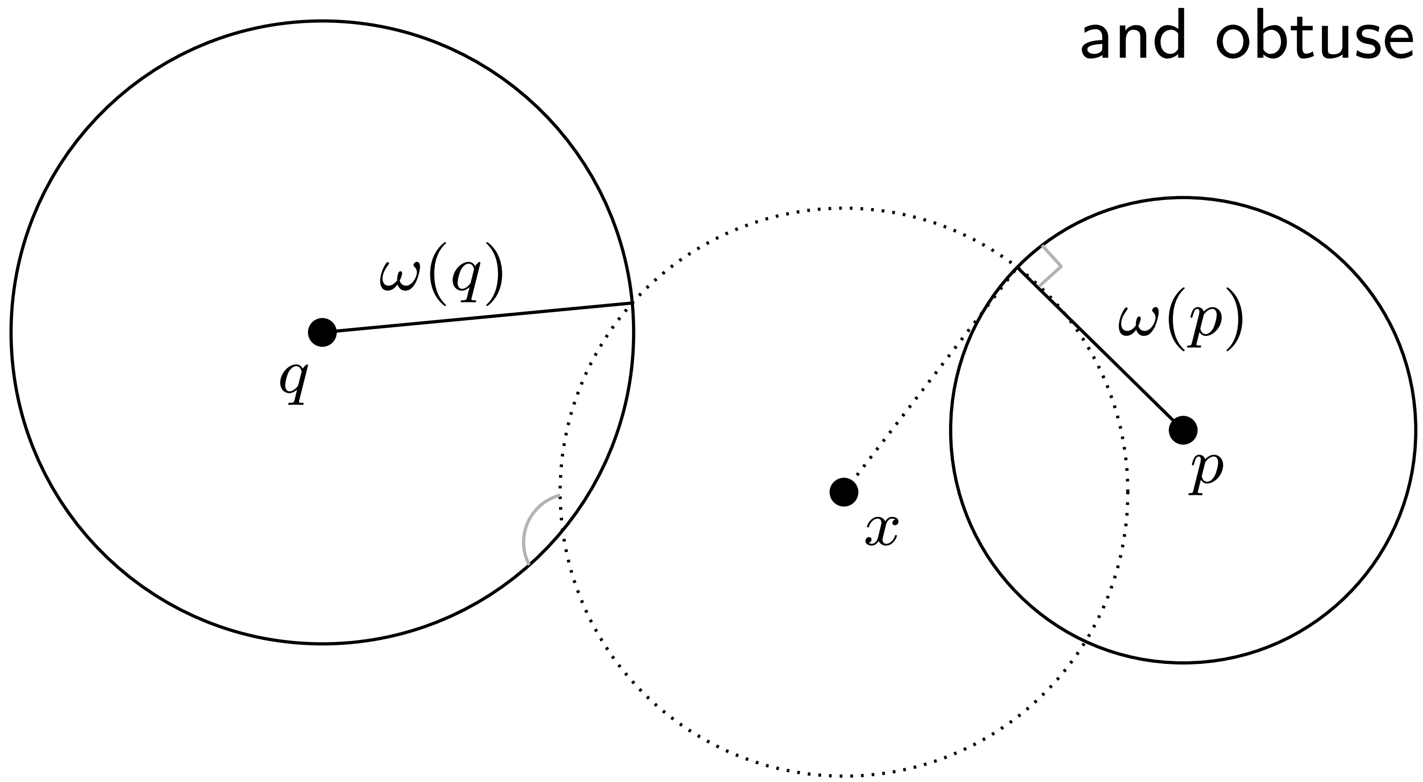
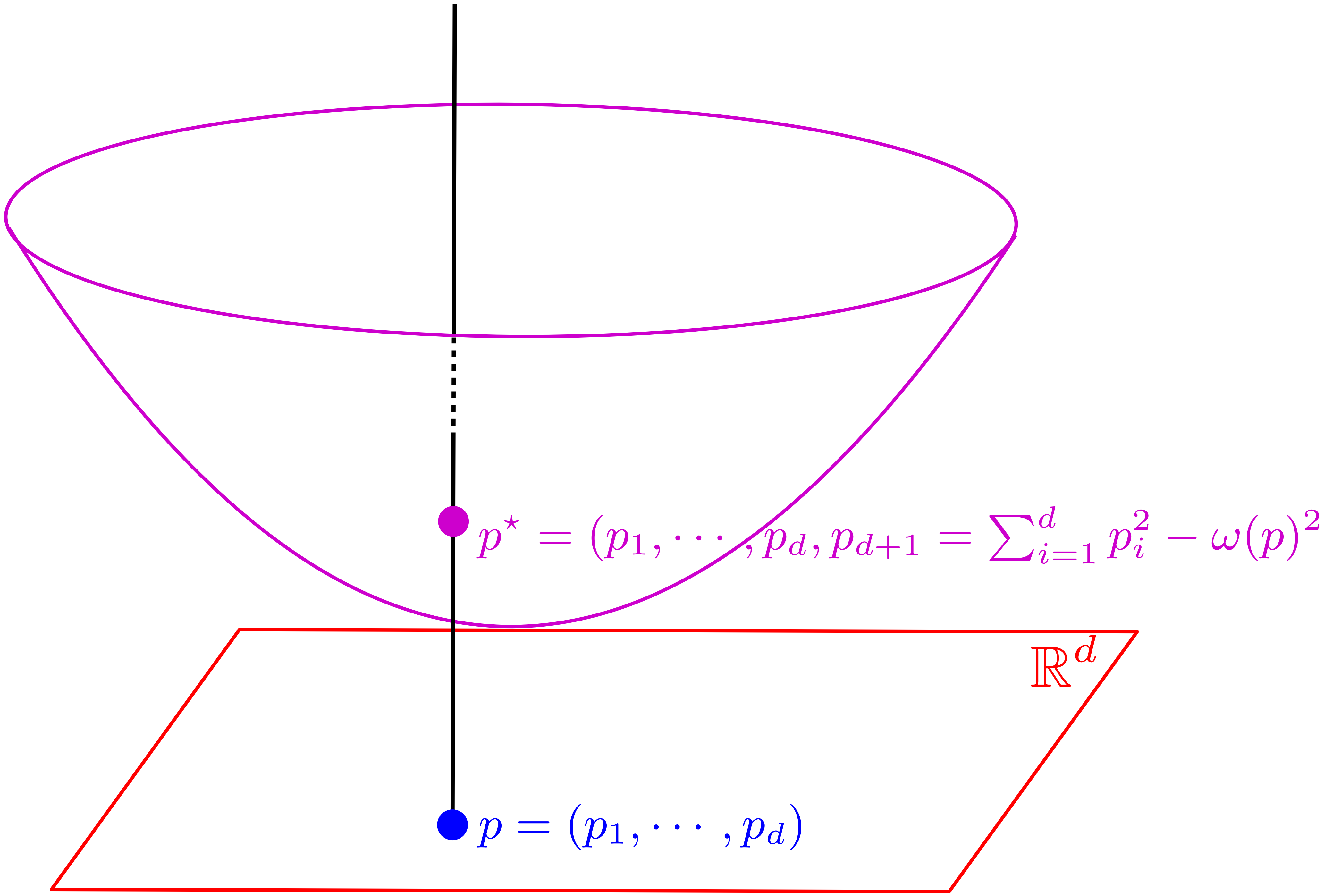
Prop: $x \in \mathcal{V}(p) \iff x$ center of sphere orthogonal to $B(p, \omega(p))$

and obtuse to $B(q, \omega(q))$ for all $q \in P \setminus \{p\}$



Point / sphere lifting



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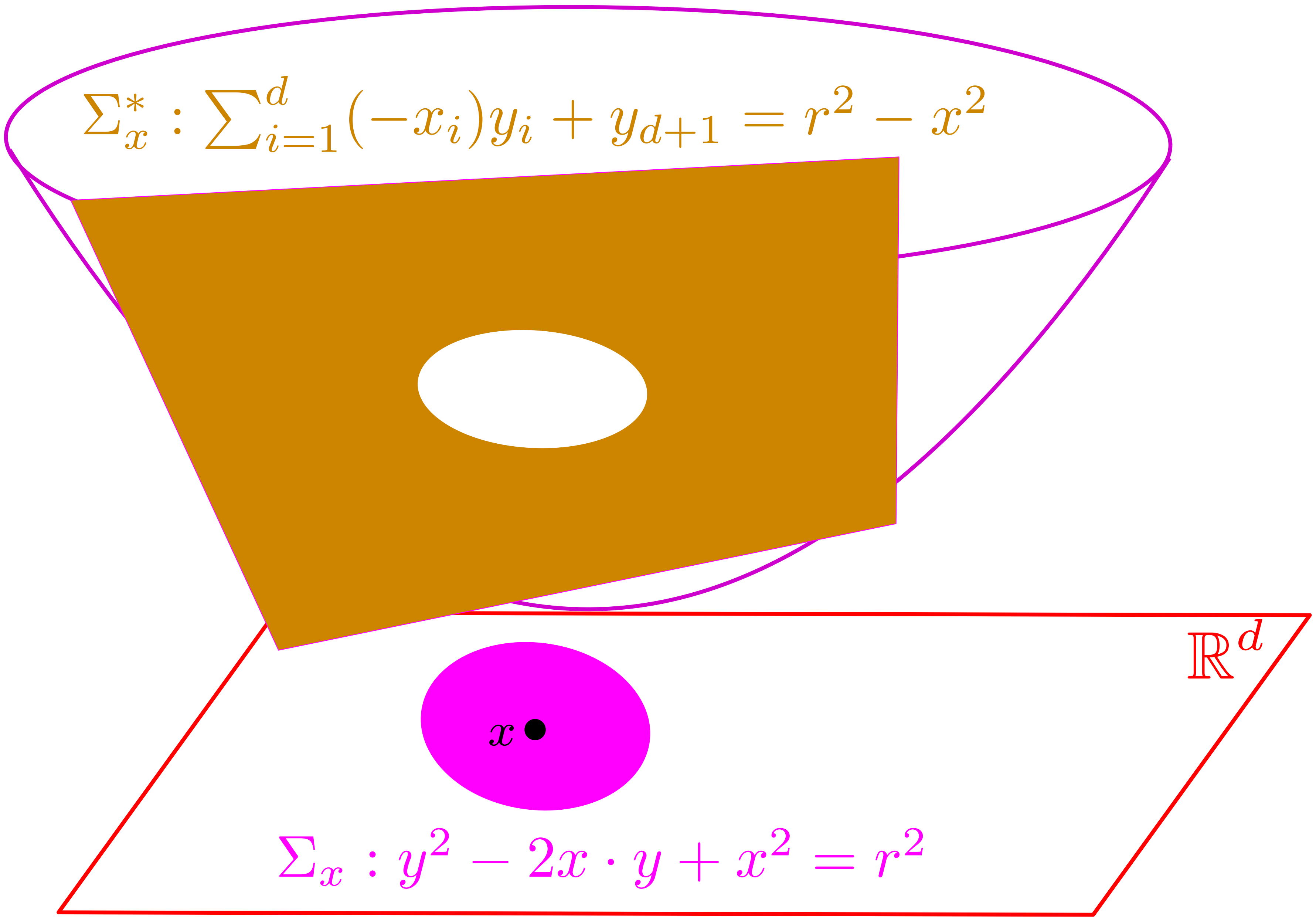

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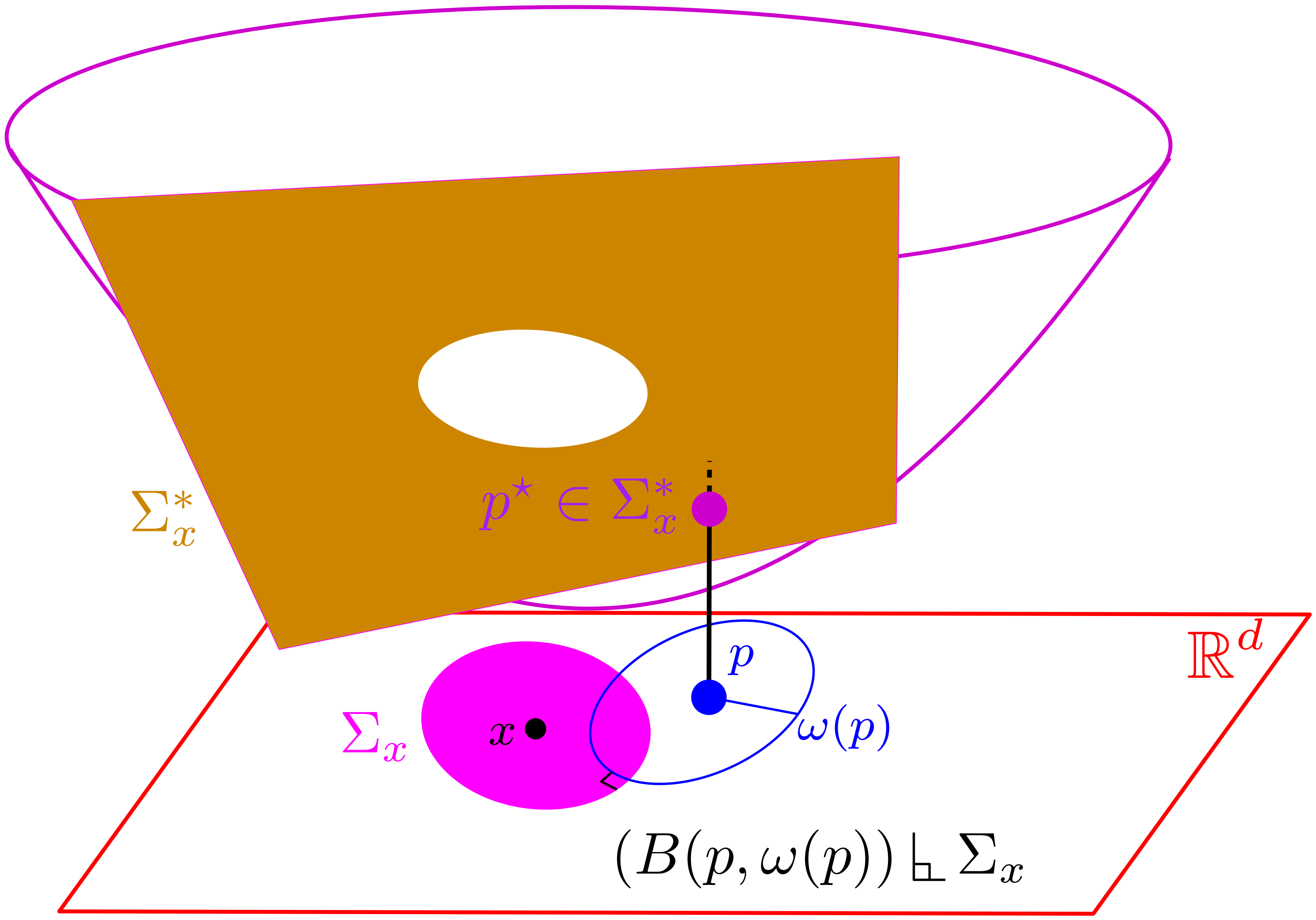

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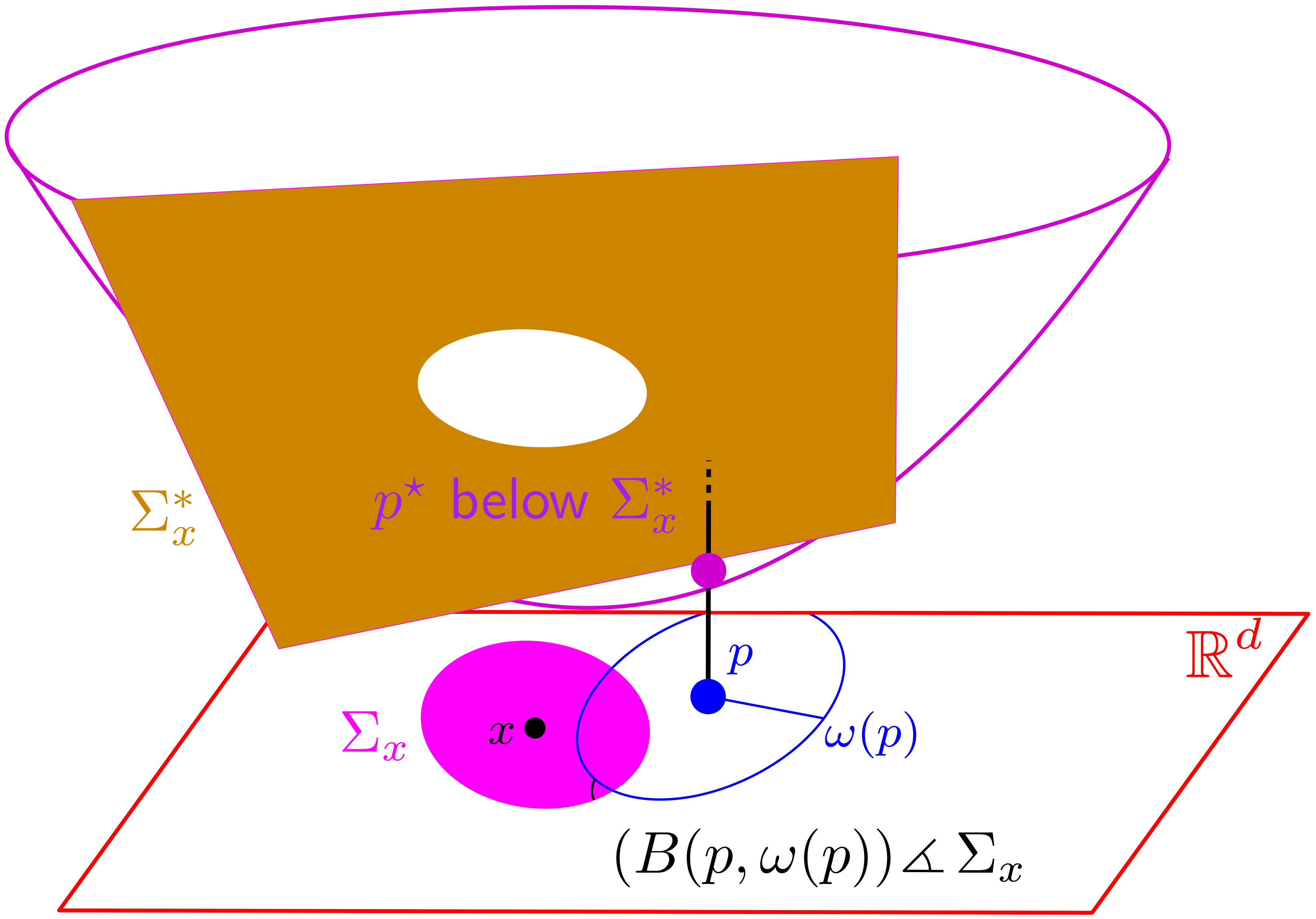

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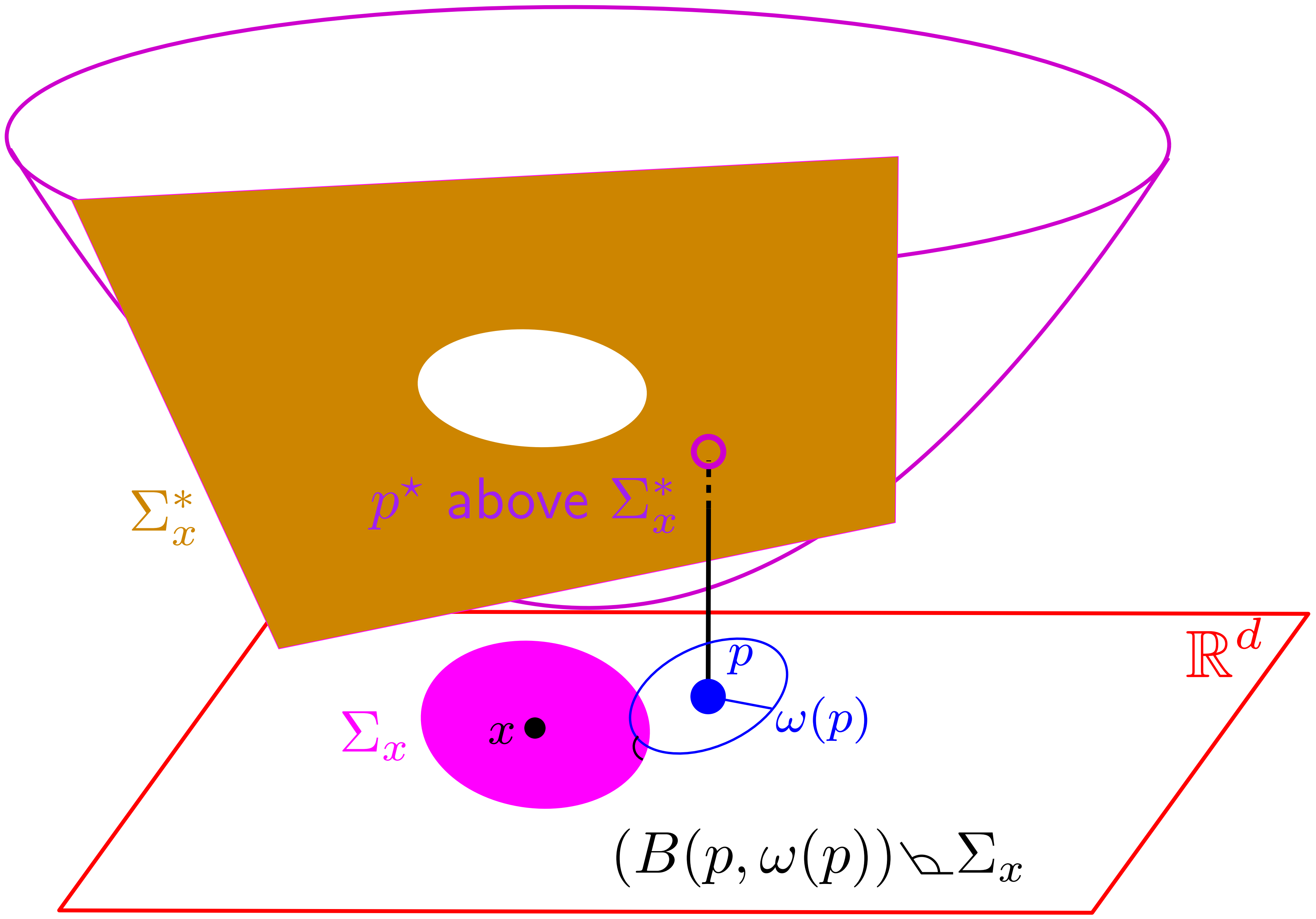

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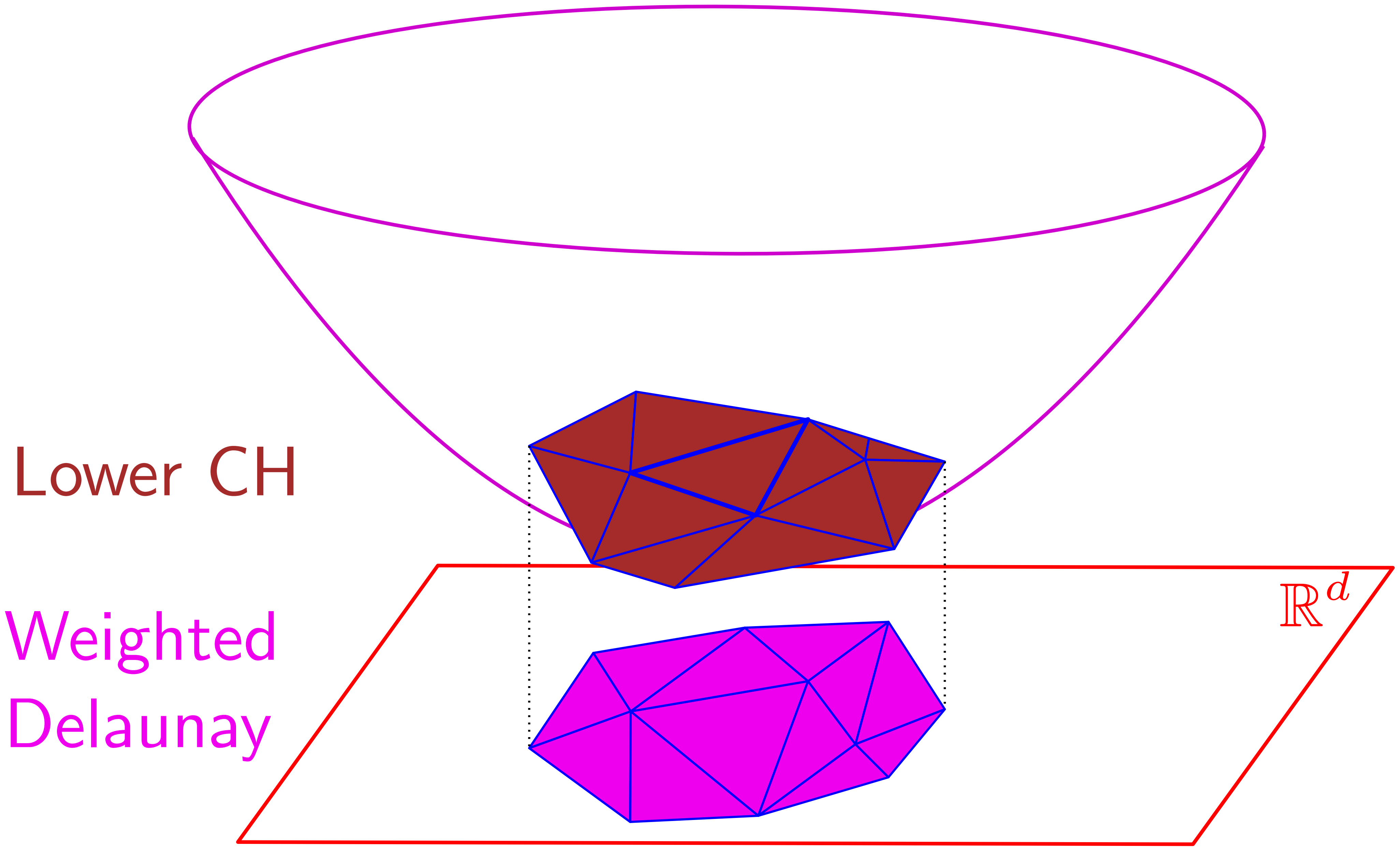
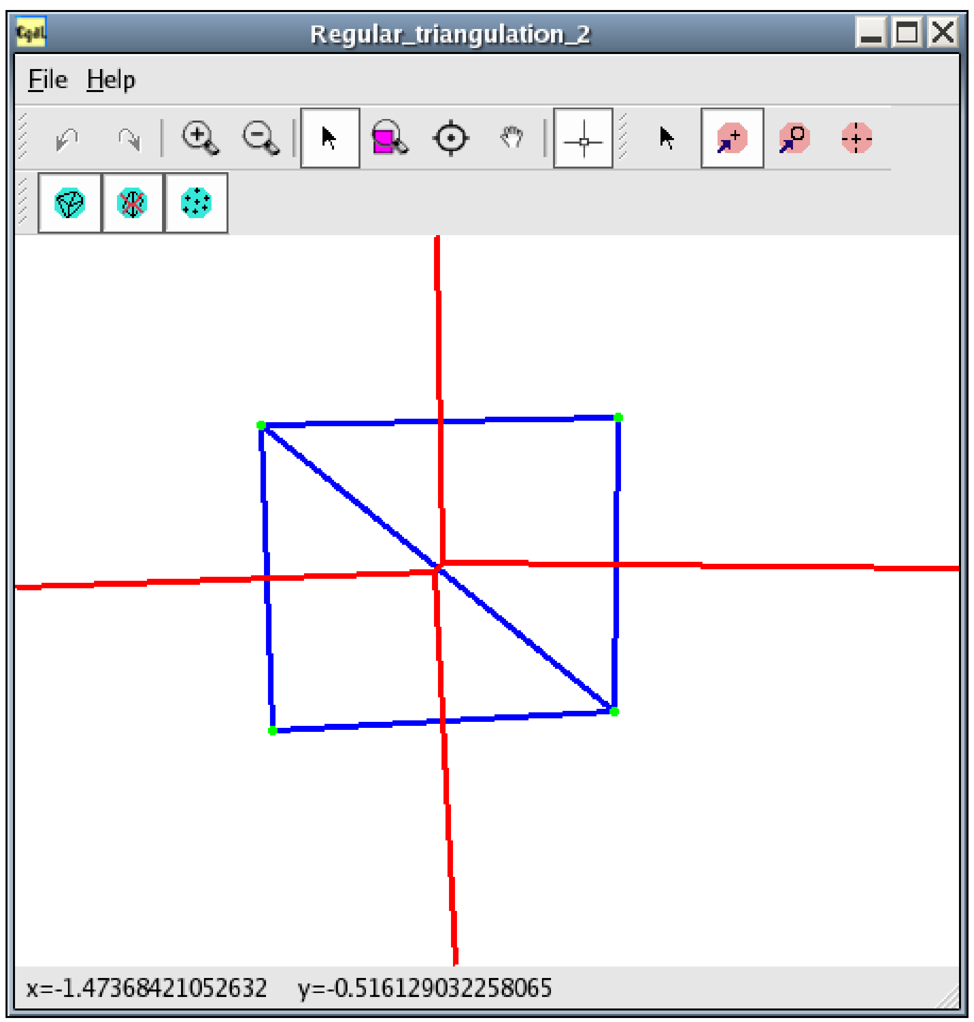
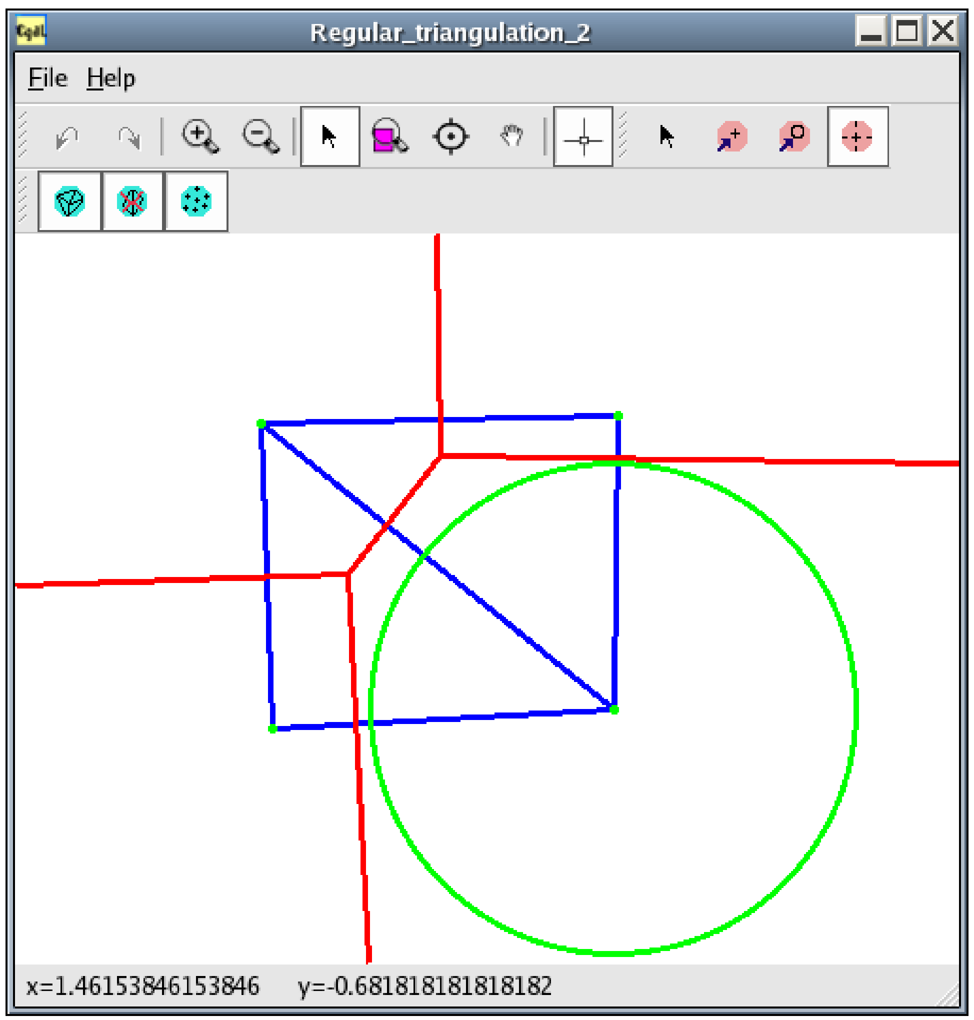


Sliver Removal [CDEFT'00]

- Each landmark $u \in L$ is assigned a weight $0 \leq \omega(u) < \frac{1}{2}\,\mathrm{d}(u, L \setminus \{u\})$.
- The Voronoi diagram of L is replaced by its weighted version, $\mathcal{V}_\omega(L)$:
  $p \in \mathrm{cell}(u)$ iff $\forall v \in L$, $\mathrm{d}(p,u)^2 - \omega(u)^2 \leq \mathrm{d}(p,v)^2 - \omega(v)^2$
- $\mathcal{V}_\omega(L)$ is an affine diagram, its dual complex $\mathcal{D}_\omega(L)$ is a triangulation.





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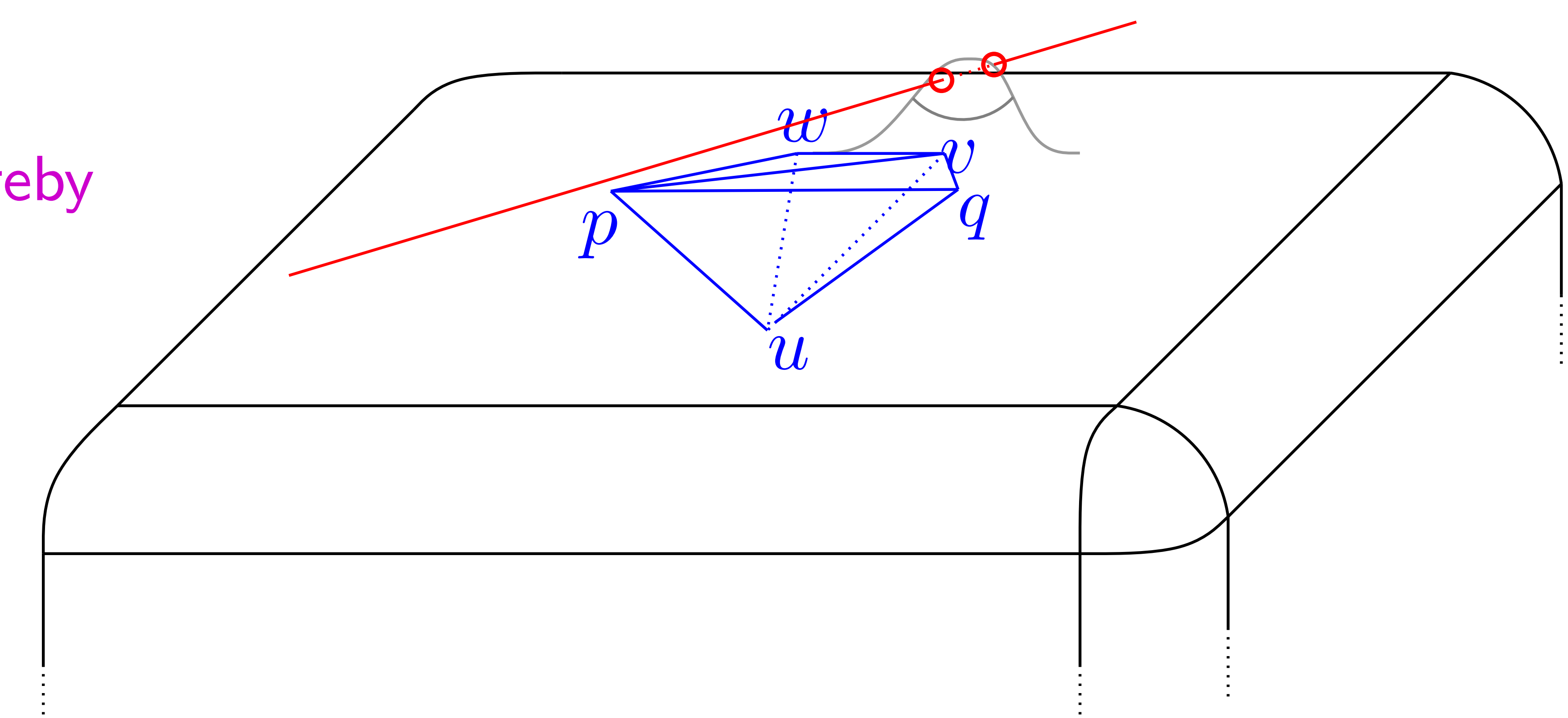
Thm [Cheng, Dey, Ramos 05] If L is an ε-sparse ε-sample of M, with $\varepsilon \ll \mathrm{rch}(M)$, then $\exists \omega_0$ that removes slivers from the vicinity of $\mathcal{D}^M_{\omega_0}(L)$.

$\Rightarrow \mathcal{D}^M_{\omega_0}(L) \simeq M$

- ω_0 removes slivers, thereby improving the normals

- Closed Ball Property



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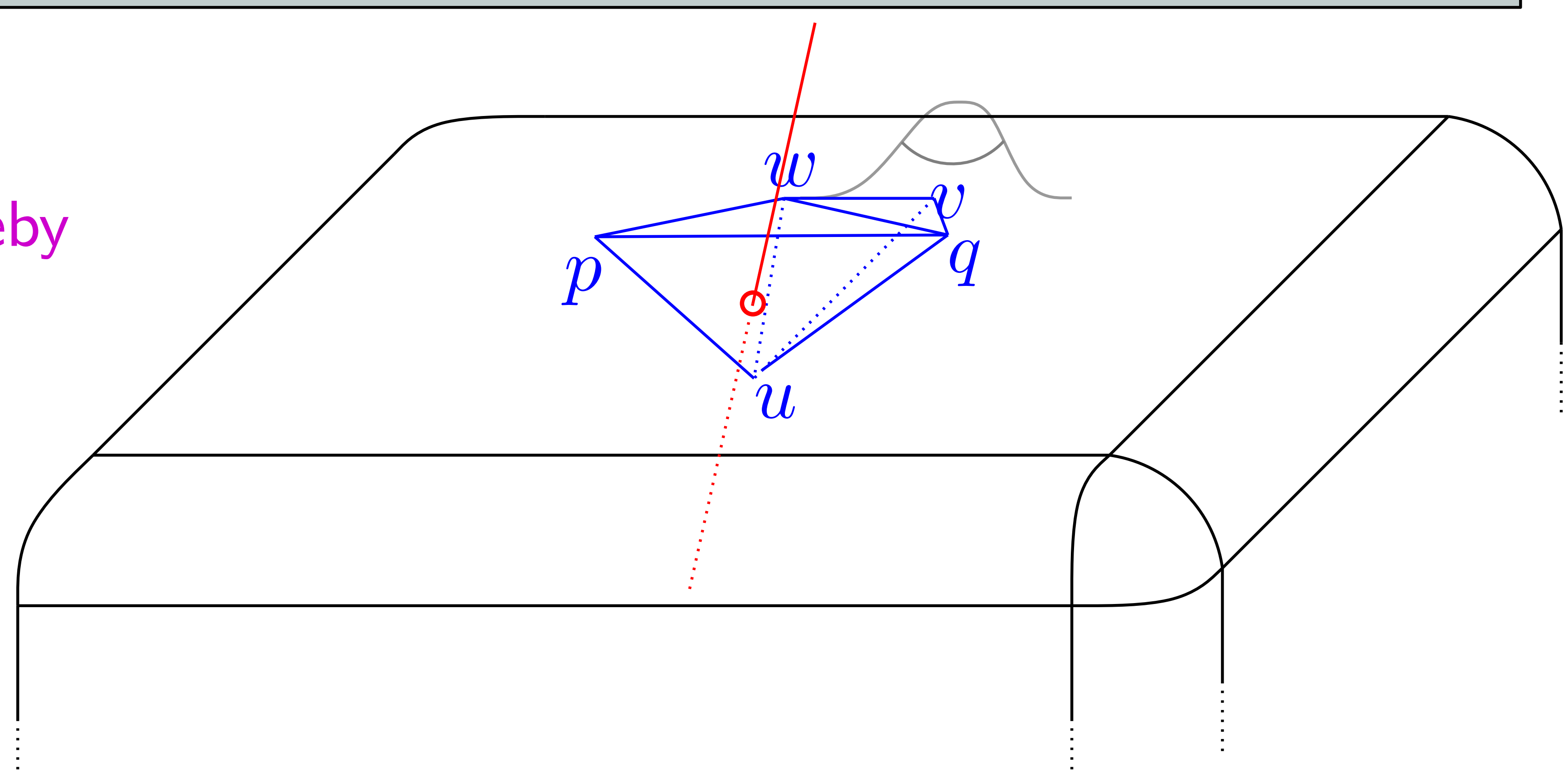

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Thm [Boissonnat, Guibas, O. 07]
[Boissonnat, Dyer, Ghosh, O. 17]

- Under the same conditions on L, one has $\mathcal{C}^W_{\omega_0}(L) \subseteq \mathcal{D}^M_{\omega_0}(L)$ for all $W \subseteq M$.

- If W is a δ-sample of M, with $\delta \ll \varepsilon$, then $\mathcal{C}^W_{\omega_0}(L) = \mathcal{D}^M_{\omega_0}(L)$.

Application to reconstruction in arbitrary dimensions

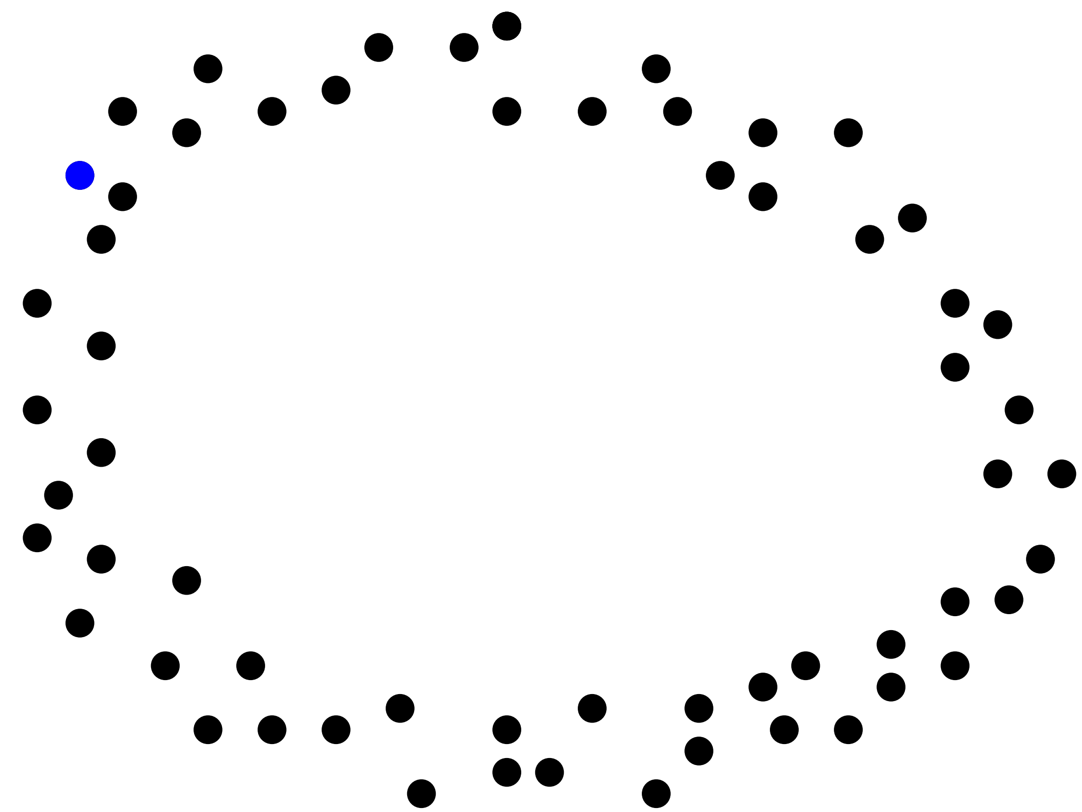
[Guibas, O. 07] [Boissonnat, Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^d$.

$\rightarrow$ greedy: furthest-point resampling of L

maintain $\mathcal{C}_\omega^W(L)$ for some carefully-chosen weight function ω.

Init: $L := \{p\}$, for some arbitrary $p \in W$;



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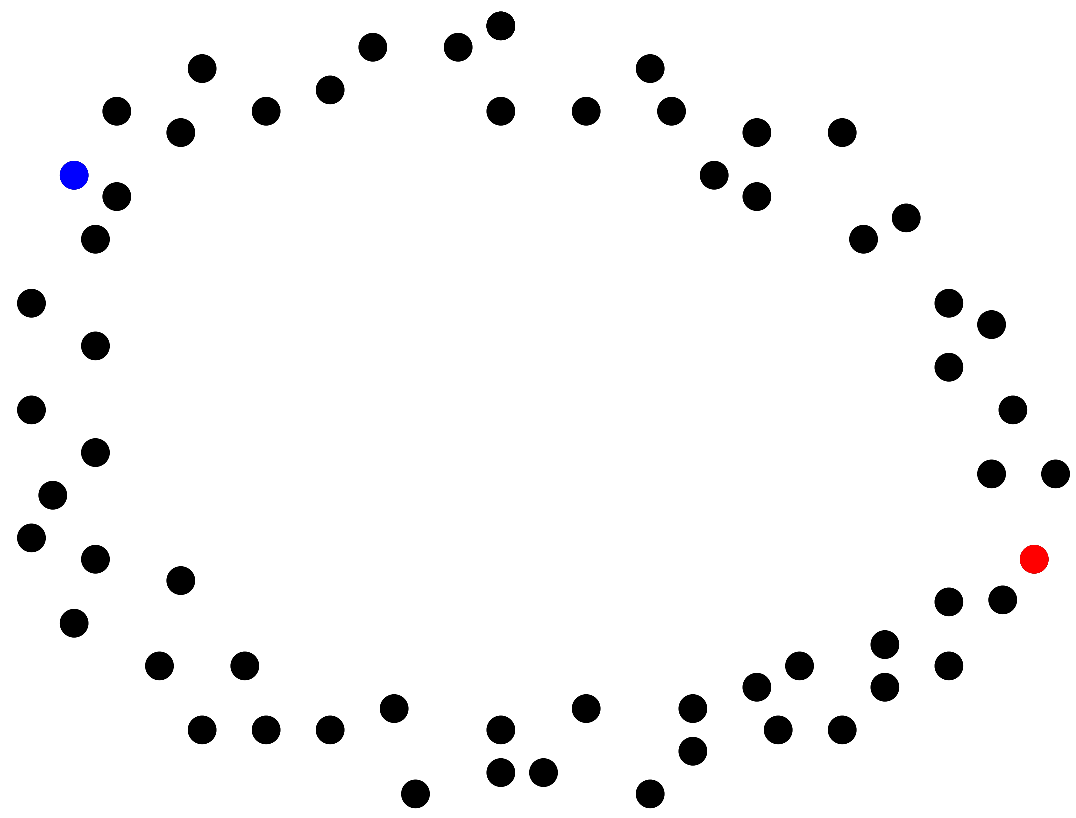

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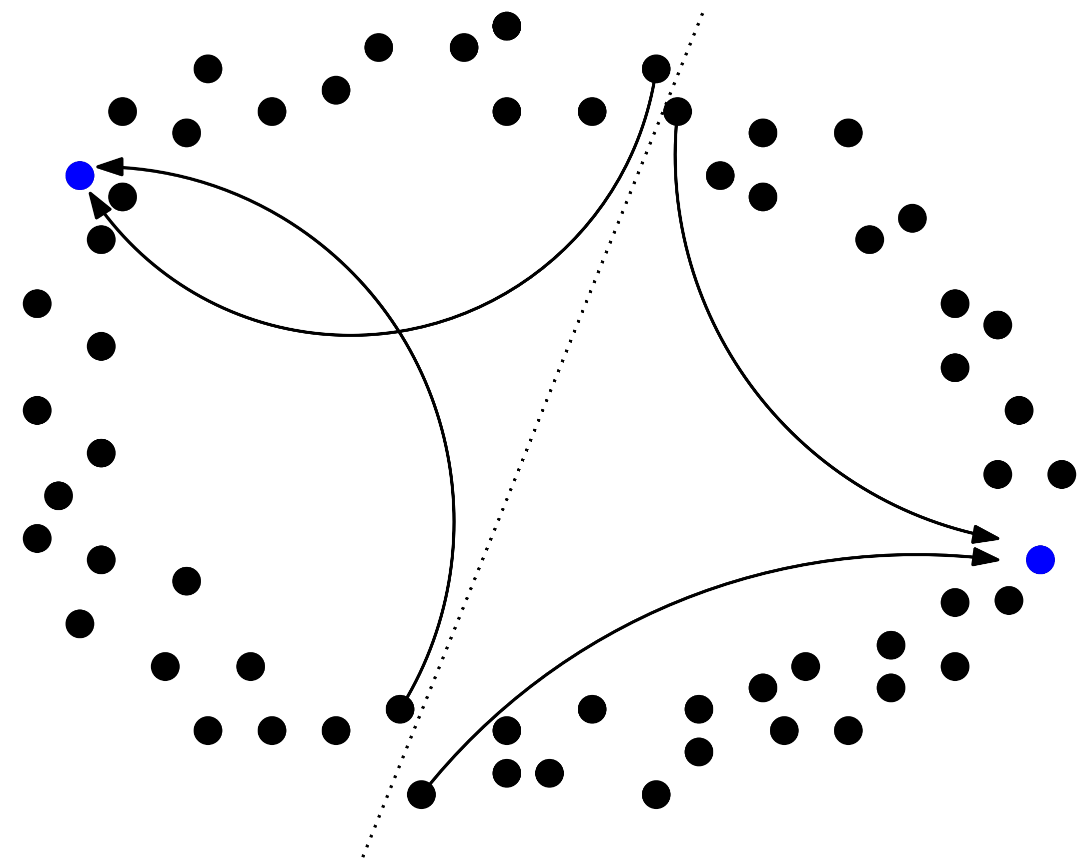

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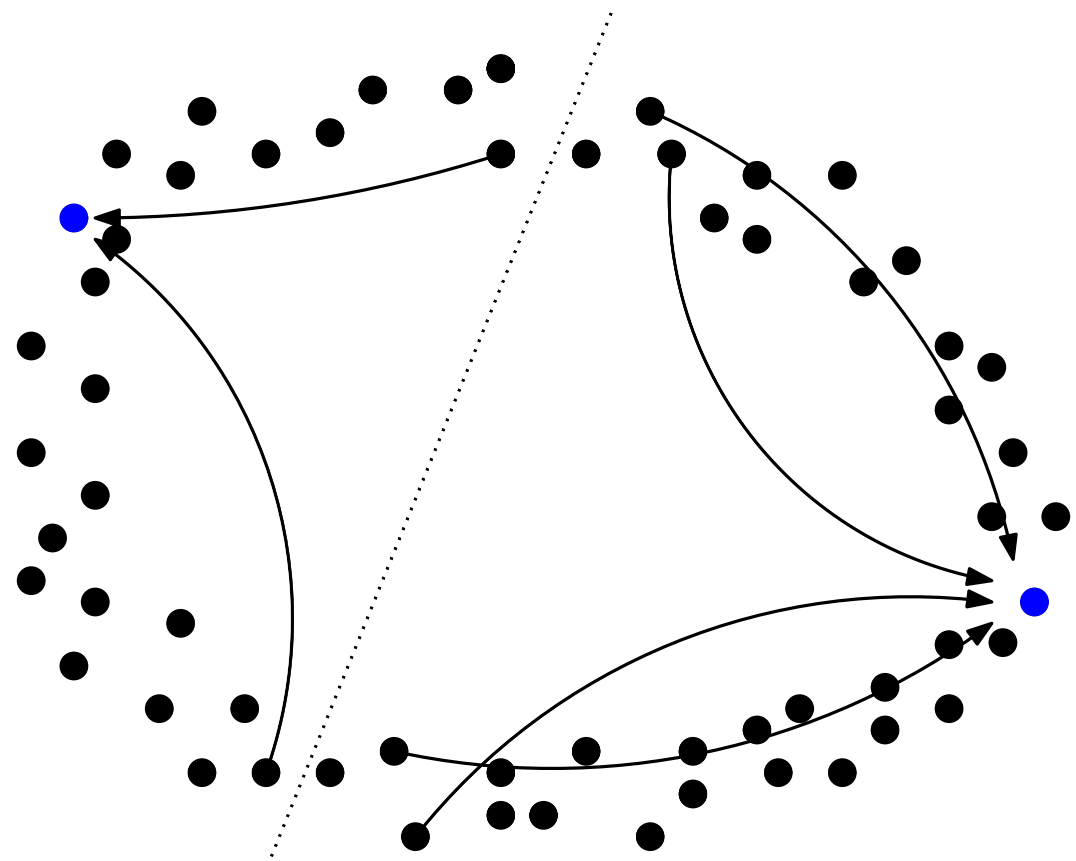

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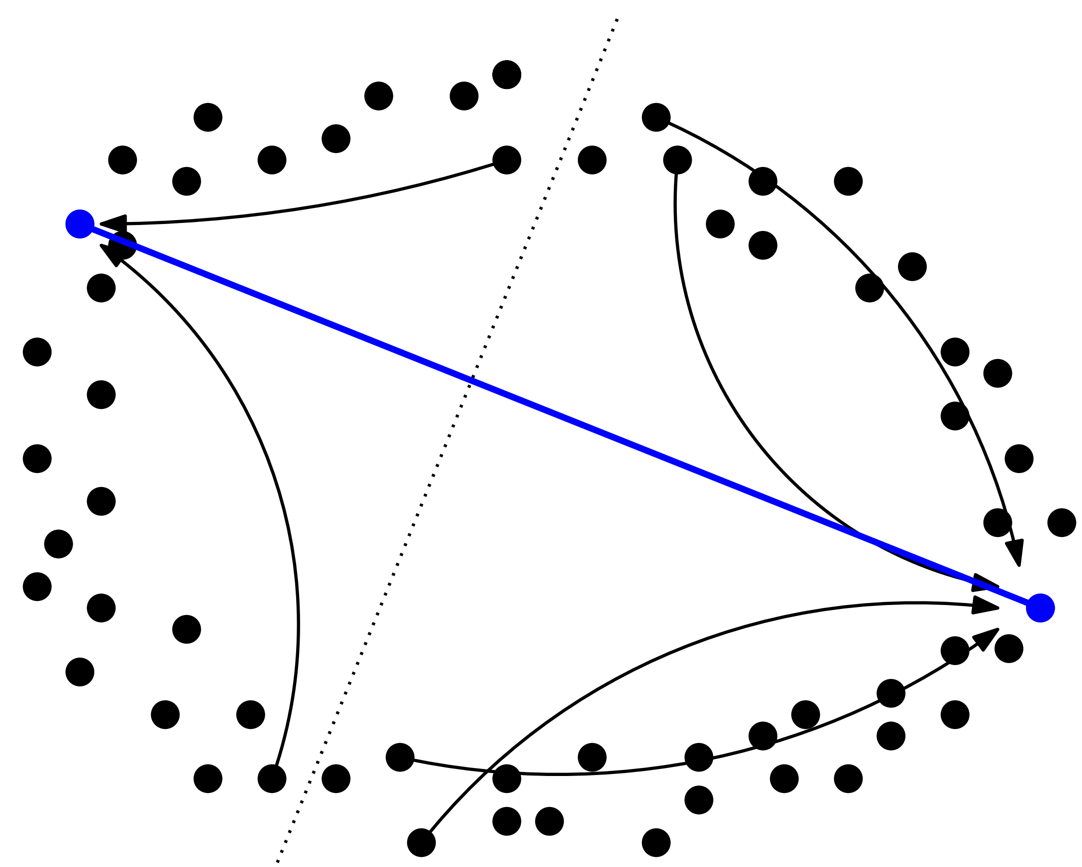

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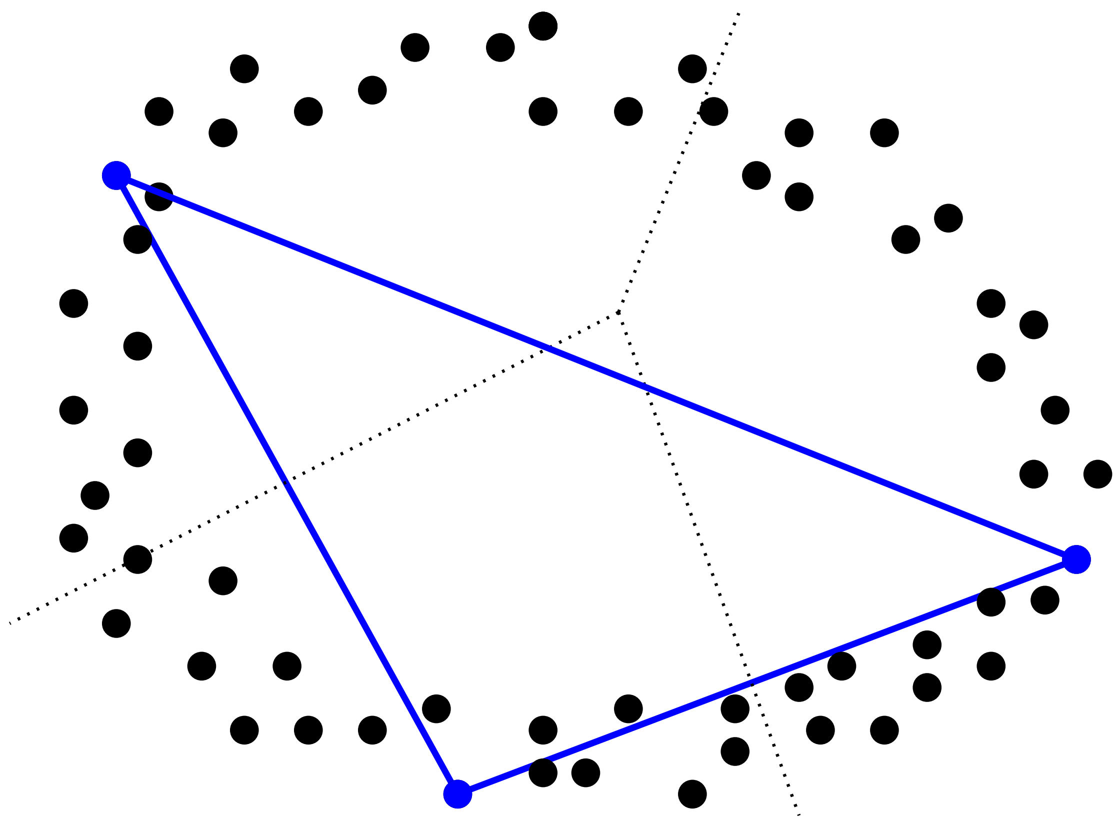

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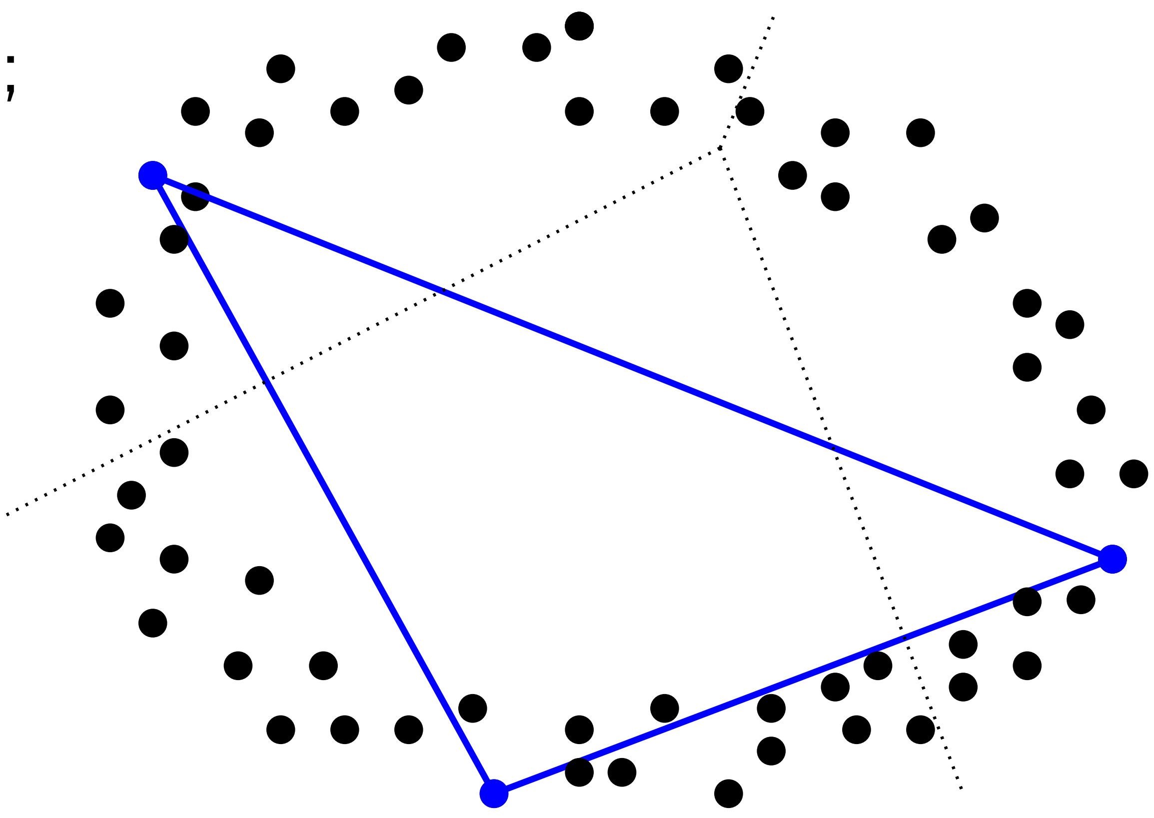

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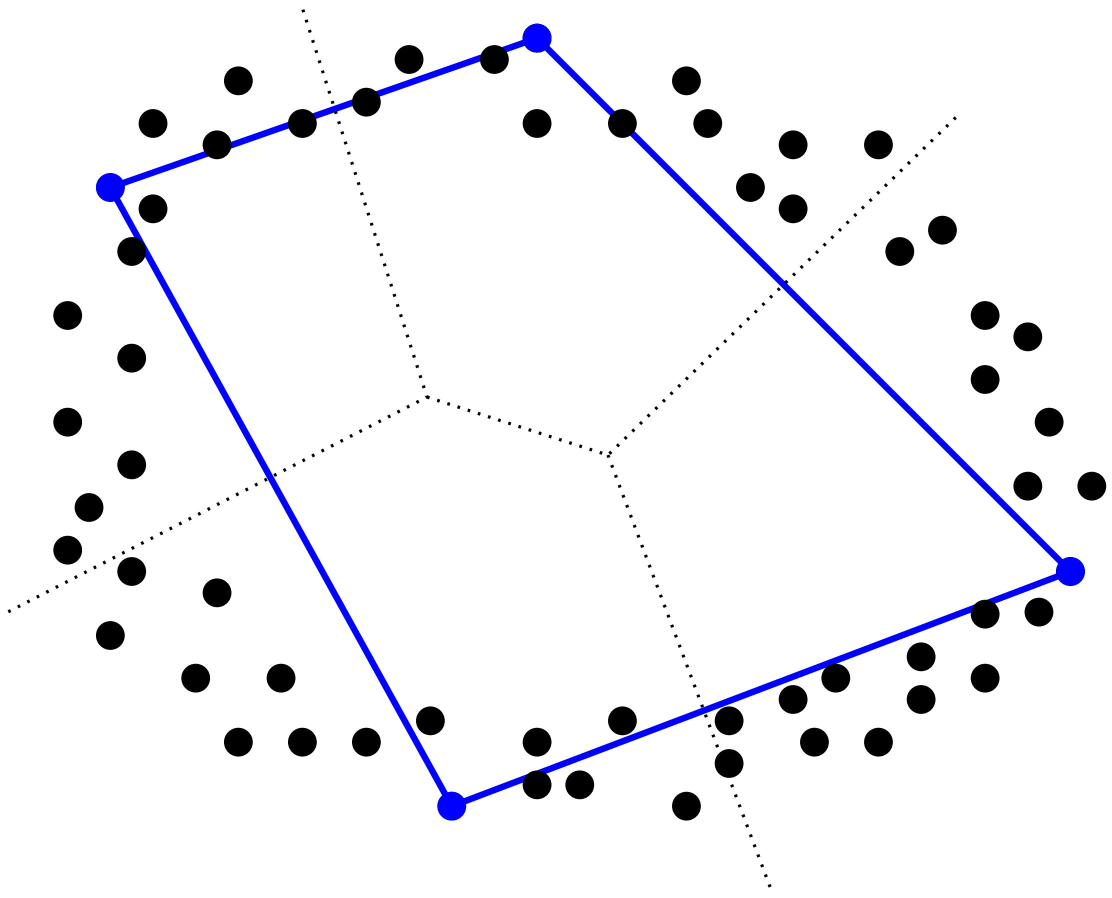

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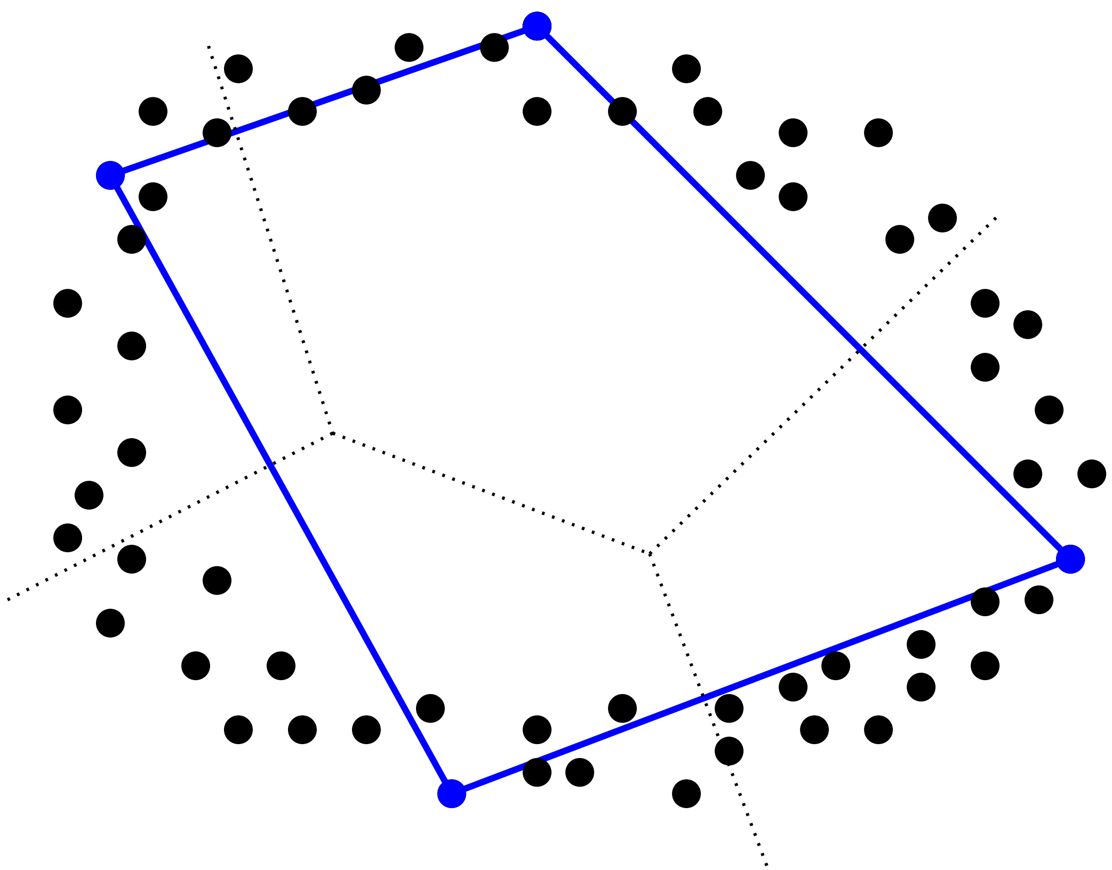

Output: sequence of simplicial complexes $\mathcal{C}_\omega^W(L)$ built throughout.

Application to reconstruction in arbitrary dimensions

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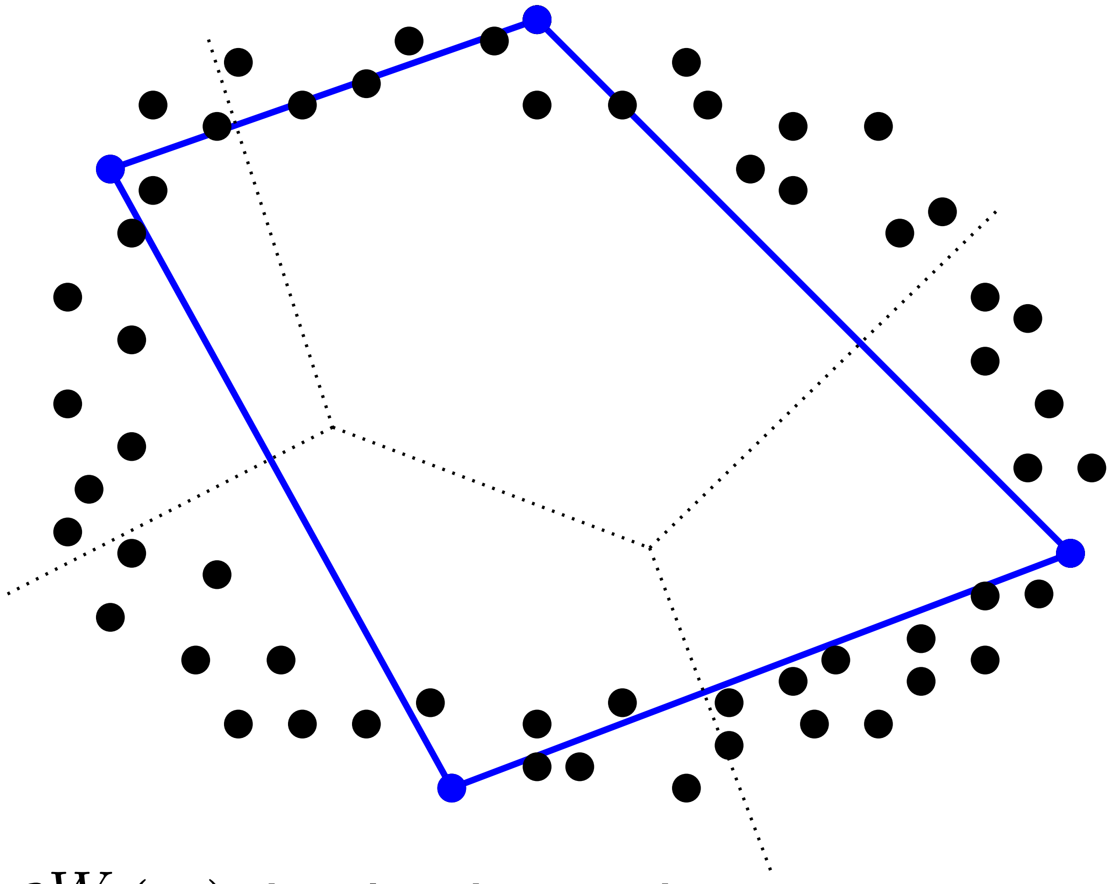

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Weight Assignment

[Boissonnat, Dyer, Ghosh, O. 17]

Candidate simplices: (requires to know the intrinsic dimension m)

$N(p) = \{k$-NN of p in $L\}$, where $k = 66^m$

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Claims:

$[0, \bar{\omega}] \setminus \bigcup_{\sigma:\text{candidate}} I_\sigma \neq \emptyset$

for every σ, I_σ depends only on weights of L and on radius & flatness of σ

(no need to maintain $\mathcal{D}(L)$)

Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07] [Boissonnat, Dyer, Ghosh, O. 17]

Input: a finite point set $W \subset \mathbb{R}^d$.

Thm If W is a δ-sample of M, with $\delta \ll \mathrm{rch}(M)$, then, at some stage of the process, the weight assignment removes all slivers from the vicinity of $\mathcal{D}_\omega^M(L)$, therefore $\mathcal{C}_\omega^W(L) = \mathcal{D}_\omega^M(L) \simeq M$.

Init: $L := \{p\}$, for some arbitrary $p \in W$;

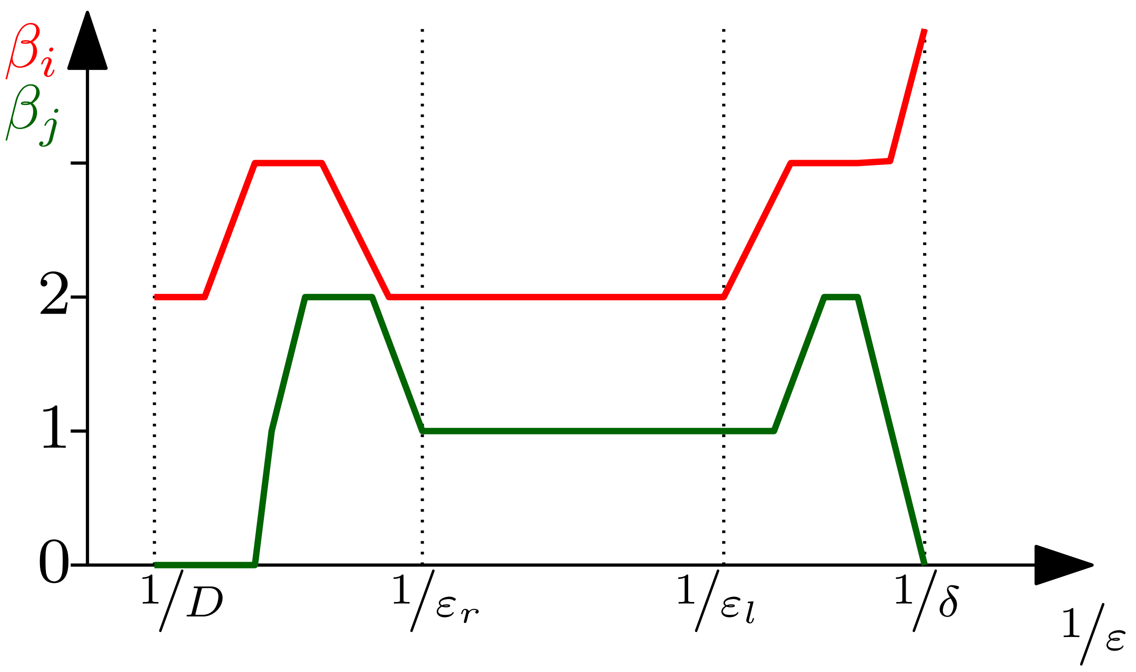
WHILE $L \subsetneq W$

insert $p = \mathrm{argmax}_{w \in W} \mathsf{d}(w, L)$ in L;

assign weight to p;

update $\mathcal{C}_\omega^W(L)$;

END_WHILE



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Input: a finite point set $W \subset \mathbb{R}^d$.

Running time: $dn(2^{O(m)}n + 2^{O(m^2)} + O(mn)) + O(m^3 n)$

Space usage: $n(d + 2^{O(m^2)}) + O(mn^2)$ ($n = |W|$, $m =$intrinsic dim.)

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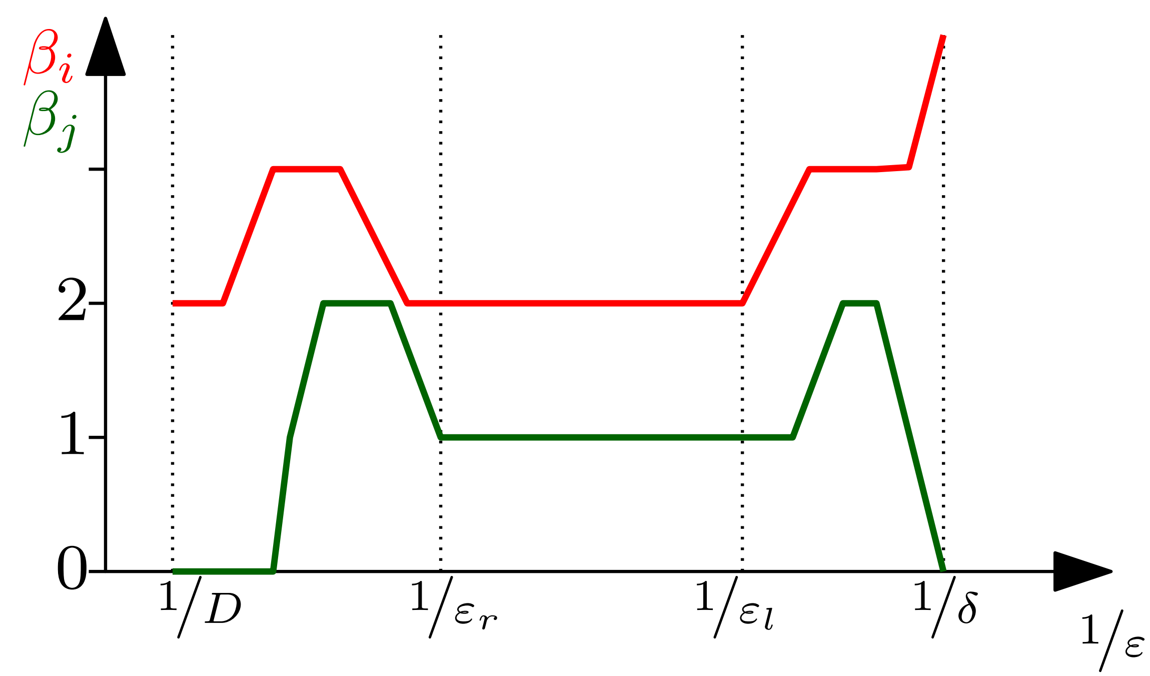

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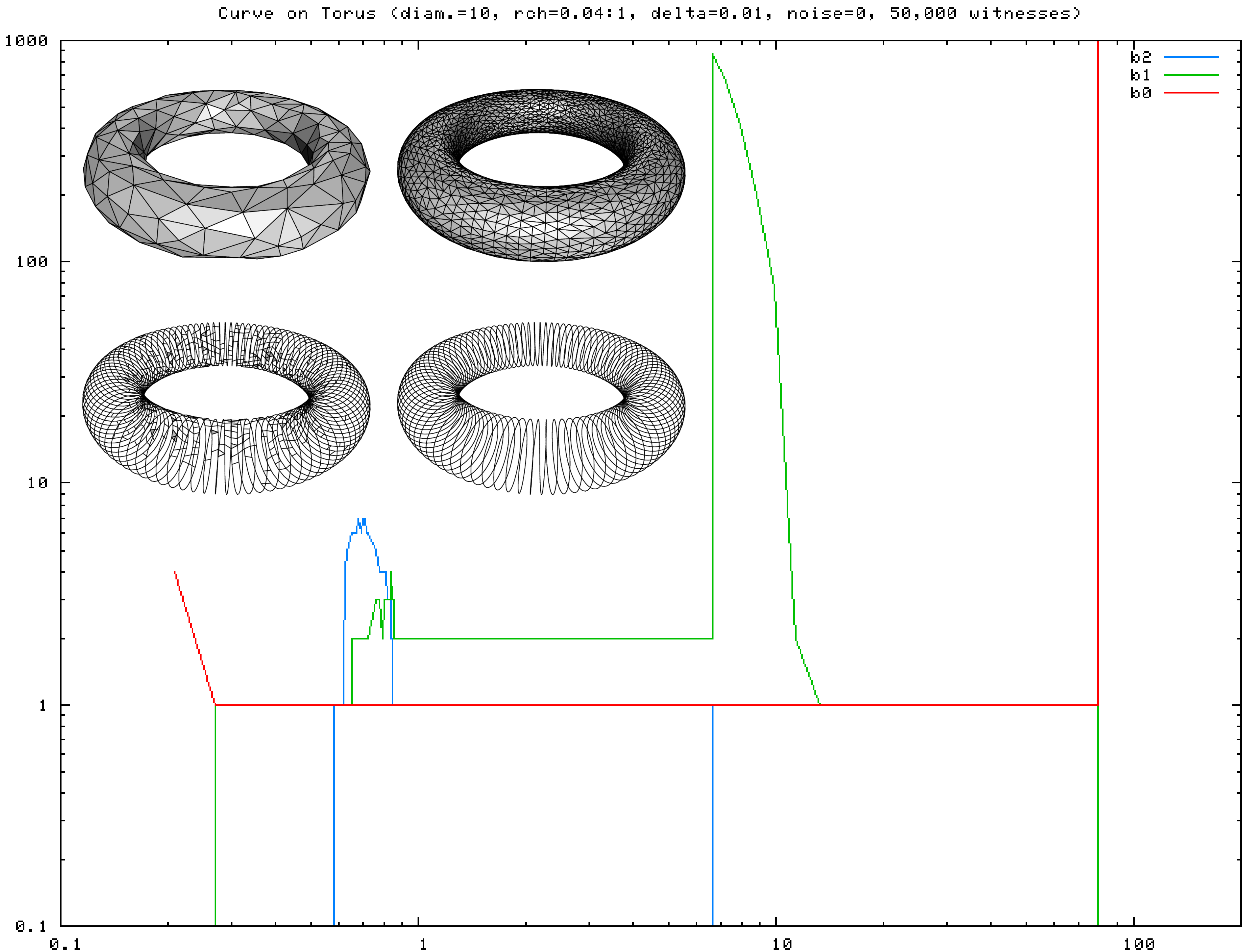
Some results



Example of application: Sensor Networks

[Gao, Guibas, O., Wang '07]

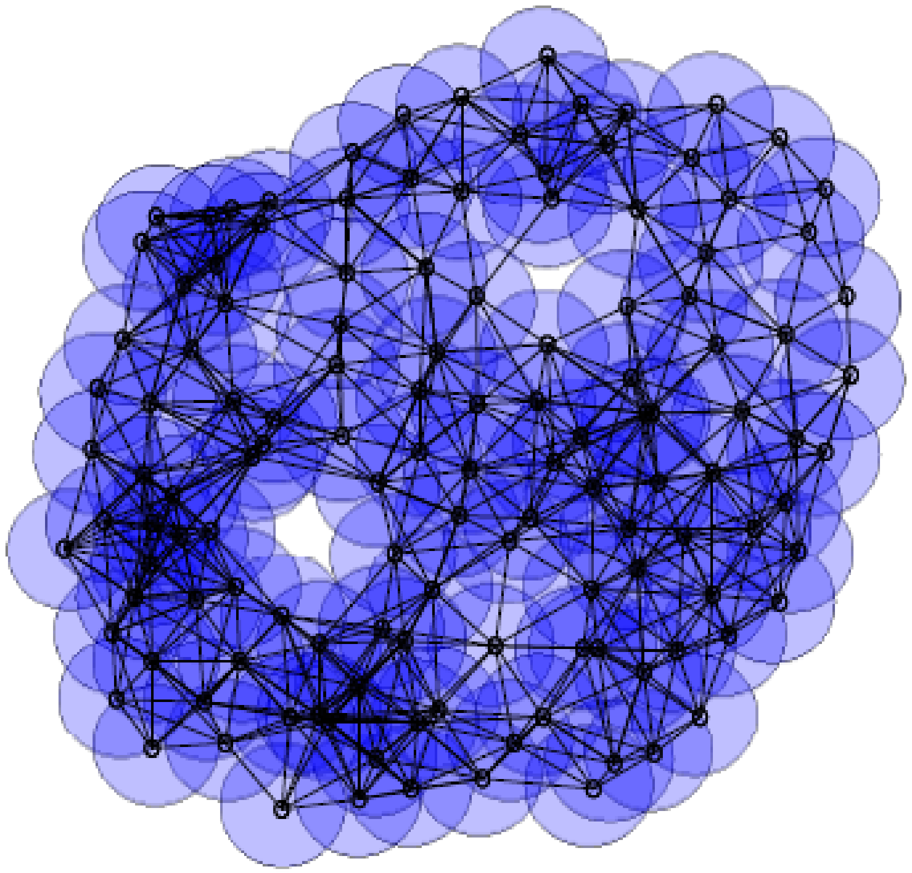
Input: a set of nodes W sampling some unknown planar domain M.

$\rightarrow$ each node has:

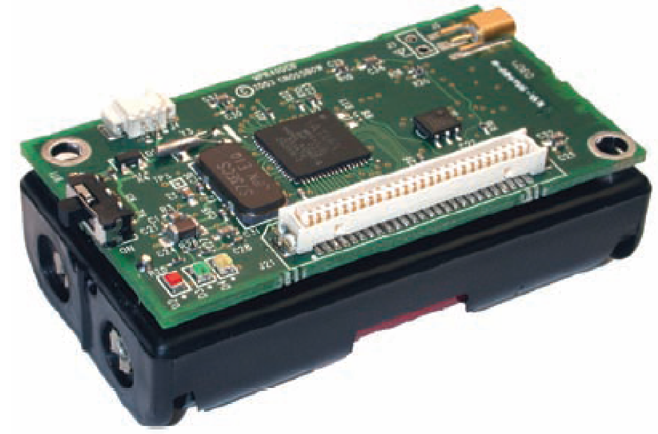
- no location capabilities,
- limited computation power,
- limited memory,
- limited battery power,
- communication radius r.

Q What is the topology of X?

How many nodes are needed to recover it?



[Ghrist, Muhammad, IPSN 05]



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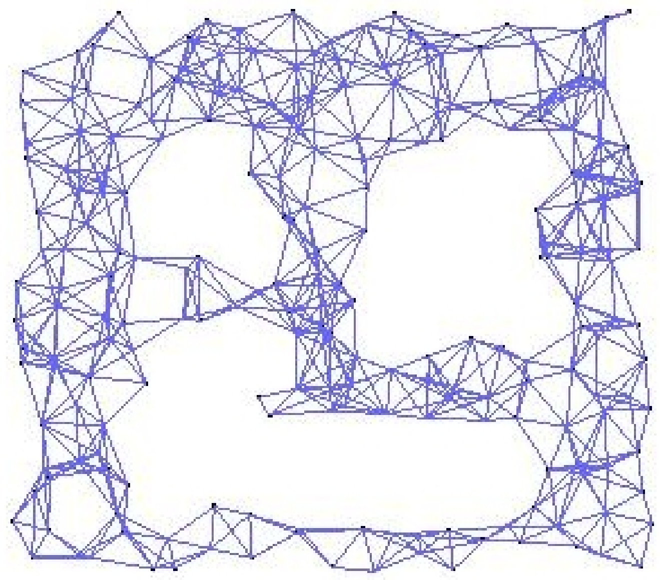

Input: a set of nodes W sampling some unknown planar domain M.

→ the witness complex disregards the embedding (only approximate geodesic distances are used)