Curve Reconstruction, the Traveling Salesman Problem and Menger's Theorem on Length

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Abstract

We give necessary and sufficient regularity conditions under which the curve reconstruction problem is solved by a Traveling Salesman path.

1 Introduction

In 1930 Karl Menger [6] proposed a new definition of arc length:

> The length of an arc be defined as the least upper bound of the set of all numbers that could be obtained by taking each finite set of points of the curve and determining the length of the shortest polygonal graph joining all the points.
> .....
> We call this the messenger problem (because in practice the problem has to be solved by every postman, and also by many travelers): finding the shortest path joining all of a finite set of points, whose pairwise distances are known.

This statement is one of the first references to the Traveling Salesman Problem.

Arc length is commonly defined as the least upper bound of the set of numbers obtained by taking each finite set of points of the curve and determining the length of the polygonal graph joining all the points in their order along the arc. In [7] Menger proves the equivalence of his definition and the common one (Menger's theorem).

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It seems natural to think that this equivalence holds due to the fact that the shortest polygonal graph coincides with the polygonal graph joining the points in their order along the arc, provided the set of points is sufficiently dense. In other words, for sufficiently dense point sets a Traveling Salesman path solves the polygonal reconstruction problem for arcs. This problem was stated by Amenta, Bern and Eppstein [2] as follows:

> Given a curve $\gamma \in \mathbf{R}^d$ and a finite set of sample points $S \subset \gamma$. A polygonal reconstruction of γ from S is a graph that connects every pair of samples adjacent along γ, and no others.

But Menger's proof does not show this at all. So we want to study the question whether the polygonal reconstruction problem is solved by a Traveling Salesman path, provided the sample points are sufficiently dense in the curve. Since a Traveling Salesman path is always simple, we cannot expect that it solves the reconstruction problem for curves with intersections. Even worse, the Traveling Salesman path may not coincide with the polygonal reconstruction for arbitrarily dense samples of simple curves. Consider the following example:

Let γ be the simple arc which consists of the unit interval on the x-axis and the graph of $y = x^2$ on this interval. That is,

$$\gamma : [0,1] \to \mathbf{R}^2, t \mapsto \begin{cases} (1-2t, 0) & : \ t \leq \frac{1}{2} \\ (2t-1, (2t-1)^2) & : \ t > \frac{1}{2} \end{cases}$$

For large n the samples

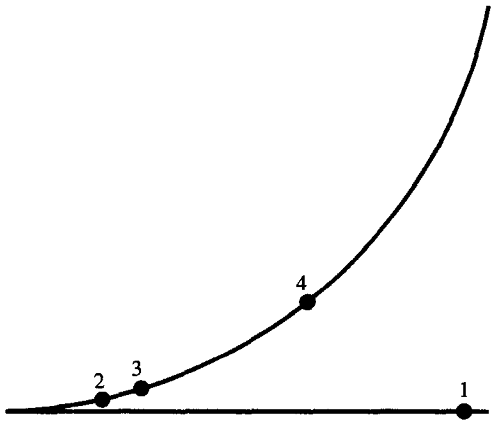
$$S_n = \{p_n^1, p_n^2, p_n^3, p_n^4\} \cup \bigcup_{i=2}^{n} \left\{ \left(\frac{i}{n}, 0\right), \left(\frac{i}{n}, \frac{i^2}{n^2}\right) \right\}$$

with

$$p_n^1 = \left(\tfrac{1}{n}, 0\right) \quad , \quad p_n^2 = \left(\tfrac{1}{n^3}, \tfrac{1}{n^6}\right),$$
$$p_n^3 = \left(\tfrac{2}{n^3}, \tfrac{4}{n^6}\right) \quad , \quad p_n^4 = \left(\tfrac{1}{n^2}, \tfrac{1}{n^4}\right)$$

become arbitrarily dense in γ. But the Traveling Salesman path through S_n is different from the polygonal reconstruction from S_n because

$$\begin{aligned}&|p_n^1 - p_n^2| + |p_n^2 - p_n^3| + |p_n^3 - p_n^4| \\ > \ &|p_n^1 - p_n^3| + |p_n^3 - p_n^2| + |p_n^2 - p_n^4|.\end{aligned}$$



1-3-2-4 is shorter than 1-2-3-4

In this example, the arc γ has finite length and finite total curvature. Thus, even finite curvature, which is a stronger property than rectifiability, is not sufficient for the polygonal reconstruction problem to be solved by a Traveling Salesman path, provided the points are sampled densely enough. The crucial point is that γ behaves quite well, but it is not regular in $(0,0) \in \gamma$. The regularity conditions necessary turn out to be only slightly stronger.

In this article we prove:

Suppose that for every point of the arc

1. the left and the right tangents exist and are non-zero,
2. the angle between these tangents is no more than π.

Under these conditions there exists a finite sampling density such that the Traveling Salesman path solves the polygonal reconstruction problem for all samples with larger sample density.

Regularity is a local property. In contrast to that, it is a global property for a path to be a shortest polygonal path through a finite point set. One of the most interesting aspects of our work is this transition from a local to a global property and the methods used therein. We first prove a local version of our global theorem by using a projection technique from Integral Geometry. We believe that this technique could be useful in many other contexts, even in the study of higher dimensional objects than curves. At a first glance it is not obvious how to derive the global version from the local one. This extension is achieved by an application of two corollaries of Menger's theorem.

Many algorithms designed for the curve reconstruction problem (for example [2, 3, 4]) work by picking a cleverly chosen subset of the edges of the Delaunay triangulation of the sample points. From the example above we can derive another example in which there is an edge of the polygonal reconstruction which is not a Delaunay edge for arbitrary dense samples. Let R_n be the radius of the unique circle through the points p_n^1, p_n^2 and p_n^3. We can calculate that:

$$\lim_{n\to\infty} R_n = \infty$$

That is, if we extend γ to the halfspace $\{(x,y) : y < 0\}$ and take sample points in this halfspace we find by the open ball criterion for Delaunay edges that the edge $\mathrm{conv}\{p_n^1, p_n^2\}$ cannot be a Delaunay edge for large n. It turns out that for curve reconstruction via the Delaunay triangulation the same regularity restrictions are necessary and sufficient as for reconstruction via a Traveling salesman path. In this article we do not want to give a proof of this, because we consider it less interesting than the proof for the Traveling Salesman path. Nevertheless we want to point out two interesting questions which result from this observation:

1. What are the necessary conditions on the regularity of a simple curve such that for dense samplings a Traveling Salesman path through a set of sample points always consists of Delaunay edges? In [5] we show that it is sufficient that in every point of the curve a non-zero tangent exists. Here we claim without giving a proof that the regularity conditions mentioned before are sufficient.
2. Is there an efficient algorithm that always computes a simple polygon, which consists only of Delaunay edges and which for sufficiently dense samples is a polygonal reconstruction? This algorithm should work for the weakest regularity conditions possible, i.e. in every point on the curve that we want to reconstruct left and right tangents have to exist and the angle between these tangents should be strictly smaller than π.

2 Basic Definitions

In this section we give the definition of regularity and the definition of a sample and its density. For the simplicity of presentation we restrict ourselves in this article on simple closed curves $\gamma : [0,1] \to \mathbf{R}^d$.

Here we abuse slightly the notions of Differential Geometry and call a curve γ regular if in every point on γ

non-zero left and right tangents exist. This is expressed in the following definition:

Definition 2.1 *Let*

$$T = \{(t_1, t_2) \,:\, t_1 < t_2, t_1, t_2 \in [0,1]\}$$

and

$$\tau : T \to \mathbf{S}^{d-1}, (t_1, t_2) \mapsto \frac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|}.$$

The curve γ is called left (right) regular at $\gamma(t_0)$ with left (right) tangent $\mathrm{t}(\gamma(t_0))$ if for every sequence (ξ_n) in T which converges to (t_0, t_0) from the left (right) in closure(T) *the sequence $\tau(\xi_n)$ converges to $\mathrm{t}(\gamma(t_0))$. We call γ regular if it is left and right regular in all points $\gamma(t), t \in [0,1]$.*

The relationship between regularity and two of the most interesting geometric properties of curves, length and total absolute curvature, was shown by Aleksandrov and Reshetnyak [1]:

Theorem 2.1 *Every curve γ of finite total absolute curvature $C(\gamma)$ is regular and every regular curve has finite length $L(\gamma)$.*

The example we gave in the introduction shows that even the finiteness of total curvature and hence regularity are not sufficient for the Traveling Salesman tour to solve the reconstruction problem. Next we want to give the definition of a sample we use in this article:

Definition 2.2 *A sample S of γ is a finite set*

$$S = \{p^1, \ldots, p^n\}$$

of points where $p^i \in \gamma$. We assume that the sample points p^i are ordered according to the order of the $\gamma^{-1}(p^i) \in [0,1]$. To every sample S its density is defined to be the inverse of the following number

$$\varepsilon(S) = \sup_{x \in \gamma} \min\{|p^i - x| \,:\, i = 0, \ldots n\}.$$

Here we want to study what conditions on the regularity of γ are necessary such that there exists a positive constant ε, which of course depends on γ, such that for every sample S of γ with $\varepsilon(S) < \varepsilon$ the Traveling Salesman tour through the sample points solves the reconstruction problem.

3 A Local Analysis

In this section we give several reformulations of our notion of regularity. We end up with a reformulation which is a local version of the theorem we want to prove in this article.

Differentiability has two aspects. The first one is algorithmical in a certain sense: We can approximate a differentiable function locally by a linear one, which allows us to make use of the apparatus of Linear Algebra. The second aspect is regularity, which is independent of a underlying linear structure on the range space of our function. It only makes use of the metric structure. The reformulation of regularity in the following lemma is a pure metric interpretation of our definition of regularity.

Lemma 3.1 *Let γ be a simple closed curve, which is left (right) regular in $p \in \gamma$. Let $(p_n), (q_n)$ and (r_n) be sequences of points from γ, that converge to p from the left (right), such that $p_n < q_n < r_n$ for all $n \in \mathbf{N}$ in an order locally around p along γ. Then the sequence of angles (α_n) converges to π, where α_n is the angle at q_n of the triangle with corner points p_n, q_n and r_n.*

PROOF. Since γ is locally homeomorphic the sequences $(\gamma^{-1}(p_n)), (\gamma^{-1}(q_n))$ and $(\gamma^{-1}(r_n))$ converge to $\gamma^{-1}(p)$. Thus by our definition of left (right) regularity asymptotically the three secants

$$\mathrm{conv}\{p_n, q_n\}, \mathrm{conv}\{q_n, r_n\} \text{ and } \mathrm{conv}\{p_n, r_n\}$$

have to point in the direction of the left (right) tangent $\mathrm{t}(p)$. That is, $\lim_{n\to\infty} \alpha_n = \pi$. □

We suppose that the metric aspect of regularity is the important one for our theorem to hold. In [5] we give a pure metric proof of this theorem under slightly stronger regularity conditions, but in the following we have to make use of the linear structure of $\mathbf{R}^{d+1}$. We exploit this linear structure by studying projections of a set of sample points on one-dimensional subspaces of $\mathbf{R}^{d+1}$. It is an interesting open question if the theorem as we want to prove it here can be proved in abstract metric spaces.

In the following let $p \in \gamma$ be a fixed point. We want to introduce the following notions. Let $\eta > 0$. The connected component of

$$\{q \in \gamma \,:\, |p - q| < \eta\}$$

which contains p is denoted by $B_\eta(p)$. The left tangent of γ in p is denoted by $\mathrm{t}_l(p)$ and the right tangent in p is denoted by $\mathrm{t}_r(p)$. We assume that every one-dimensional subspace ℓ of $\mathbf{R}^{d+1}$ not perpendicular to $\mathrm{t}_l(p)$ is oriented according to the orientation induced by the orthogonal projection $\pi_\ell(\mathrm{t}_l(p))$ of $\mathrm{t}_l(p)$ on ℓ and that every one-dimensional subspace ℓ perpendicular to $\mathrm{t}_l(p)$ carries an arbitrary orientation.

We want to compare the ordering of a set of sample points close to p on γ with the ordering of the projections of these points on one-dimensional subspaces of $\mathbf{R}^{d+1}$. The following reformulation of regularity states to which extent these orderings can be different:

Lemma 3.2 *Assume there exists a sequence* (ℓ_n) *of one-dimensional subspaces of* $\mathbf{R}^{d+1}$ *and sequences* $(p_n), (q_n), (r_n) \in B_{1/n}(p)$ *with* $p_n < q_n < r_n \leq p$ *in the order along* γ, *but*

$$\pi_{\ell_n}(q_n) < \pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n)$$

or

$$\pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n) < \pi_{\ell_n}(q_n)$$

in the order on ℓ_n. *Then the limit of every convergent subsequence of* (ℓ_n) *has to be perpendicular to* $\mathrm{t}_l(p)$.

PROOF. We want to do the proof by contradiction. That is, we can assume that (ℓ_n) converges and that

$$0 \leq \alpha := \lim_{n\to\infty} \angle(\mathrm{t}_l(p), \ell_n) < \frac{\pi}{2}.$$

For the proof we assume without loss of generality that $\pi_{\ell_n}(q_n) < \pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n)$, and by the continuity of γ and the continuity of the projection maps π_{ℓ_n} we can assume that there exists $0 < \lambda < 1$ such that

$$|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)| = \lambda \cos(\alpha)\, |\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|$$

if we move q_n or r_n a little bit on γ. From the regularity of γ we find that

$$\begin{aligned} &\lim_{n\to\infty} \angle(\mathrm{conv}\{q_n, r_n\}, \mathrm{t}_l(p)) \\ &= \lim_{n\to\infty} \angle(\mathrm{conv}\{p_n, r_n\}, \mathrm{t}_l(p)) \\ &= 0 \end{aligned}$$

and together with the triangle inequality for spherical triangles

$$\begin{aligned} &\lim_{n\to\infty} \angle(\mathrm{conv}\{q_n, r_n\}, \pi_{\ell_n}(\mathrm{conv}\{q_n, r_n\})) \\ &\leq \lim_{n\to\infty} \Big(\angle(\mathrm{conv}\{q_n, r_n\}, \mathrm{t}_l(p)) \\ &\qquad + \angle(\mathrm{t}_l(p), \pi_{\ell_n}(\mathrm{conv}\{q_n, r_n\}))\Big) \\ &= \lim_{n\to\infty} \angle(\mathrm{conv}\{q_n, r_n\}, \mathrm{t}_l(p)) \\ &\qquad + \lim_{n\to\infty} \angle(\mathrm{t}_l(p), \pi_{\ell_n}(\mathrm{conv}\{q_n, r_n\})) \\ &= 0 + \lim_{n\to\infty} \angle(\mathrm{t}_l(p), \ell_n) \\ &= \alpha < \frac{\pi}{2} \end{aligned}$$

and analogously

$$\lim_{n\to\infty} \angle(\mathrm{conv}\{p_n, r_n\}, \pi_{\ell_n}(\mathrm{conv}\{p_n, r_n\})) \leq \alpha < \frac{\pi}{2}.$$

Hence

$$0 < \cos(\alpha) \leq \lim_{n\to\infty} \frac{|\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|}{|q_n - r_n|} \leq 1$$

and

$$0 < \cos(\alpha) \leq \lim_{n\to\infty} \frac{|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)|}{|p_n - r_n|} \leq 1.$$

Therefore we find

$$\begin{aligned} &\frac{1}{\cos(\alpha)} \\ &\geq \lim_{n\to\infty} \frac{|\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|}{|q_n - r_n|} \frac{|p_n - r_n|}{|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)|} \\ &= \lim_{n\to\infty} \frac{|\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|}{|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)|} \frac{|p_n - r_n|}{|q_n - r_n|} \\ &= \lim_{n\to\infty} \frac{|\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|}{|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)|} \lim_{n\to\infty} \frac{|p_n - r_n|}{|q_n - r_n|} \\ &= \frac{1}{\lambda \cos(\alpha)} \lim_{n\to\infty} \frac{|p_n - r_n|}{|q_n - r_n|} \end{aligned}$$

and thus

$$\lim_{n\to\infty} \frac{|p_n - r_n|}{|q_n - r_n|} \leq \lambda < 1.$$

Hence there exists $N \in \mathbf{N}$ such that $|p_n - r_n| \leq |q_n - r_n|$ for all $n \geq N$. Now we consider the triangles with corner points p_n, q_n and r_n. From the law of cosines together with $|p_n - r_n| \leq |q_n - r_n|$ we find

$$\cos(\alpha_n) = -\frac{|p_n - r_n|^2 - |p_n - q_n|^2 - |q_n - r_n|^2}{2|p_n - q_n||q_n - r_n|} > 0.$$

Thus the angle at q_n, denoted by α_n, has to be smaller or equal than $\frac{\pi}{2}$ for all $n \geq N$. Since γ is left regular in p that is a contradiction to Lemma 3.1. □

The collection of all one-dimensional subspaces of $\mathbf{R}^{d+1}$ forms the d-dimensional projective space $\mathbf{P}^d$. In the following the elements of $\mathbf{P}^d$ are called lines. From a standard construction in Integral Geometry [8] we get a probability measure μ_d on $\mathbf{P}^d$. The next reformulation of regularity makes use of this measure:

Lemma 3.3 *For all* $n \in \mathbf{N}$ *let* $p_n, q_n, r_n \in B_{1/n}(p)$ *with* $p_n < q_n < r_n \leq p$ *in the order along* γ. *Let*

$$\begin{aligned} L_n = \{\ell \in \mathbf{P}^d : &\ \pi_\ell(q_n) < \pi_\ell(p_n) < \pi_\ell(r_n) \ \textit{or} \\ &\ \pi_\ell(p_n) < \pi_\ell(r_n) < \pi_\ell(q_n) \ \textit{in the order on } \ell\}, \end{aligned}$$

where π_ℓ *denotes the orthogonal projection on the line* ℓ. *Then* $\lim_{n\to\infty} \mu_d(L_n) = 0$.

PROOF. In [8] Reshetnyak shows: Let V be a d-dimensional subspace of $\mathbf{R}^{d+1}$ and $E \subset \mathbf{P}^d$ be the set of all one-dimensional subspaces contained in V. Then $\mu_d(E) = 0$.

The proof of the lemma follows directly from this theorem of Reshetnyak and Lemma 3.2. □

Obviously an analogous result for sample points larger than p in the order along γ also holds.

Now we are prepared to prove a local version of our theorem. For a given sample S of a neighborhood of p we fix the smallest and the largest sample point along

γ and consider paths through S which connect these points. One such path is the polygonal reconstruction $P(S)$ which connects the sample points in their order along γ. We distinguish three types of lines $\ell \in \mathbf{P}^d$:

1. There exists a path through the sample points different from $P(S)$ which has a shorter projection on ℓ than $P(S)$.
2. Every path through the sample points different from $P(S)$ has a larger projection on ℓ than $P(S)$.
3. There exist paths through the sample points different from $P(S)$ which projections on ℓ have the same length as the projection of $P(S)$, but there is no path with shorter projection.

The proof is subdivided in three parts. First, we show that the measure of the first set of lines tends to zero as the neighborhood of p shrinks to p itself. Second, there exists a constant larger than zero such that the measure of the second set of lines is larger than this constant for arbitrary small neighborhoods of p. Finally we conclude from the first two steps by integrating over the length of all projections that for small neighborhoods of p the polygonal reconstruction $P(S)$ has to be the shortest path through the sample points.

Theorem 3.1 *Assume*

$$\alpha = \sup\{\angle(\mathrm{t}_l(q), \mathrm{t}_r(q)) \,:\, q \in \gamma\} < \pi$$

and let (S_n) *be a sequence of samples of* $B_{1/n}(p)$. *Then there exists* $N \in \mathbf{N}$ *such that*

$$TSP^*(S_n) = P(S_n),$$

for all $n \geq N$. *Where* $TSP^*(S_n)$ *is a path of minimal length through the sample points* S_n *with fixed startpoint* $\min S_n$ *and fixed endpoint* $\max S_n$. *Here* min *and* max *are taken with respect to the order induced on* S_n *by* γ. *Furthermore* $TSP^*(S_n)$ *is unique for all* $n \geq N$.

PROOF. **First Step.** We show that

$$\lim_{n\to\infty} \mu_d(L_n) = 0.$$

Here $L_n \subset \mathbf{P}^d$ is the set of lines for which the projection $\pi_\ell(P(S_n))$ is not a shortest path through the projected sample points $\pi_\ell(S_n)$ and μ_d is the probability measure on $\mathbf{P}^d$ introduced in [8]. We use the following abbreviations

$$\begin{aligned} m_1 &= \pi_\ell(\min S_n) \\ m_2 &= \min_\ell\{\pi_\ell(\min S_n), \pi_\ell(\max S_n)\} \\ m_3 &= \max_\ell\{\pi_\ell(\min S_n), \pi_\ell(\max S_n)\} \\ m_4 &= \pi_\ell(\max S_n) \end{aligned}$$

Path of minimal length

Take $\ell \in \mathbf{P}^d$ together with its orientation. A path of minimal length through the points $\pi_\ell(S_n)$ which connects m_1 with m_4 consists of

$$\mathrm{conv}\{m_1, m_2\} \cup \mathrm{conv}\{m_2, m_3\} \cup \mathrm{conv}\{m_3, m_4\}$$

That is, the points of ℓ between m_1 and m_2 are covered twice by a path of minimal length through the points $\pi_\ell(S_n)$, the points between m_1 and m_4 are covered once and the points between m_3 and m_4 are covered twice again. If $\pi_\ell(P(S_n))$ is not a path of minimal length through the points $\pi_\ell(S_n)$, then there has to exists an interval

$$I = [\min_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\}, \max_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\}]$$

on ℓ with $p_n^i, p_n^{i+1} \in S_n$, which is covered by $\pi_\ell(P(S_n))$

1. $2 + 2k$ times, $k \geq 1$, if
$$\begin{aligned} m_2 &\leq \min_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} \\ &< \max_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} \leq m_1 \end{aligned}$$
or
$$\begin{aligned} m_4 &\leq \min_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} \\ &< \max_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} \leq m_3 \end{aligned}$$
2. $1 + 2k$ times, $k \geq 1$, if
$$\begin{aligned} m_1 &\leq \min_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} \\ &< \max_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} \leq m_4. \end{aligned}$$

For all $p_n^j \in S_n - \{p_n^{|S_n|}\}$ we call the interval

$$[\min_\ell\{\pi_\ell(p_n^j), \pi_\ell(p_n^{j+1})\}, \max_\ell\{\pi_\ell(p_n^j), \pi_\ell(p_n^{j+1})\}]$$

positive oriented if $\pi_\ell(p_n^j) < \pi_\ell(p_n^{j+1})$ in the order on ℓ and negative oriented otherwise. The intervals

$$[\min_\ell\{\pi_\ell(p_n^j), \pi_\ell(p_n^{j+1})\}, \max_\ell\{\pi_\ell(p_n^j), \pi_\ell(p_n^{j+1})\}]$$

which we get from the projection of $P(S_n)$ and which cover the interval I must have alternating orientations. That is, if $\pi_\ell(P(S_n))$ is not a shortest path through the points $\pi_\ell(S_n)$ we find, using that $P(S_n)$ connects the $p^j \in S_n$ in their order along γ and using the continuity of γ and of the projection map π_ℓ, three points $p_n, q_n, r_n \in B_{1/n}(p)$ with $p_n < q_n < r_n \leq p$ in the order along γ, but

$$\pi_\ell(q_n) < \pi_\ell(p_n) < \pi_\ell(r_n) \;\text{ or }\; \pi_\ell(p_n) < \pi_\ell(r_n) < \pi_\ell(q_n)$$

in the order on ℓ. Or we find three points $p_n, q_n, r_n \in B_{1/n}(p)$ with $p_n > q_n > r_n \geq p$ in the order along γ, but

$$\pi_\ell(q_n) > \pi_\ell(p_n) > \pi_\ell(r_n) \text{ or } \pi_\ell(p_n) > \pi_\ell(r_n) > \pi_\ell(q_n)$$

in the order on ℓ. The points p_n, q_n and r_n need not be sample points. From Lemma 3.3 we can conclude that

$$\lim_{n\to\infty} \mu_d(L_n) = 0.$$

Second Step. We show that there exist $c > 0$ and $N' \in \mathbf{N}$ such that for all $n \geq N'$ we have

$$\mu_d(M_n) \geq c.$$

Here $M_n \subset \mathbf{P}^d$ is the set of lines ℓ for which we have

1. the order of S_n along γ and the order of S_n induced by the order of $\pi_\ell(S_n)$ on ℓ coincide.
2. for all $\text{conv}\{p^i, p^{i+1}\} \subset P(S_n)$, $p^i, p^{i+1} \in S_n$ we have

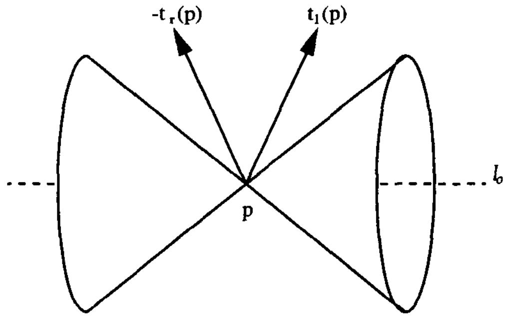
   $$|\pi_\ell(p^{i+1}) - \pi_\ell(p^i)| \geq \frac{c_d}{2}\cos(\frac{\pi+\alpha}{4})|p^{i+1} - p^i|,$$

   with
   $$c_d = \frac{\Gamma((d+1)/2)}{\sqrt{\pi}\,\Gamma((d+2)/2)}.$$

For the proof we construct a set of lines $C \subset \mathbf{P}^d$ with $\mu_d(S) > 0$ and show that there exists $N' \in \mathbf{N}$ such that $C \subset M_n$ for all $n \geq N'$. The set C is defined as follows: Let ℓ_0 be the line, $\ell_0 \subset \text{span}\{t_l(p), t_r(p)\}$ such that $\ell_0^\perp \subset \text{span}\{t_l(p), t_r(p)\}$ halves the angle $\angle(t_l(p), -t_r(p))$ between the lines determined by $t_l(p)$ and $t_r(p)$. Now we define

$$C = \left\{\ell \in \mathbf{P}^d \,:\, \angle(\ell, \ell_0) \leq \frac{1}{4}(\pi - \alpha)\right\}.$$

By $\angle(\ell, \ell')$ for $\ell, \ell' \in \mathbf{P}^d$ we denote the value of of the minimum of the two angles determined by ℓ and ℓ'. The set $\bigcup_{\ell\in C}\{x \in \ell\} \subset \mathbf{R}^{d+1}$ is a double cone.



The double cone C

Since $\alpha < \pi$ we have $\mu_d(C) > 0$. It remains to check conditions 1. and 2. to prove that for sufficiently large n we have $C \subset M_n$.

By construction $\pi_\ell(t_l(p))$ and $\pi_\ell(t_r(p))$ point in the same direction on every $\ell \in C$. Using the triangle inequality for spherical triangles and $\alpha < \pi$ we find for all $\ell \in C$,

$$\begin{aligned}\angle(\ell, t_l(p)) &\leq \angle(\ell, \ell_0) + \angle(\ell_0, t_l(p)) \\ &< \frac{1}{4}(\pi - \alpha) + \left(\frac{\pi}{2} - \frac{\pi-\alpha}{2}\right) \\ &= \frac{\pi+\alpha}{4} < \frac{\pi}{2}\end{aligned}$$

and analogously

$$\angle(\ell, t_r(p)) < \frac{\pi}{2}.$$

That is, every $\ell \in C$ is neither perpendicular to $t_l(p)$ nor to $t_r(p)$.

1. Assume that for arbitrary large n we find $\ell_n \in C$ such that the first condition is violated. Then we can find three points $p_n, q_n, r_n \in B_{1/n}(p)$ with $p_n < q_n < r_n \leq p$ in the order along γ, but

   $$\pi_{\ell_n}(q_n) < \pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n)$$

   or

   $$\pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n) < \pi_{\ell_n}(q_n)$$

   in the order on ℓ_n. Or we find three points $p_n, q_n, r_n \in B_{1/n}(p)$ with $p_n > q_n > r_n \geq p$ in the order along γ, but

   $$\pi_{\ell_n}(q_n) > \pi_{\ell_n}(p_n) > \pi_{\ell_n}(r_n)$$

   or

   $$\pi_{\ell_n}(p_n) > \pi_{\ell_n}(r_n) > \pi_{\ell_n}(q_n)$$

   in the order on ℓ_n. By Lemma 3.1 the limit of every convergent subsequence of (ℓ_n) has to be perpendicular to either $t_l(p)$ or $t_r(p)$. Since C is compact we find $\ell \in C$ as the limit of a convergent subsequence of (ℓ_n) which is perpendicular to either $t_l(p)$ or $t_r(p)$. But using the triangle inequality for spherical triangles we find for all $\ell \in C$:

   $$\begin{aligned}\angle(\ell, t_l(p)) &\leq \angle(\ell, \ell_0) + \angle(\ell_0, t_l(p)) \\ &< \frac{1}{4}(\pi - \alpha) + \left(\frac{\pi}{2} - \frac{\pi-\alpha}{2}\right) \\ &= \frac{\pi+\alpha}{4} < \frac{\pi}{2}\end{aligned}$$

   and analogously

   $$\angle(\ell, t_r(p)) < \frac{\pi}{2}$$

   That is, every $\ell \in C$ is neither perpendicular to $t_l(p)$ nor to $t_r(p)$. So we got a contradiction. That means, for all sufficiently large n the first condition cannot be violated.

2. From the regularity of γ we have for $p_n^i, p_n^{i+1} \in S_n$ with $p_n^i < p_n^{i+1} \leq p$ and all $\ell \in C$,

$$\begin{aligned}
&\lim_{n\to\infty} \angle(\ell, \operatorname{conv}\{p_n^i, p_n^{i+1}\}) \\
&= \angle(\ell, \lim_{n\to\infty} \operatorname{conv}\{p_n^i, p_n^{i+1}\})) \\
&= \lim_{n\to\infty} \angle(\ell, \mathrm{t}_l(p)) \leq \frac{\pi+\alpha}{4}.
\end{aligned}$$

That is, for all sufficiently large n and all $\ell \in C$ we have

$$|\pi_\ell(p^i) - \pi_\ell(p^{i+1})| \geq c_d \cos(\frac{\pi+\alpha}{4})|p^i - p^{i+1}|.$$

For $p_n^i, p_n^{i+1} \in S_n$ with $p \leq p_n^i < p_n^{i+1}$ we get the same result using $\mathrm{t}_r(p)$ instead of $\mathrm{t}_l(p)$ in the triangle inequality for spherical triangles. Now choose $\eta > 0$ such that

$$\cos\left(\frac{\pi+\alpha}{4} + \eta\right) > \frac{1}{2}\cos\left(\frac{\pi+\alpha}{4}\right).$$

For $p_n^i \leq p < p_n^{i+1}$ or $p_n^i < p \leq p_n^{i+1}$ and sufficiently large n we find

$$\begin{aligned}
\angle(\ell_0, \operatorname{conv}\{p_n^i, p_n^{i+1}\}) &\leq \angle(\ell_0, \mathrm{t}_l(p)) + \eta \\
&= \angle(\ell_0, \mathrm{t}_r(p)) + \eta \\
&= \frac{\alpha}{2} + \eta,
\end{aligned}$$

because the regularity of γ implies

$$\begin{aligned}
\limsup\left(\angle(\ell_0, \pi(\operatorname{conv}\{p_n^i, p_n^{i+1}\})\right) &\leq \angle(\ell_0, \mathrm{t}_l(p)) \\
&= \angle(\ell_0, \mathrm{t}_r(p)),
\end{aligned}$$

where π is the projection on $\operatorname{span}\{\mathrm{t}_l(p), \mathrm{t}_r(p)\}$, and

$$\lim_{n\to\infty} \angle(\operatorname{span}\{\mathrm{t}_l(p), \mathrm{t}_r(p)\}, \operatorname{conv}\{p_n^i, p_n^{i+1}\}) = 0.$$

That means, we have for sufficiently large n,

$$\begin{aligned}
&\angle(\ell, \operatorname{conv}\{p_n^i, p_n^{i+1}\}) \\
&\leq \angle(\ell, \ell_0) + \angle(\ell_0, \operatorname{conv}\{p_n^i, p_n^{i+1}\}) \\
&\leq \angle(\ell, \ell_0) + \angle(\ell_0, \mathrm{t}_l(p)) \\
&\leq \frac{\pi-\alpha}{4} + \left(\frac{\alpha}{2} + \eta\right) \\
&= \frac{\pi+\alpha}{4} + \eta.
\end{aligned}$$

Hence for all sufficiently large n and all $\ell \in C$ we have

$$|\pi_\ell(p^i) - \pi_\ell(p^{i+1})| \geq \frac{c_d}{2} \cos(\frac{\pi+\alpha}{4})|p^i - p^{i+1}|$$

for all $p_n^i, p_n^{i+1} \in S_n$.

That is, there exists $N' \in \mathbf{N}$ such that for all $n \geq N'$ we have $C \subset M_n$, hence

$$\mu_d(M_n) \geq \mu_d(C) > 0.$$

Third step. In the first two steps we considered the measures of the subspaces L_n and M_n of $\mathbf{P}^d$, the space on which we want to integrate. In this step we want to compare the integrands. For $\ell \in L_n$ we define we define permutations $\rho_{\ell,n}$ of $\{1, \ldots, |S_n|\}$ such that

$$\rho_{\ell,n}(i) \geq |\{j : \pi_\ell(p_n^j) < \pi_\ell(p_n^i) \text{ in the order of } \ell\}|.$$

and $\rho_{\ell,n}(j) = \rho(i) + 1$ if $\pi_\ell(p_n^i) = \pi_\ell(p_n^j)$ and $i < j$. If all $\pi_\ell(S_n)$ are distinct $\rho_{\ell,n}(i)$ is the position of $\pi_\ell(p_n^i)$ in the order on ℓ. For any path $\tilde{P}(S_n)$ through the points S_n, which connects $p_n^1 = \min S_n$ with $p_n^{|S_n|} = \max S_n$, with shorter projection $\pi_\ell(\tilde{P}(S_n))$ than $\pi_\ell(P(S_n))$ there has to exist at least one non-degenerate interval

$$I = [\pi_\ell(p_n^{\rho(i)}), \pi_\ell(p_n^{\rho(i)+1})]$$

which is covered less often (at least two times) by $\pi_\ell(\tilde{P}(S_n))$ than by $\pi_\ell(P(S_n))$. That is, there has to exist $j \in \{1, \ldots, |S_n| - 1\}$ such that the oriented segment $\operatorname{conv}\{p_n^j, p_n^{j+1}\} \notin \tilde{P}(S_n)$ and the interval

$$[\min_\ell\{\pi_\ell(p_n^j), \pi_\ell(p_n^{j+1})\}, \max_\ell\{\pi_\ell(p_n^j), \pi_\ell(p_n^{j+1})\}]$$

cover I. It is still possible that the same segment $\operatorname{conv}\{p_n^{j+1}, p_n^j\}$ with opposite orientation is part of $\tilde{P}(S_n)$, e.g. if $\rho_{\ell,n}(i) = j + 1$ and $\rho_{\ell,n}(i) + 1 = j$! We distinguish two cases.

1. $j \neq 1, |S_n| - 1$: For all $\ell' \in M_n$ the interval $[\pi_{\ell'}(p_n^j), \pi_{\ell'}(p_n^{j+1})]$ has to be covered by $\pi_{\ell'}(\tilde{P}(S_n))$ at least two times more than by $\pi_{\ell'}(P(S_n))$.

2. $j = 1$ or $j = |S_n| - 1$: We want to reduce this case to the first one and assume $j = 1$. For $j = |S_n| - 1$ there is an analogous reasoning. Since I has to be covered at least three times by $\pi_\ell(P(S_n))$ and the covering intervals have alternating orientations (see first step) we find $k \in \{2, \ldots |S_n| - 2\}$ such that

   $$[\min_\ell\{\pi_\ell(p_n^k), \pi_\ell(p_n^{k+1})\}, \max_\ell\{\pi_\ell(p_n^k), \pi_\ell(p_n^{k+1})\}]$$

   also covers I and has opposite orientation than $[\min_\ell\{\pi_\ell(p_n^1), \pi_\ell(p_n^2)\}, \max_\ell\{\pi_\ell(p_n^1), \pi_\ell(p_n^2)\}]$.
   That is,

   $$\rho_{\ell,n}(k) < \rho_{\ell,n}(1) \text{ and } \rho_{\ell,n}(k+1) > \rho_{\ell,n}(2)$$

   if $\rho_{\ell,n}(1) > \rho_{\ell,n}(2)$,

   $$\rho_{\ell,n}(k) > \rho_{\ell,n}(1) \text{ and } \rho_{\ell,n}(k+1) < \rho_{\ell,n}(2)$$

   if $\rho_{\ell,n}(1) < \rho_{\ell,n}(2)$. If the oriented segment $\operatorname{conv}\{\pi_\ell(p_n^k), \pi_\ell(p_n^{k+1})\} \not\subset \tilde{P}(S_n)$ we have the same situation as in the first case. Otherwise $\tilde{P}(S_n)$ has to contain a segment s on the way from p_n^1 to p_n^k which projection covers the interval I. Since $\pi_\ell(P(S_n))$ covers I more often than $\tilde{P}(S_n)$ there has to exist $k' \in \{2, \ldots |S_n| - 1\}$ such that the interval

$$[\min_\ell\{\pi_\ell(p_n^{k'}), \pi_\ell(p_n^{k'+1})\}, \max_\ell\{\pi_\ell(p_n^{k'}), \pi_\ell(p_n^{k'+1})\}]$$

covers I and has the same orientation than $[\min_\ell\{\pi_\ell(p_n^1), \pi_\ell(p_n^2)\}, \max_\ell\{\pi_\ell(p_n^1), \pi_\ell(p_n^2)\}]$ and the oriented segment $\text{conv}\{\pi_\ell(p_n^{k'}), \pi_\ell(p_n^{k'+1})\} \notin P$. If $k' \neq |S_n| - 1$ we are in the first case. Otherwise $\tilde{P}(S_n)$ has to contain a segment s' on the way from p_n^{k+1} to $p_n^{|S_n|}$ which projection covers the interval I, because we have from the orientations of the covering intervals

$$\rho_{\ell,n}(|S_n|) > \rho_{\ell,n}(k+1) \quad \text{if} \quad \rho_{\ell,n}(1) < \rho_{\ell,n}(2)$$
$$\rho_{\ell,n}(|S_n|) > \rho_{\ell,n}(k+1) \quad \text{if} \quad \rho_{\ell,n}(1) > \rho_{\ell,n}(2).$$

By construction we have $s' \neq s$. That is, $\tilde{P}(S_n)$ covers I at least three times. Since $\pi_\ell(P(S_n))$ covers I at more often than $\tilde{P}(S_n)$ there has to exist $k'' \in \{2, \ldots |S_n| - 2\}$ such that the interval

$$[\min_\ell\{\pi_\ell(p_n^{k''}), \pi_\ell(p_n^{k''+1})\}, \max_\ell\{\pi_\ell(p_n^{k''}), \pi_\ell(p_n^{k''+1})\}]$$

also covers I and we have for the oriented segment $\text{conv}\{\pi_\ell(p_n^{k''}), \pi_\ell(p_n^{k''+1})\} \not\subset \tilde{P}(S_n)$. So we are finally in the first case.

Using this property of the coverings and using that

$$|\pi_{\ell'}(p^{i+1}) - \pi_{\ell'}(p^{i+1})| \;\geq\; \frac{c_d}{2}\cos(\frac{\pi+\alpha}{4})|p^{i+1} - p^i|$$

for all $p_n^i \in S_n - \{p_n^{|S_n|}\}$ and all $\ell' \in M_n$ we find that the increase of length of the projections on $\ell' \in M_n$ is bounded from below by the decrease of length of the projection on ℓ as follows,

$$\begin{aligned} & L(\pi_{\ell'}(P(S_n))) - L(\pi_{\ell'}(\tilde{P}(S_n))) \\ & \geq \frac{c_d}{2}\cos(\frac{\pi+\alpha}{4}) \\ & \quad \Big(L(\pi_\ell(\tilde{P}(S_n)) - L(\pi_\ell(P(S_n)))\Big). \end{aligned}$$

This inequality is valid for all $\ell \in L_n$ and all $\ell' \in M_n$. That is, we get for the increase of length on M_n

$$\begin{aligned} & \int_{M_n} \Big(L(\pi_\ell(P(S_n))) - L(\pi_\ell(\tilde{P}(S_n)))\Big) \mathrm{d}\mu_d(\ell) \\ & \geq \frac{c_d}{2}\cos(\frac{\pi+\alpha}{4})\,\mu_d(M_n) \\ & \quad \sup_{\ell \in L_n} \Big(L(\pi_\ell(\tilde{P}(S_n))) - L(\pi_\ell(P(S_n)))\Big) \end{aligned}$$

and for the decrease of length on L_n

$$\begin{aligned} & \int_{L_n} \Big(L(\pi_\ell(\tilde{P}(S_n))) - L(\pi_\ell(P(S_n)))\Big) \mathrm{d}\mu_d(\ell) \\ & \leq \mu_d(L_n) \sup_{\ell \in L_n} \Big(L(\pi_\ell(\tilde{P}(S_n))) - L(\pi_\ell(P(S_n)))\Big). \end{aligned}$$

Since $\lim_{n\to\infty} \mu_d(L_n) = 0$ there exists $N \geq N'$ such for all $n \geq N$ we have

$$\begin{aligned} \mu_d(L_n) & < \frac{c_d}{2}\cos(\frac{\pi+\alpha}{4})\mu_d(C) \\ & \leq \frac{c_d}{2}\cos(\frac{\pi+\alpha}{4})\mu_d(M_n). \end{aligned}$$

Using another theorem of Reshetnyak [8] which states for regular curves γ:

$$\int_{\mathbf{P}^d} L(\pi_\ell(\gamma))\mathrm{d}\mu_d(\ell) = c_d L(\gamma)$$

we find that for all $n \geq N$ there is no shortcut possible and that $P(S_n)$ is the unique path of minimal length through the points S_n with fixed start- and endpoint, because the polygon connecting the points in the order induced by γ has a shorter projection on all $\ell \in C$ than every other polygon through the points S_n with fixed start- and endpoint. □

4 From Local to Global

In this section we finally want to prove the promised theorem. That is, here we achieve the transition from the local results of the last section to the global. In doing so we make heavy use of two corollaries of Menger's theorem [7].

To formulate these corollaries let (S_n) be a sequence of samples with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. The first corollary states

$$\lim_{n\to\infty} L(TSP(S_n)) = L(\gamma).$$

The second one states, if π_n is a permutation of S_n such that for all $n \in \mathbf{N}$

$$\sum_{i=1}^{|S_n|-1} |p_n^{\pi_n(i+1)} - p_n^{\pi_n(i)}| \leq L(\gamma)$$

Then

$$\lim_{n\to\infty} \max\{|p_n^{\pi_n(i+1)} - p_n^{\pi_n(i)}| : p_n^i \in S_n\} = 0.$$

That is, the maximal length of a segment in the Traveling Salesman tour tends to zero as the density of the samples goes to infinity.

Furthermore we need another two definitions. Let $S = \{p^1, \ldots, p^n\}$ be a sample of γ. We write

$$i \lhd j \;\text{ if }\; L(\gamma(p^i : p^j)) \leq L(\gamma(p^j : p^i)),$$

where $\gamma(p^i : p^j) \subset \gamma$ is the arc connecting p^i and p^j in the order along γ. We write $i \trianglelefteq j$ if we want to include the possibility that $i = j$. These notions are well defined if γ is regular, because regular curves have finite length according to Theorem 2.1.
We call $r \in S$ a return point, if r is connected to $p, q \in S$ and $r \lhd p, q$ or $r \rhd p, q$ in the order along γ. In the first case we call the return point positive and in the second case we call it negative.

Now we are prepared to prove our main theorem. The proof is done by contradiction and it is subdivided in three steps. We show in the third step that the local version of the theorem does not hold if the global one does not hold. That is, the local version implies the global version.

Theorem 4.1 *Assume*

$$\alpha = \sup\{\angle(\mathrm{l}(q), \mathrm{r}(q)) \,:\, q \in \gamma\} < \pi$$

and let (S_n) *be a sequence of samples of* γ *with* $\lim_{n\to\infty} \varepsilon(S_n) = 0$. *Then there exists* $N \in \mathbf{N}$ *such that* $TSP(S_n) = P(S_n)$ *for all* $n \geq N$. *Here* $TSP(S_n)$ *is a shortest tour through the points* S_n. *Furthermore* $TSP(S_n)$ *is unique for all* $n \geq N$.

PROOF. We want to do the proof by contradiction and assume without loss of generality that $TSP(S_n) \neq P(S_n)$ for all $n \in \mathbf{N}$.

First Step. We show that there has to exist a return point for large n. Assume the contrary. That is, there does not exist a return point in S_n for arbitrary large n. By turning to a subsequence we can assume without loss of generality that there does not exist a return point for all $n \in \mathbf{N}$. Since $TSP(S_n) \neq P(S_n)$ there exists $p_n^i \in S_n$ which is not connected to p_n^{i-1} in $TSP(S_n)$. We cut $TSP(S_n)$ in two polygonal arcs P_n^1, with startpoint p_n^i and endpoint p_n^{i-1}, and P_n^2, with startpoint p_n^{i-1} and endpoint p_n^i. By our assumption that there does not exist a return point in S_n the sample points in both polygonal arcs are connected in their order along γ. From the two corollaries of Menger's theorem we can conclude that

$$\liminf L(P_n^1),\ \liminf L(P_n^2) \geq L(\gamma).$$

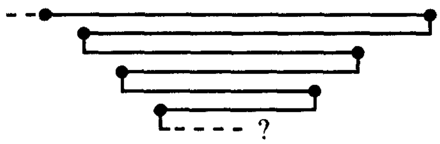
That is, $\liminf L(TSP(S_n) \geq 2L(\gamma)$. Which is a contradiction.

Second Step. We show that there must exist two return points $r_n^1 \lhd r_n^2$ incident along $TSP(S_n)$ such that the other return points $\tilde{r}_n^1$ incident to r_n^1 and $\tilde{r}_n^2$ incident to r_n^2 along $TSP(S_n)$ are not in between r_n^1 and r_n^2. That is, we do not have the following situation

$$(1) \qquad r_n^1 \lhd \tilde{r}_n^1 \lhd r_n^2 \ \text{ or } \ r_n^1 \lhd \tilde{r}_n^2 \lhd r_n^2.$$

But it is possible that $\tilde{r}_n^1 = r_n^2$ and $\tilde{r}_n^2 = r_n^1$!

We observe that the sum of the signs of the return points in S_n always has to sum up to zero and that return points incident along $TSP(S_n)$ always have different signs. So we can conclude from the first step that for sufficiently large n there exist at least two return points. Assume that for all incident return points we find situation (4.1) then all return points have to accumulate in between two return points.



Accumulating return points

That is impossible since $TSP(S_n)$ is closed.

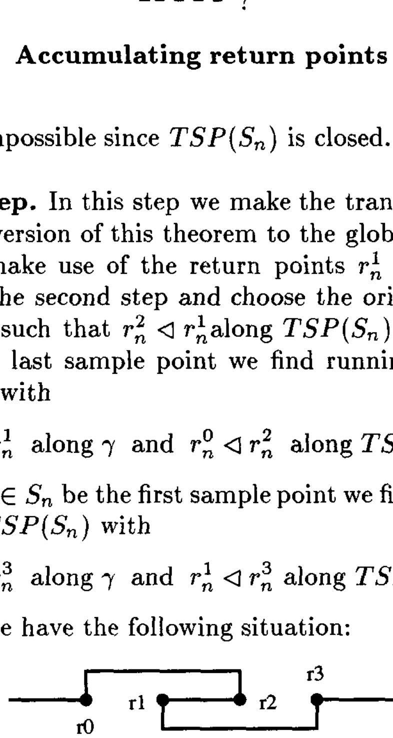
Third Step. In this step we make the transition from the local version of this theorem to the global one. We want to make use of the return points r_n^1 and r_n^2 we found in the second step and choose the orientation of $TSP(S_n)$ such that $r_n^2 \lhd r_n^1$along $TSP(S_n)$. Let $r_n^0 \in S_n$ be the last sample point we find running through $TSP(S_n)$ with

$$r_n^0 \lhd r_n^1 \text{ along } \gamma \text{ and } r_n^0 \lhd r_n^2 \text{ along } TSP(S_n)$$

and let $r_n^3 \in S_n$ be the first sample point we find running through $TSP(S_n)$ with

$$r_n^2 \lhd r_n^3 \text{ along } \gamma \text{ and } r_n^1 \lhd r_n^3 \text{ along } TSP(S_n).$$

That is, we have the following situation:



Shortcut through return points

By the compactness of γ we can assume by turning to convergent subsequences that $(r_n^0), (r_n^1), (r_n^2)$ and (r_n^3) converge to $r^0, r^1, r^2, r^3 \in \gamma$. Let $s_n \in S_n$ be the successor of r_n^0 and let $p_n \in S_n$ be the successor of r_n^3 along $TSP(S_n)$. By construction we have

$$r_n^0 \lhd r_n^1 \lhd s_n \text{ and } p_n \lhd r_n^2 \lhd r_n^3 \text{ along } \gamma.$$

From the second corollary of Menger's theorem we conclude

$$\lim_{n\to\infty} |r_n^0 - s_n| = \lim_{n\to\infty} |p_n - r_n^3| = 0.$$

That is, $r^0 = r^1$ and $r^2 = r^3$. Now assume $r^1 \lhd r^2$. We consider three sets of sample points

$$M_n^1 = \{p \in S_n : p \trianglelefteq r_n^2 \text{ along } TSP(S_n)\}$$

$$M_n^2 = \{p \in S_n : r_n^2 \trianglelefteq p \trianglelefteq r_n^1 \text{ along } TSP(S_n)\}$$
$$M_n^3 = \{p \in S_n : p \trianglerighteq r_n^1 \text{ along } TSP(S_n)\}.$$

We have using the first corollary of Menger's theorem

$$\begin{aligned}
&\lim_{n\to\infty} L(TSP(S_n)) \\
&= \lim_{n\to\infty} \Big(L(TSP(M_n^1)) + L(TSP(M_n^2)) + \\
&\qquad L(TSP(M_n^3))\Big) \\
&= \lim_{n\to\infty} L(TSP(M_n^1)) + \lim_{n\to\infty} L(TSP(M_n^2)) + \\
&\qquad \lim_{n\to\infty} L(TSP(M_n^3)) \\
&= L(\gamma|_{[0,\gamma^{-1}(r^2)]}) + L(\gamma|_{[\gamma^{-1}(r^1),\gamma^{-1}(r^2)]}) + \\
&\qquad L(\gamma|_{[\gamma^{-1}(r^1),1]}) \\
&= L(\gamma) + 2L(\gamma|_{[\gamma^{-1}(r^1),\gamma^{-1}(r^2)]}) > L(\gamma).
\end{aligned}$$

That is a contradiction. Hence we have

$$r^0 = r^1 = r^2 = r^3 =: r \in \gamma.$$

By turning to an appropriate subsequence of (S_n) we can assume without loss of generality that

$$r_n^0, r_n^1, r_n^2, r_n^3 \in S_n \cap B_{1/n}(r).$$

That is a contradiction to Theorem 3.1, which is the local version of this theorem. □

The example in the introduction shows that the regularity conditions required to prove this theorem are necessary. That is, this theorem is best possible.

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References

[1] A.D. Aleksandrov, Yu.G. Reshetnyak *Integral Curvature of a curve in n-dimensional Euclidean space*, Sib. Math. J. 29(1), pp. 1-16 (1988)

[2] N. Amenta, M. Bern, D. Eppstein *The Crust and the β-Skeleton: Combinatorial Curve Reconstruction*, Graphical Models and Image Processing 60/2:2, pp. 125-135 (1998)

[3] F. Bernardini, C. L. Bajaj *Sampling and Reconstructing Manifolds Using Alpha-Shapes*, Proc. of the Ninth Canadian Conference on Computational Geometry 1997, pp. 193–198 (1997)

[4] T.K. Dey, P. Kumar *A simple provable algorithm for curve reconstruction*, Proc. 10th SIAM Sympos. Discrete Algorithms (1999)

[5] J. Giesen *Curve Reconstruction in Arbitrary Dimension and the Traveling Salesman Problem*, Proceedings of DGCI 99 to appear in LNCS (1999)

[6] K. Menger (ed.) Ergebnisse eines Mathematischen Kolloquiums 2, Kolloquium 5.II.1930, Teubner Leipzig (1932)

[7] K. Menger *Untersuchungen über eine allgemeine Metrik. Vierte Untersuchung. Zur Metrik der Kurven*, Math. Ann. 103, pp. 467-501 (1932)

[8] Yu.G. Reshetnyak *Some Applications of Integral Geometry to the Theory of Curves of Finite Rotation*, Sib. Math. J. 29(1), pp. 109-116 (1988)