INF562, Lecture 4: Geometric and combinatorial properties of planar graphs

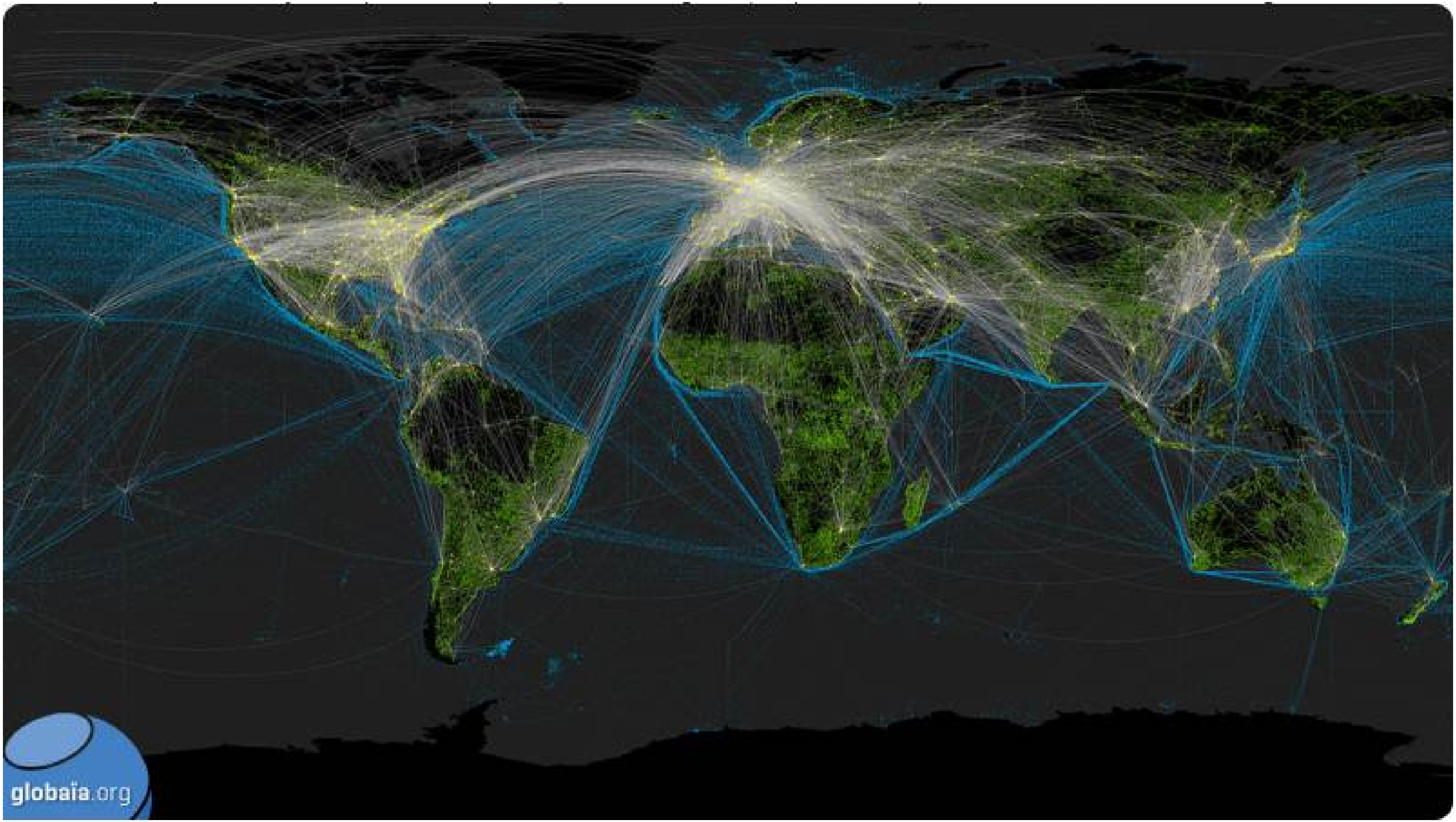
5 jan 2016

Luca Castelli Aleardi

Intro
Graph drawing: motivations and applications

Graph drawing and data visualization

Global transportation system

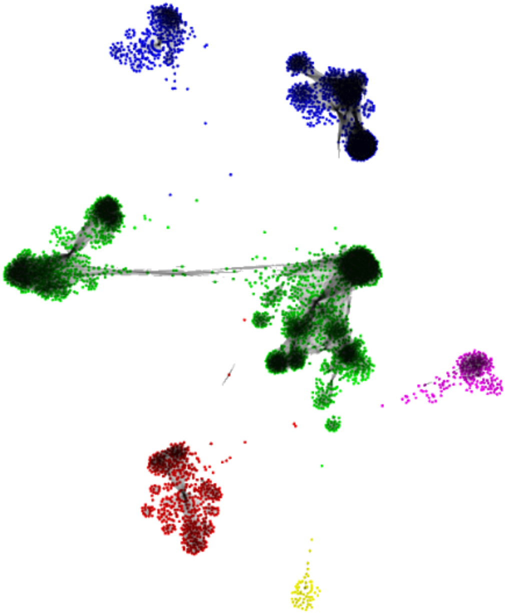


Graph drawing and data visualization

Roads, railways, ...

Graph drawing and data visualization

Social network graph

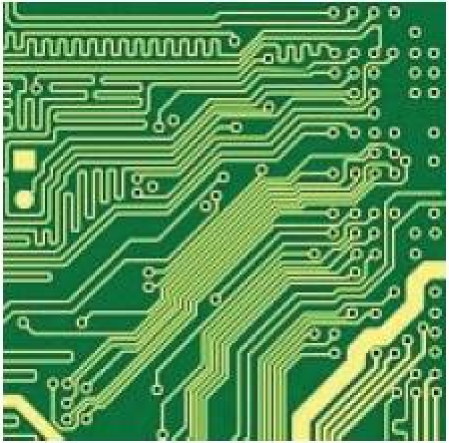


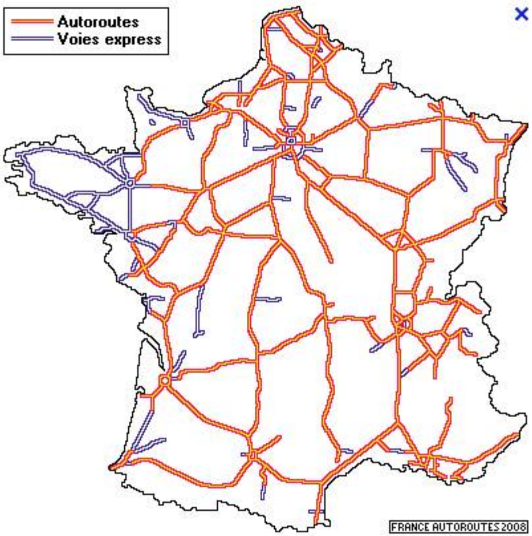
Community detection (via graph clustering)

Planar graphs

french roads network

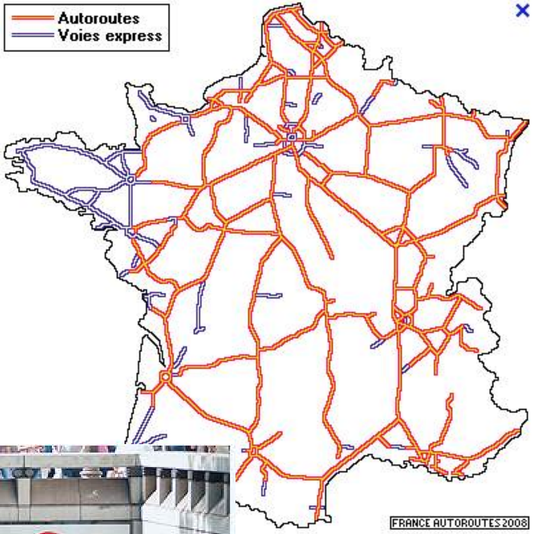
Design of integrated circuits (VLSI)



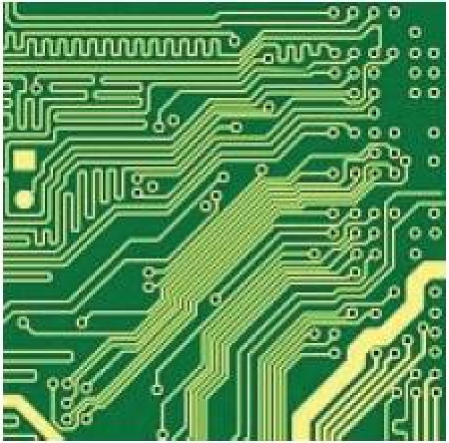


Planar graphs

french roads network



Design of integrated circuits (VLSI)

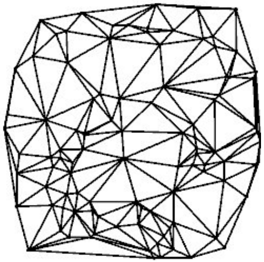


www.2m40.com

9 accidents en 2012 (last one, on 28th sptember)

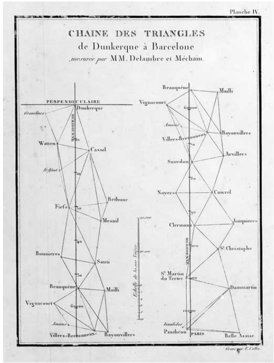
Meshes and graphs in computational geometry

Delaunay triangulations, Voronoi diagrams, planar meshes, ...

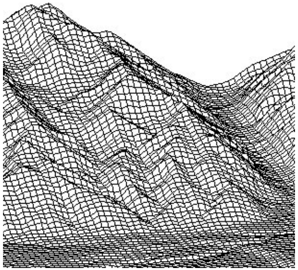


Delaunay triangulation

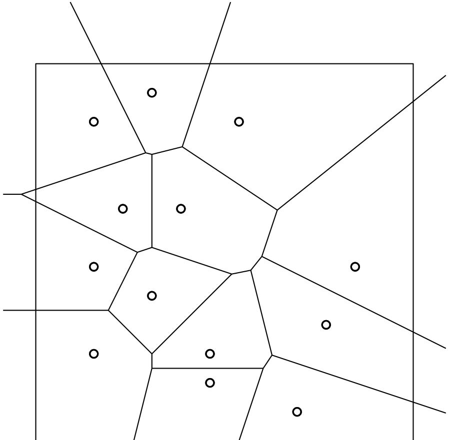
triangles meshes already used in early 19th century (Delambre et Mchain)



GIS Technology

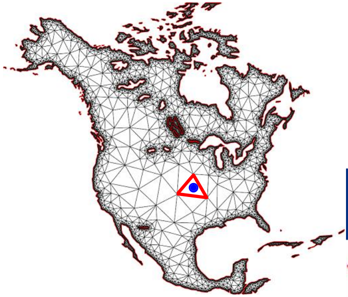


Terrain modelling

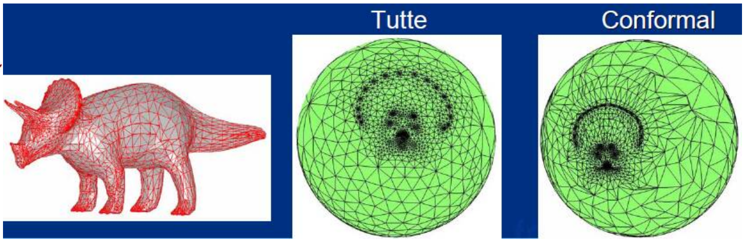


Voronoi diagram

Planar mesh by L. Rineau, M. Yvinec



Spherical Parameterization (Sheffer Gotsman)

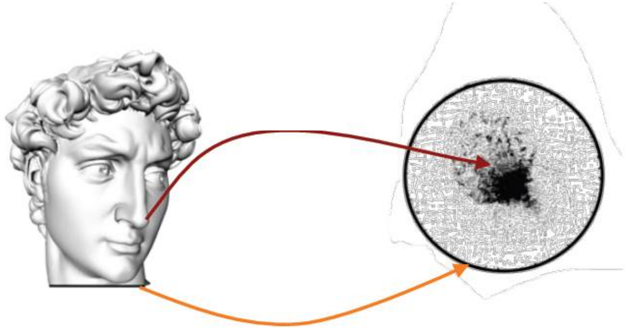


Mesh parameterization (and straight line drawing)

given a (planar) graph G compute a mapping $\rho : (V_G) \longrightarrow R^2$
s.t. edges are straight line segments without crossings

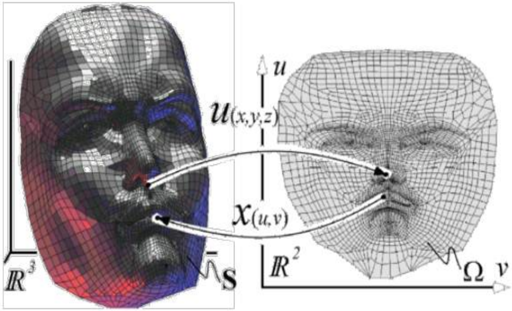
(given a 3D mesh M compute a bijective mapping $\rho : (V_G) \longrightarrow R^2$)

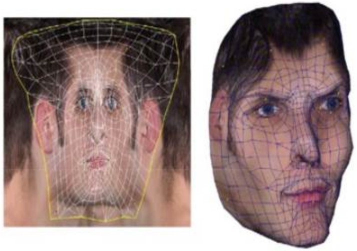
$$\rho(v_i) = (x_i, y_i)$$

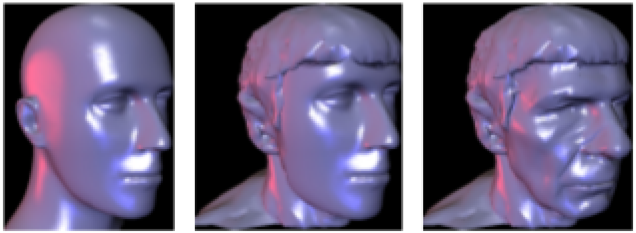


Mesh parameterization: applications

Texture mapping and morphing







(pictures by A. Sheffer)

Bennis et al., 1991
Maillot et al., 1993

Graph drawing: motivation

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Challenge: what kind of graph does A_G represent?

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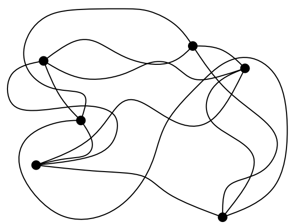

adjacency matrix

$$A_G[i,j] = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Graph drawing: motivation

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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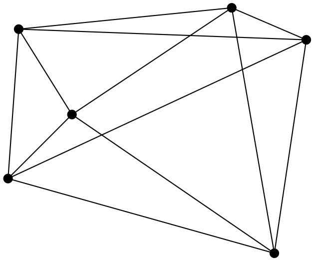

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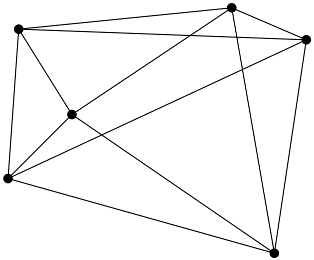

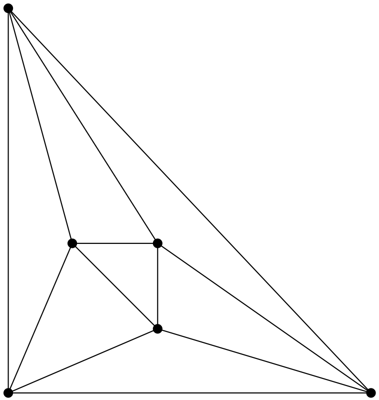

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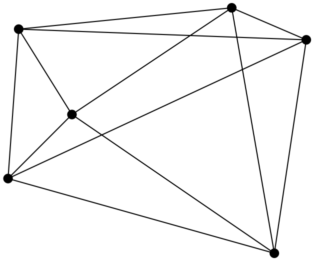

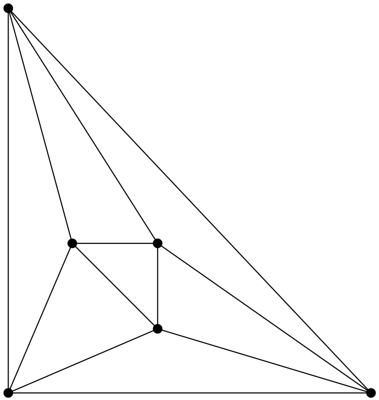

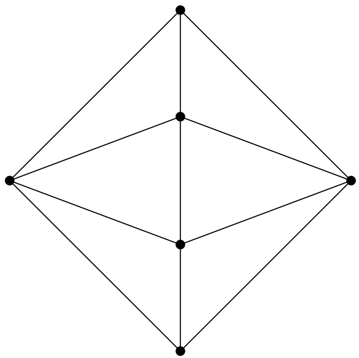

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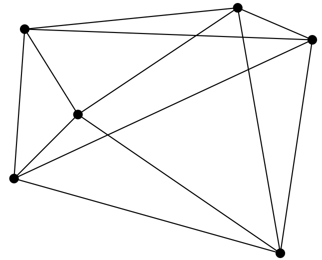

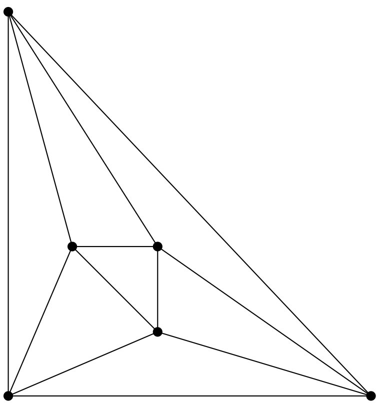

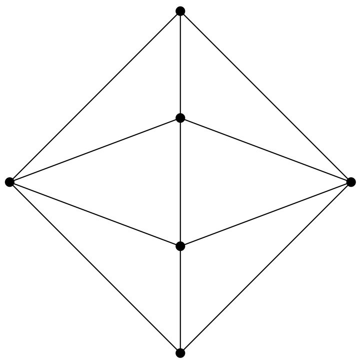

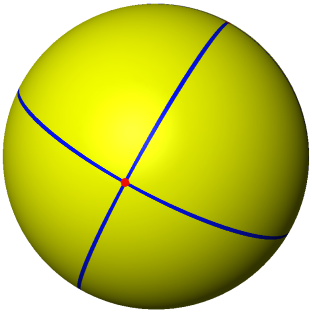

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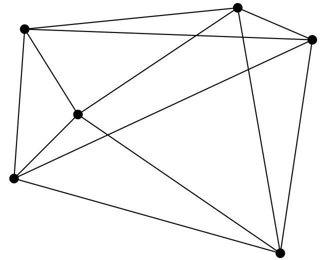

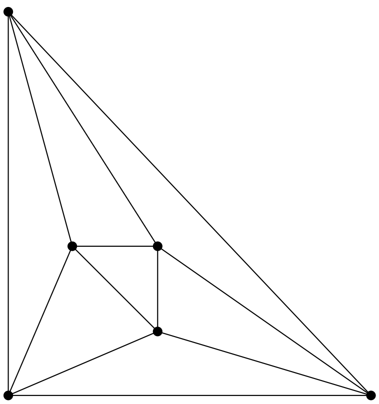

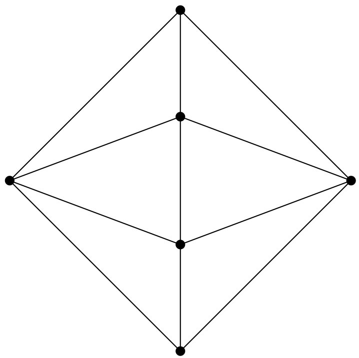

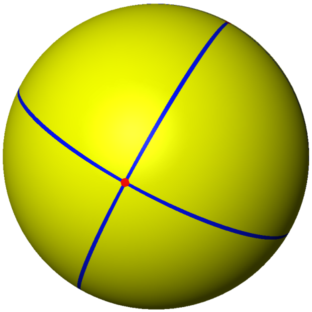

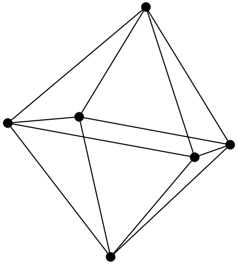

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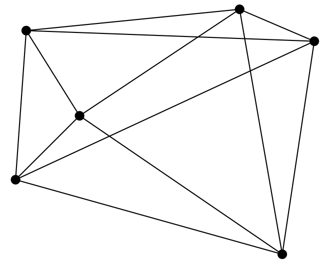

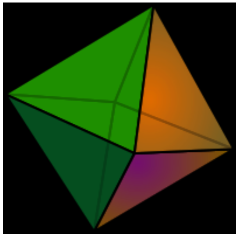

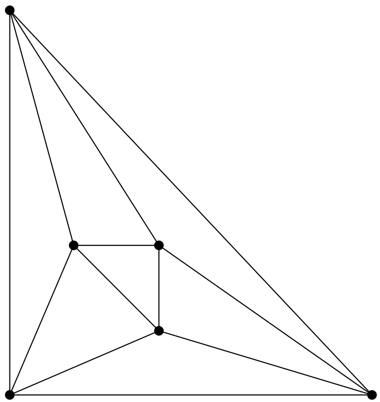

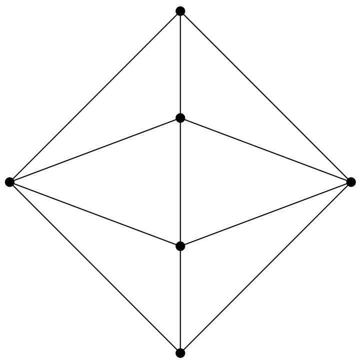

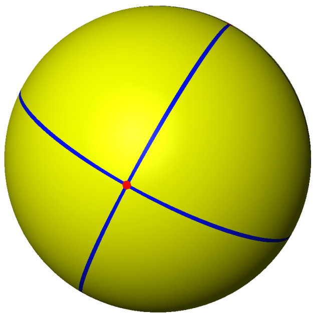

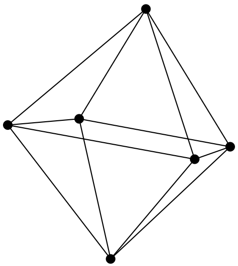

Part I

Major results in graph theory

Major results (on planar graphs) in graph theory

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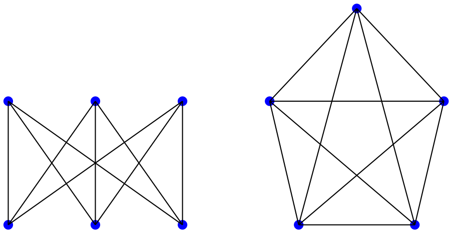
Kuratowski theorem (1930) (cfr Wagner's theorem, 1937)

- G contains neither K_5 nor $K_{3,3}$ as minors



Major results (on planar graphs) in graph theory

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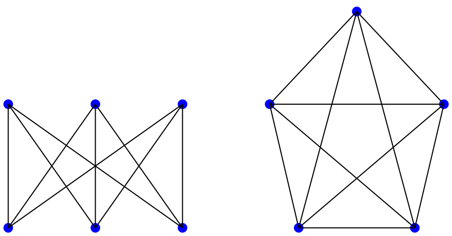
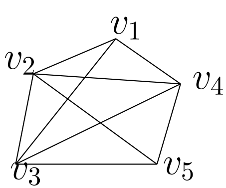
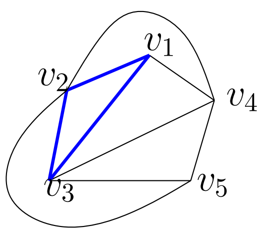


Fáry theorem (1947)

- Every (simple) planar graph admits a straight line planar embedding (no edge crossings)



dessin non planaire de G



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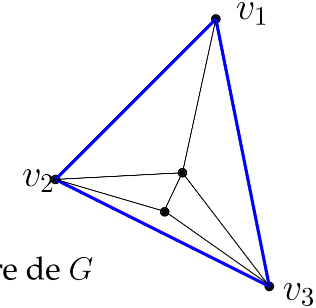
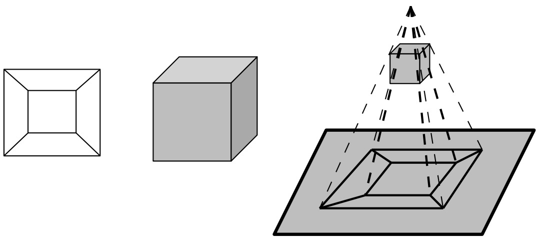


Major results (on planar graphs) in graph theory

Thm (Steinitz, 1916)



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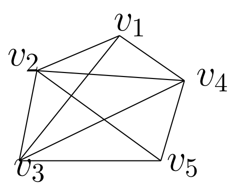

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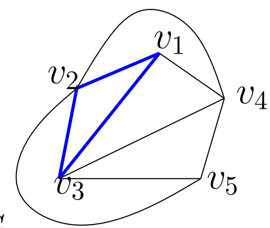

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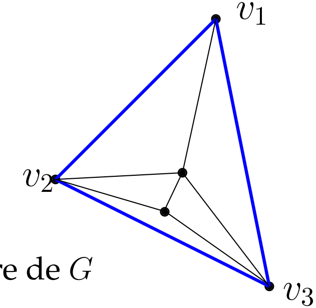
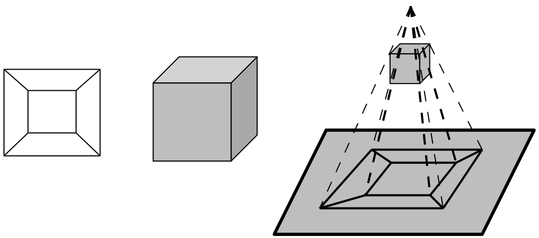


Major results (on planar graphs) in graph theory

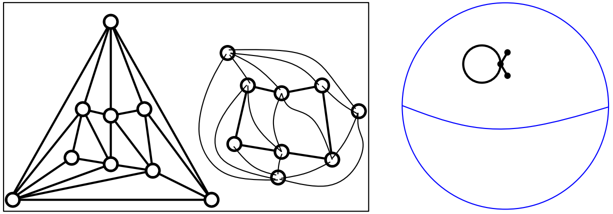
Thm (Steinitz, 1916)

3-connected planar graphs are the 1-skeletons of convex polyhedra



Thm (Whitney, 1933)

3-connected planar graphs admit a unique planar embedding (up to homeomorphism and inversion of the sphere).



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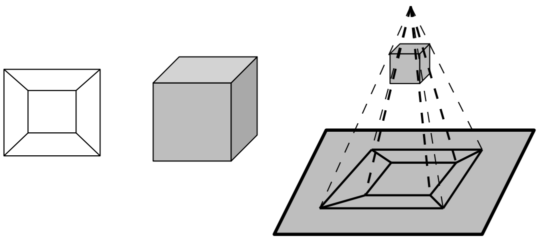
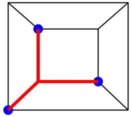


Def *G is 3-connected if*

is connected and
the removal of one or two vertices does not disconnect G



at least 3 vertices are required to disconnect the graph

Thm (Whitney, 1933)

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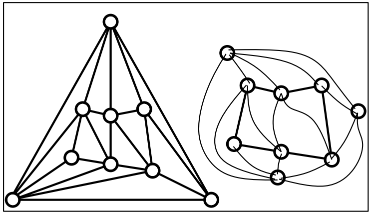
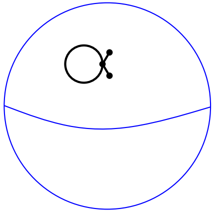
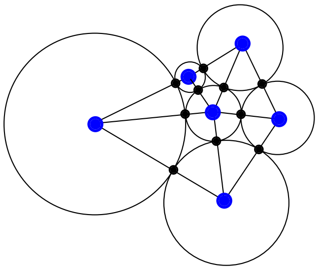


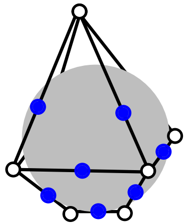


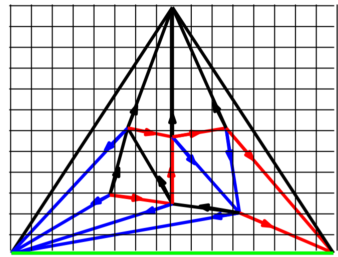
Major results (on planar graphs) in graph theory

Thm (Koebe-Andreev-Thurston)

Every planar graph with n vertices is isomorphic to the intersection graph of n disks in the plane.







Schnyder woods (via dimension of partial orders)

- $dim(G) \leq 3$

Thm (Colin de Verdière, 1990)

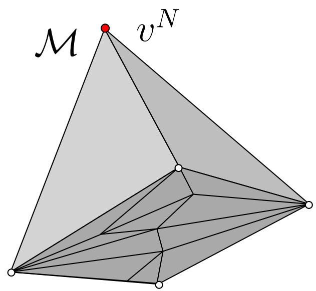
Colin de Verdiere invariant (multiplicity of λ_2 eigenvalue of a generalized laplacian)

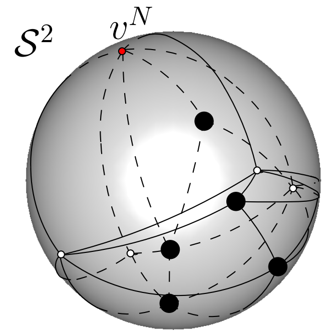
- $\mu(G) \leq 3$

$$\begin{bmatrix} 4 & -1 & \dots & \dots & 0 \\ -1 & 5 & \dots & & \\ \dots & & \dots & & \\ \dots & & & & \dots \\ 0 & \dots & & & 3 \end{bmatrix} \quad L_G[i,k] = \begin{cases} deg(v_i) \\ -A_G[i,j] \end{cases}$$

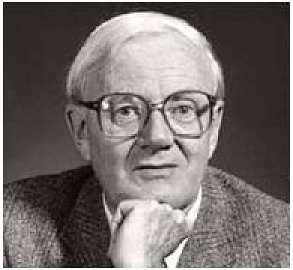
Theorem (Lovasz Schrijver '99)

Given a 3-connected planar graph G, the eigenvectors ξ_2, ξ_3, ξ_4 of a CdV matrix defines a convex polyhedron containing the origin..





Major results (on planar graphs) in graph theory



Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a barycentric representation ρ in R^2.

$$\boxed{\rho(v_i) = \sum_{j \in N(i)} w_{ij}\rho(v_j)}$$

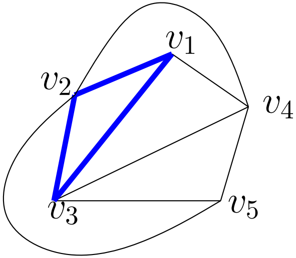
($\sum_j w_{ij} = 1$ and $w_{ij} > 0$

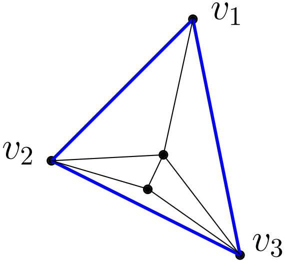
$\rho : (V_G) \longrightarrow R^2$ is barycentric iff for each inner node v_i, $\rho(v_i)$ is the barycenter of the images of its neighbors

Get a straight line drawing solving a system a linear equations

$$N(v_4) = \{v_1, v_2, v_3, v_5\}$$

$$N(v_5) = \{v_2, v_3, v_4\}$$





$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

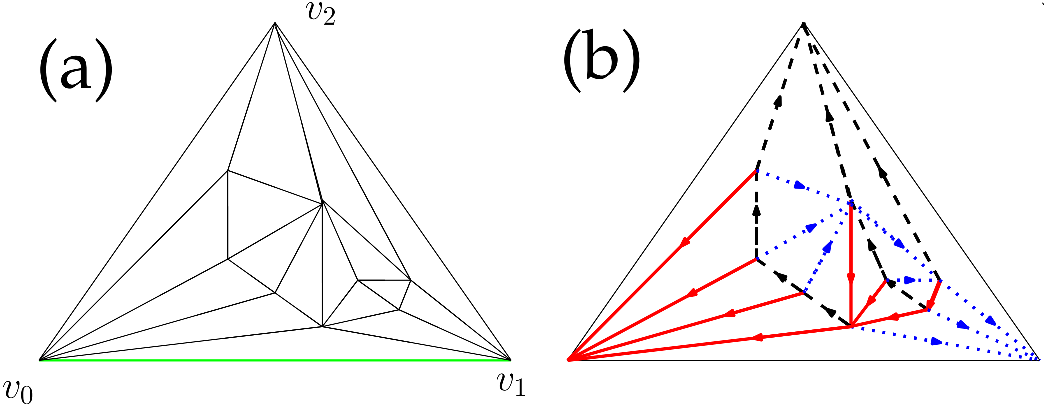
$$\begin{cases} M \cdot \underline{x} = \underline{a_x} \\ M \cdot \underline{y} = \underline{a_y} \end{cases}$$

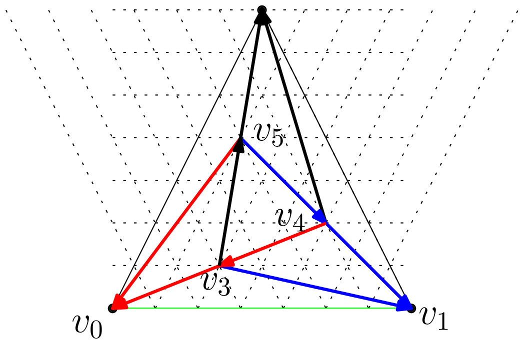
laplacian matrix

Major results (on planar graphs) in graph theory

Theorem (Schnyder '89)
A graph G is planar if and only if the dimension of its incidence poset is at most 3

Theorem (Schnyder, Soda '90)
For a triangulation $\mathcal{T}$ having n vertices, we can draw it on a grid of size $(2n-5)\times(2n-5)$, by setting $v_0=(2n-5,0)$, $v_1=(0,0)$ and $v_2=(0,2n-5)$.



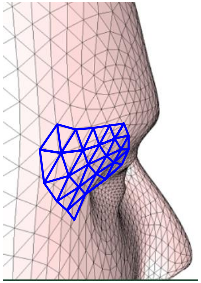


v_3 $(1,2,4)$

v_4 $(2,4,1)$

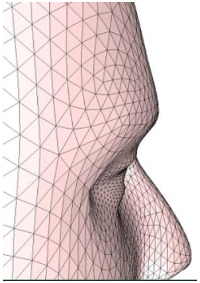
v_5 $(4,1,2)$

Part II
What is a surface mesh?



(a short digression on embedded graphs, simplicial complexes and topological and combinatorial maps)

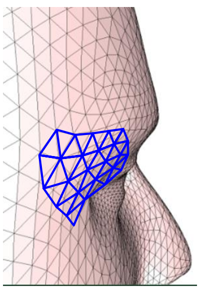
What is a (surface) mesh?

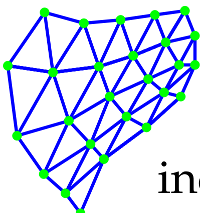


surface mesh: set of vertices, edges and faces (polygons) defining a polyhedral surface in embedded in 3D (discrete approximation of a shape)

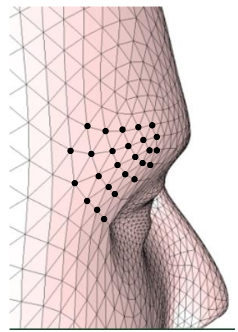
Combinatorial structure + geometric embedding

”Connectivity”: the underlying *map*



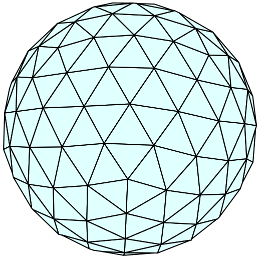


incidence relations between triangles, vertices and edges



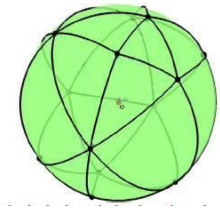
vertex coordinates

Planar and surface meshes: definition

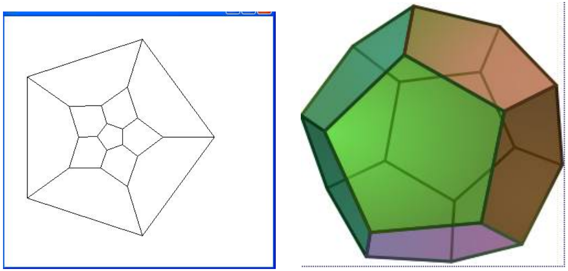


planar triangulation
embedded in R^3

triangle mesh

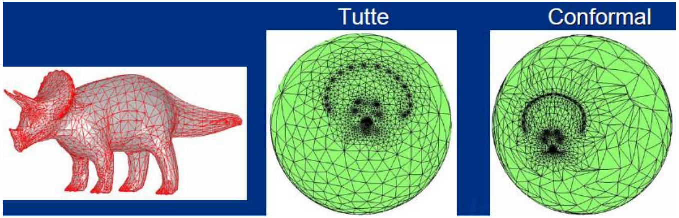


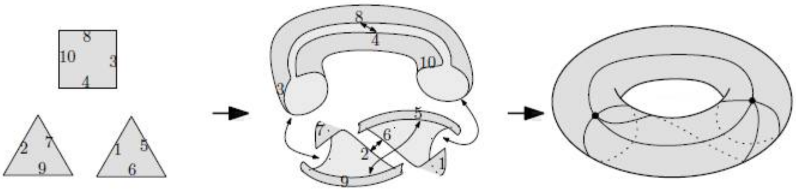
planar triangulation
spherical drawing



planar map
straight line drawing of a dodecahedron

spherical parameterizations of a
triangle mesh (Gotsman, Gu Sheffer, 2003)





toroidal map (Eric Colin de Verdière)

Surface meshes as *simplicial complexes*

abstract simplicial complex K (set of simplices)

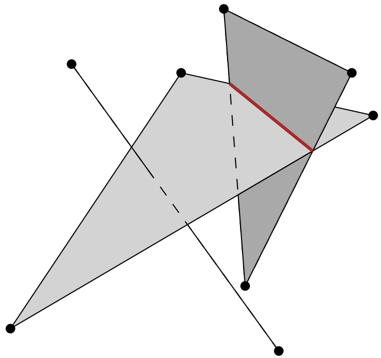
$$V = \{v_0, v_1, \ldots, v_{n-1}\}$$
$$E = \{\{i,j\}, \{k,l\}, \ldots\}$$
$$F = \{\{i,j,k\}, \{j,i,l\}, \ldots\}$$

inclusion property:

$$\rho \in K \text{ and } \sigma \subset \rho \longrightarrow \sigma \in K$$

intersection property:

given two simplices σ_1, σ_2 of K, the intersection $\sigma_1 \cup \sigma_2$ is a face of both



not valid simplicial complex

Surface meshes as *(topological) maps*

(geomettric realizations of maps)

A *graph* $G = (V, E)$ is a pair of:

- a set of *vertices* $V = (v_1, \ldots, v_n)$
- a collection of $E = (e_1, \ldots, e_m)$ elements of the cartesian product $V \times V = \{(u, v) \mid u \in V, v \in V\}$ (*edges*).

a *planar drawing* is a cellular embedding of G into R^2, satisfying:

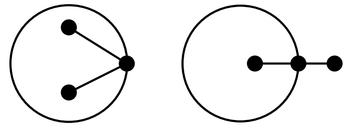
(i) graph vertices are represented as points ;

(ii) edges are represented as no croissing curves ;

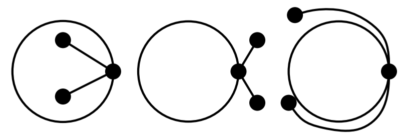
(iii) faces are simply connected.

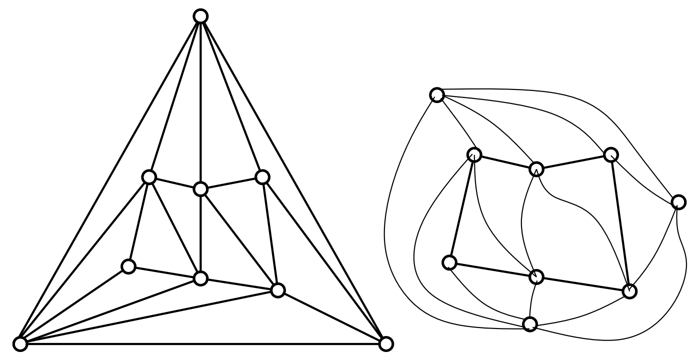
(topologica) map: cellular embedding up to homeomorphism (equivalence class)

two different embeddings of the same graph



cellular embeddings of a graph defining the same (planar) map

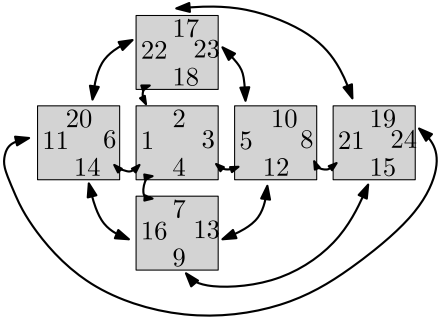


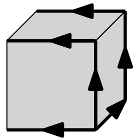


two cellular embeddings defining the same planar map

Surface meshes as *combinatorial maps*

(geomettric realizations of maps)





3 permutations on the set H of the $2n$ half-edges

(i) α involution without fixed point;

(ii) $\alpha\sigma\phi = Id$;

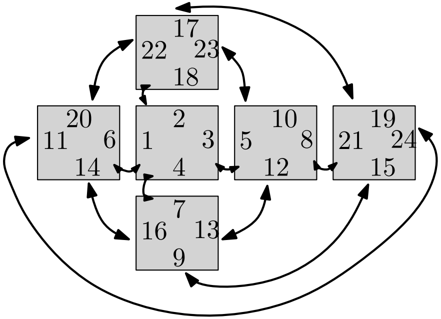
(iii) the group generated by σ, α et ϕ transitively on H.

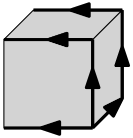
$$\phi = (1,2,3,4)(17,23,18,22)(5,10,8,12)(21,19,24,15)\ldots$$

$$\alpha = (2,18)(4,7)(12,13)(9,15)(14,16)(10,23)\ldots$$

Surface meshes as *combinatorial maps*

(geomettric realizations of maps)





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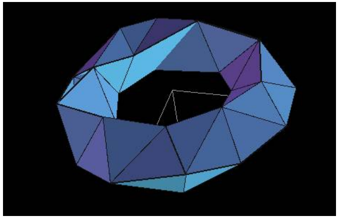
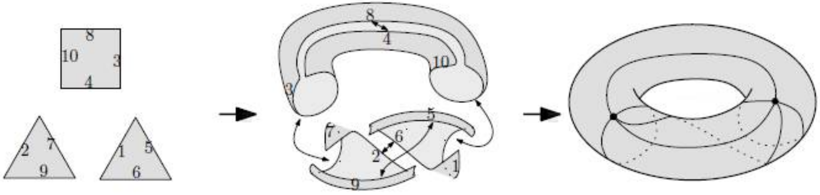
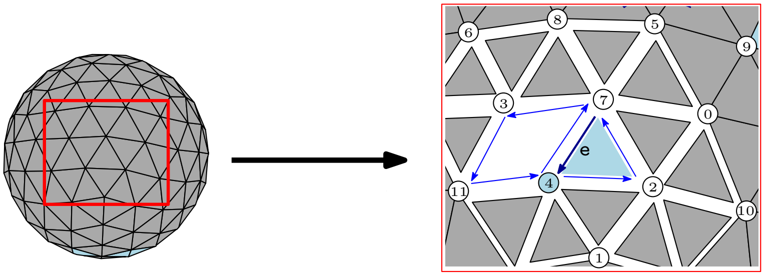
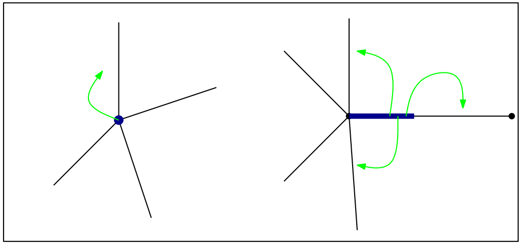
$$\alpha = (2, 18)(4, 7)(12, 13)(9, 15)(14, 16)(10, 23)\ldots$$





Half-edge data structure: polygonal (orientable) meshes



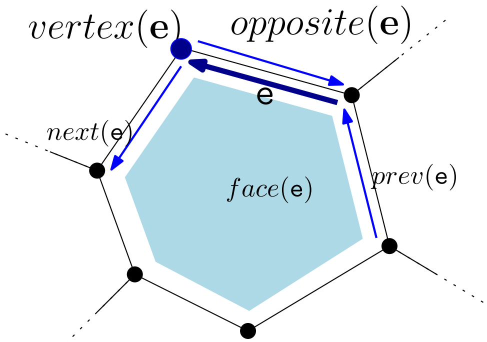


```
class Point{
  double x;
  double y;
}
```

geometric information

$$f + 5 \times h + n \;\approx\; 2n + 5 \times (2e) + n \;=\; 32n + n$$

Size (number of references)



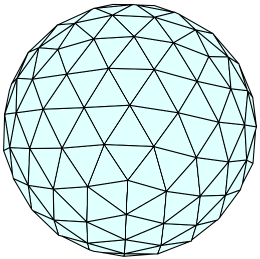

```
public int degree() {
    Halfedge<X> e,p;
    if(this.halfedge==null) return 0;

    e=halfedge; p=halfedge.next;
    int cont=1;
    while(p!=e) {
        cont++;
        p=p.next;
    }
    return cont;
}
```

```
class Halfedge{
  Halfedge  prev, next, opposite;
  Vertex v;
  Face f;
}
class Vertex{
  Halfedge  e;
  Point p;
}
class Face{
  Halfedge  e;
}
```

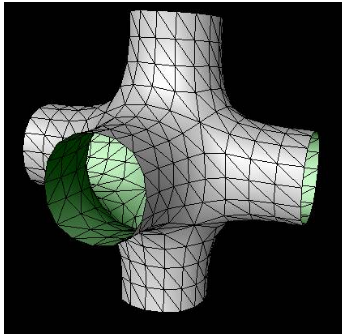
combinatorial information

Mesh representations: classification

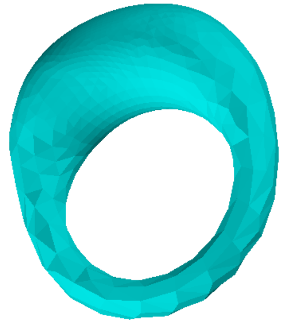
Manifold meshes



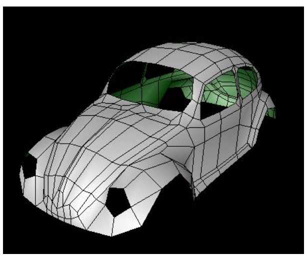
triangle meshes
no boundary



with boundaries



genus 1 mesh

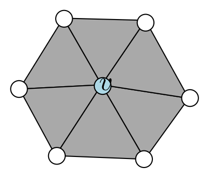


quad meshes

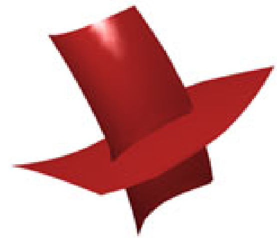
polygonal meshes

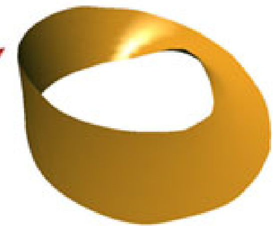
Every edge is shared by at most 2 faces

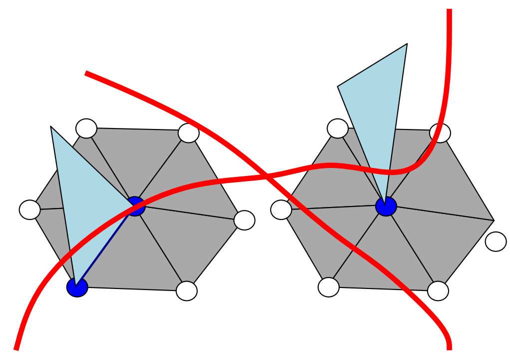
For every vertex v, the incident faces form an open or closed *fan*



non manifold or non orientable meshes





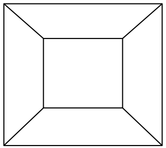


Part III

Euler formula and its consequences

Euler-Poincaré characteristic: topological invariant

$$\boxed{\chi := n - e + f}$$



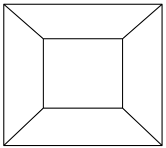
planar map

$$n - e + f = 2$$

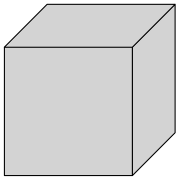
Euler's relation

Euler-Poincaré characteristic: topological invariant

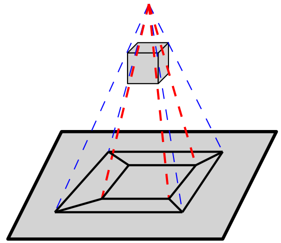
$$\boxed{\chi := n - e + f}$$



planar map



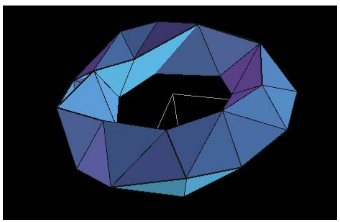
(convex) polyhedron



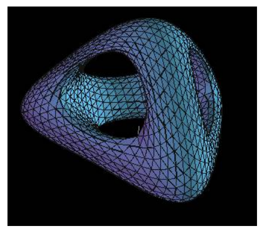
$$n - e + f = 2$$

Euler's relation

$\chi = 0$

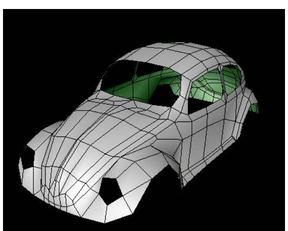


$\chi = -4$



$n = 1660$
$e = 4992$
$f = 3328$
$g = 3$

$$n - e + f = 2 - 2g$$



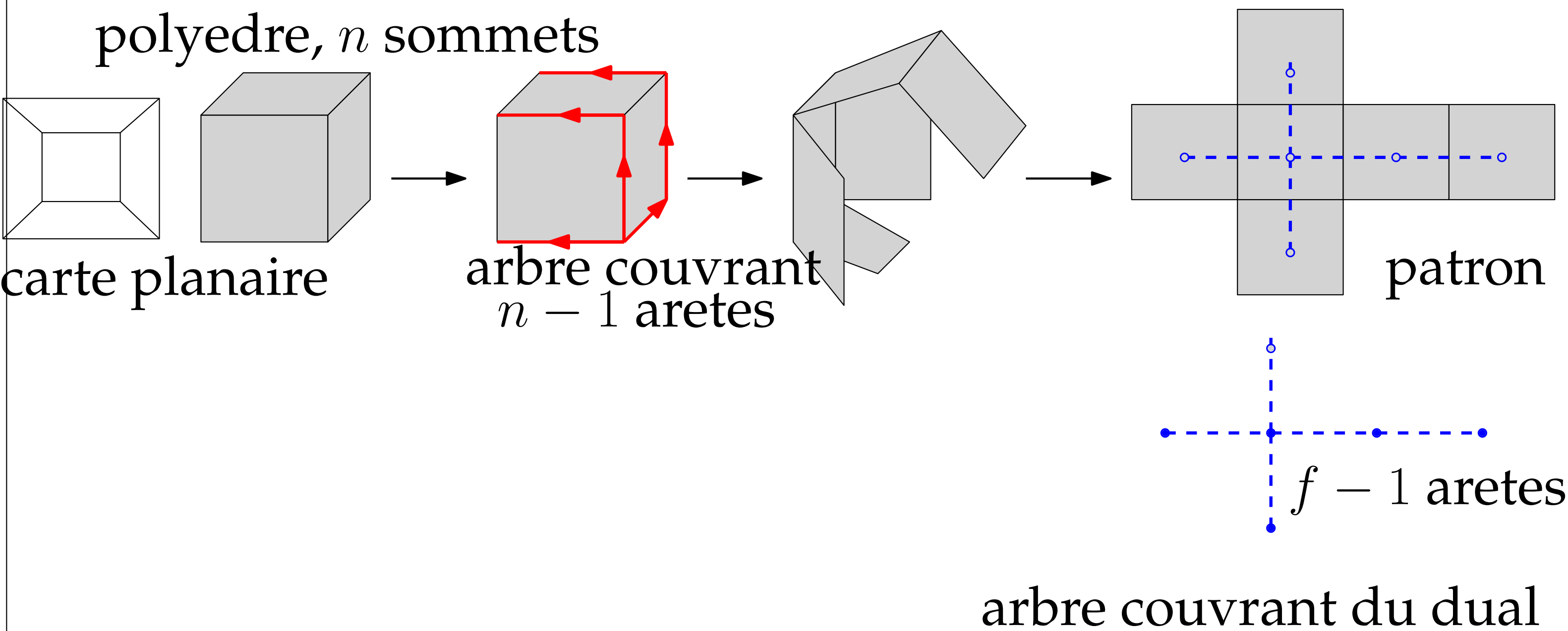
$n = 364$
$e = 675$
$f = 302$
$b = 11$
$g = 0$

$$n - e + f = 2 - b$$

Euler's relation for polyhedral surfaces

Overview of the proof

$$\boxed{n - e + f = 2}$$



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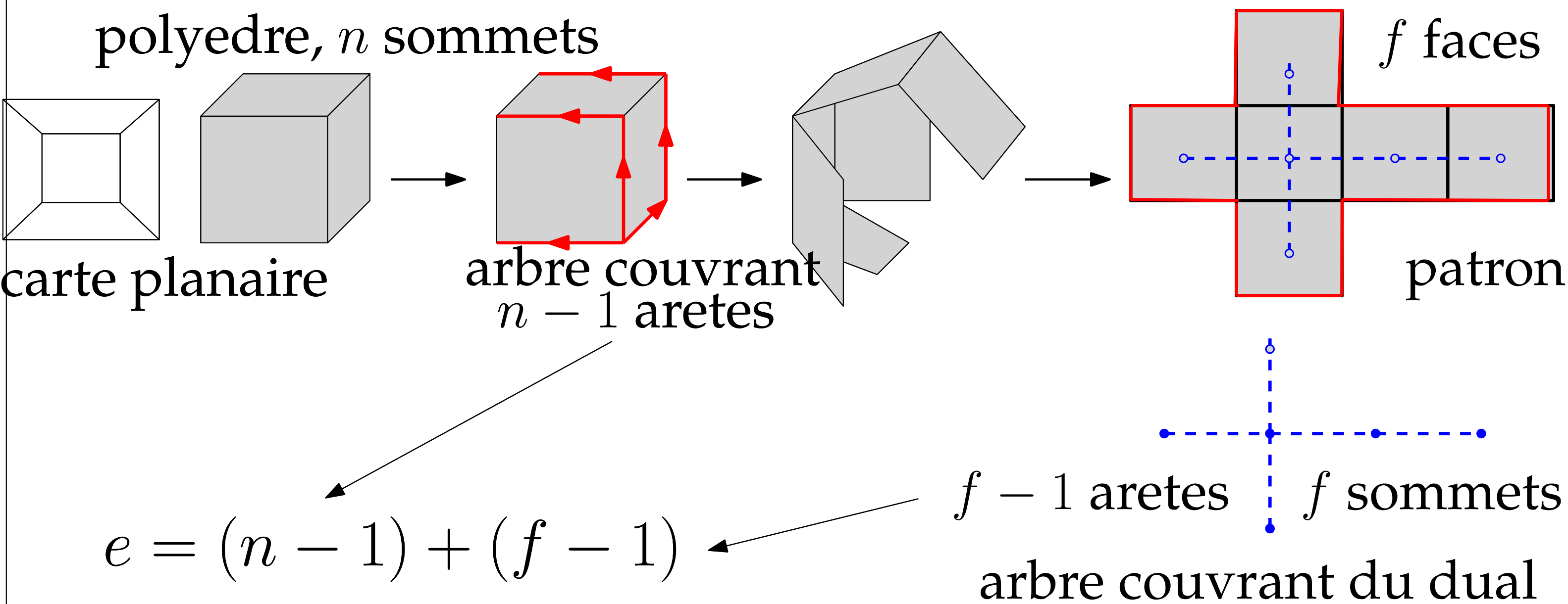
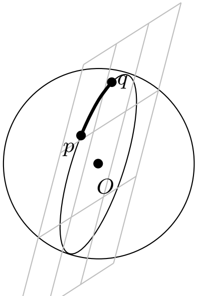


$$e = (n-1) + (f-1)$$

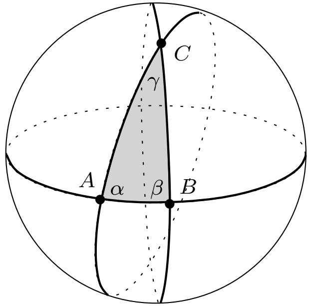
Using spherical geometry

An alternative geometric proof of Euler Formula

$$n - e + f = 2$$

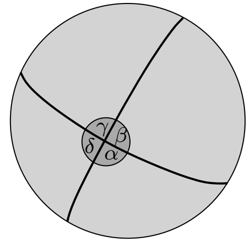


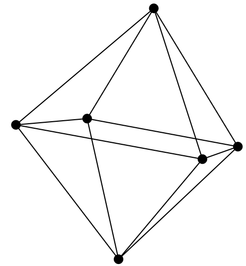
Geodesic: arc of great circle (between p and q)



Girard Theorem

$\alpha + \beta + \gamma = \pi + \frac{area(A,B,C)}{r^2}$





Spherical drawing of an octahedron

$\alpha + \beta + \gamma + \delta = 2\pi$

Theorem (Girard)

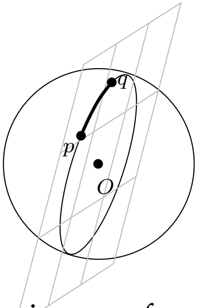
Given a spherical polygon P_i of size k_i, the sum of internal angles α_i satisfy

$$\sum_{i \leq k} \alpha_i = (k_i - 2)\pi + \frac{area(P_i)}{r^2}$$

Using spherical geometry

An alternative geometric proof of Euler Formula

$$n - e + f = 2$$



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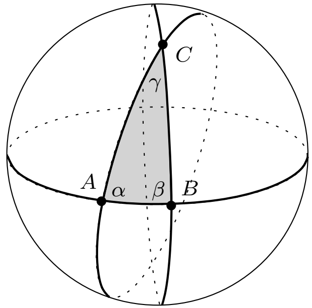

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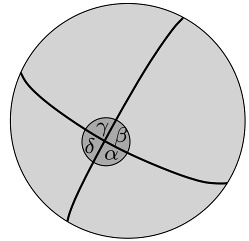

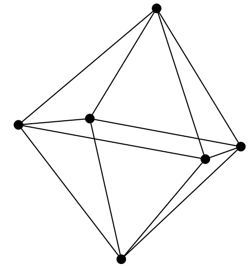

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$\sum_{j=1}^{d} \alpha_j = 2\pi$ for each v

$$\sum_i area(P_i) = 4\pi r^2$$

Euler's relation for polyhedral surfaces

Corollary: linear dependence between edges, vertices and faces

$f \leq 2n - 4$

$$\boxed{e \leq 3n - 6}$$

proof (double counting argument)

$$f = f_1 + f_2 + f_3 + \ldots$$
$$n = n_1 + n_2 + n_3 + \ldots$$

all faces have degree at least 3 ($\mathcal{G}$ simple simple), then we get

$$f = f_3 + f_4 + \ldots$$

every edge appears twice

$$2e = 3 \cdot f_3 + 4 \cdot f_4 + \ldots$$

then we get

$$2e - 3f \geq 0$$

Euler's relation for polyhedral surfaces

Corollary: linear dependence between edges, vertices and faces

$f \leq 2n - 4$

$$\boxed{e \leq 3n - 6}$$

given $2e - 3f \geq 0$

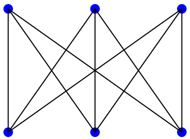
by applying Euler formula, we obtain

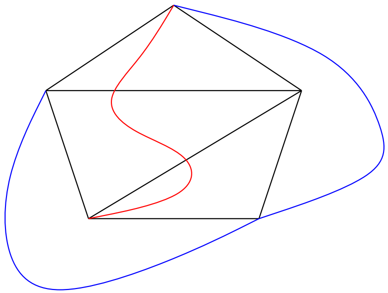
$$3n - 6 = 3(e - f + 2) = 3e - 3f \geq 0$$

Major results: Kuratowski theorem

Kuratowski theorem (1930) (cfr Wagner's theorem, 1937)
G is planar iff it does not contain K_5 nor $K_{3,3}$ as minors

proof (one direction)





$K_{3,3}$ bipartite:

no cycle of length 3

$$\boxed{e \leq 2n - 4 = 8 < 9}$$

$$e \leq 3n - 6 = 9$$

but we have $e(K_5) = \binom{5}{2} = 10$

Part IV

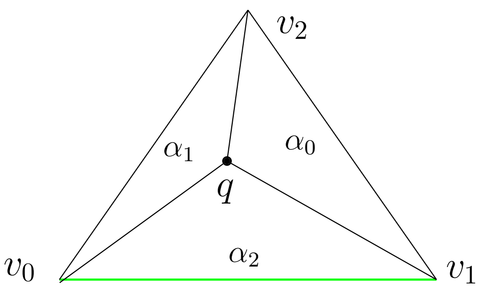
Tutte's planar embedding

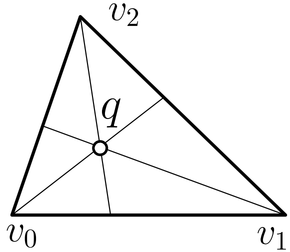
Preliminaries: barycentric coordinates

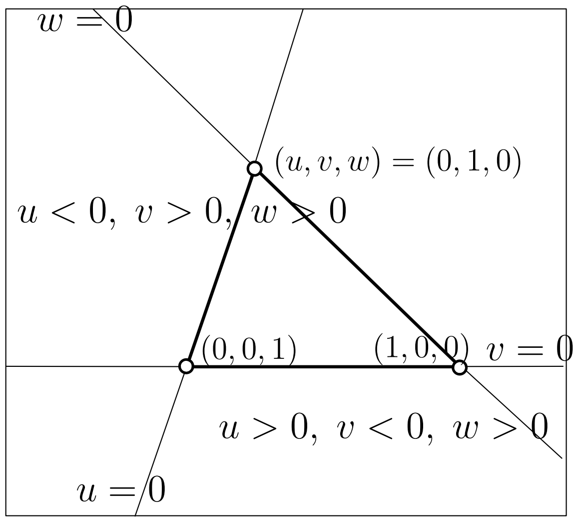
$$q = \boxed{\sum_i^n \alpha_i v_i \text{ (avec } \sum_i \alpha_i = 1)}$$

coefficients $(\alpha_1, \ldots, \alpha_n)$ are called *barycentric coordinates* of q (relative to $v_1, \ldots, v_n$)

Geometric interpretation of barycentric coordinates



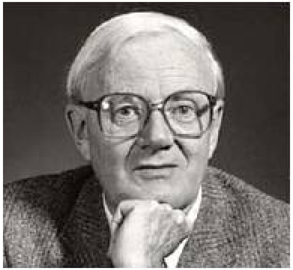




$$q = \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2$$

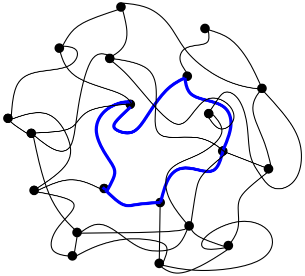
$$q = \frac{area(v, v_1, v_2)v_0 + area(v_0, v, v_2)v_1 + area(v_0, v_1, v)v_2}{area(v_0, v_1, v_2)}$$

Tutte's theorem

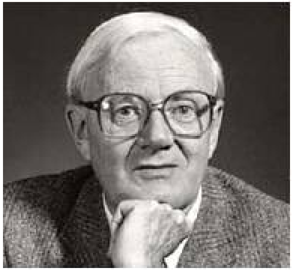


Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a convex representation ρ in R^2.



Tutte's theorem

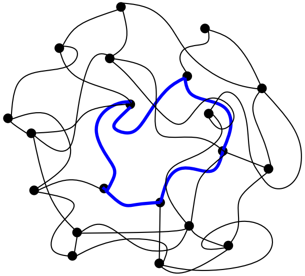


Thm (Tutte barycentric method, 1963)

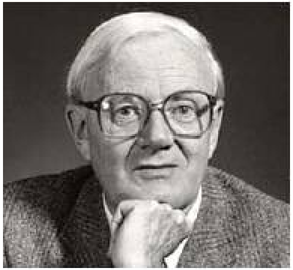
Every 3-connected planar graph G admits a convex representation ρ in R^2.

$$\rho : (V_G) \longrightarrow R^2$$

ρ is *convex* the images of the faces of G are convex polygons



Tutte's theorem



Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a convex representation ρ in R^2.

$$\rho : (V_G) \longrightarrow R^2$$

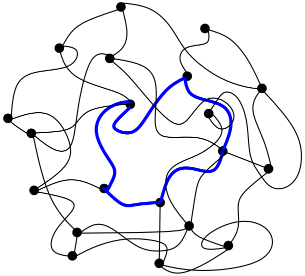
ρ is *barycentric* the images of interior vertices are barycenters of their neighbors

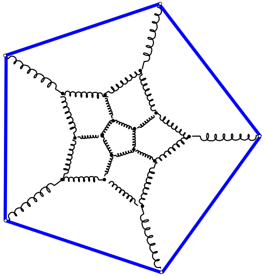
$$\boxed{\rho(v_i) = \sum_{j \in N(i)} w_{ij}\rho(v_j)}$$

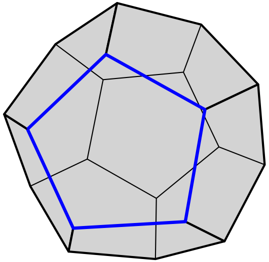
where w_{ij} satisfy $\sum_j w_{ij} = 1$, and $w_{ij} > 0$

according to Tutte: $w_{ij} = \frac{1}{deg(v_i)}$

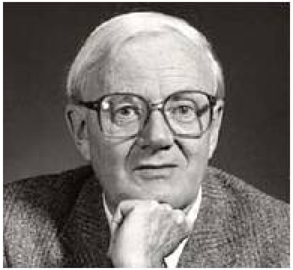
$N(v_4) = \{v_1, v_2, v_3, v_5\}$







Tutte's spring embedder (physical interpretation)



The representation ρ with minimum energy is uniquely determined (in R^d).: we require G connected and F not empty

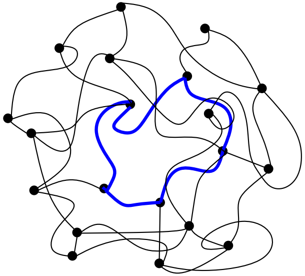
$$\rho : (V_G) \longrightarrow R^2$$

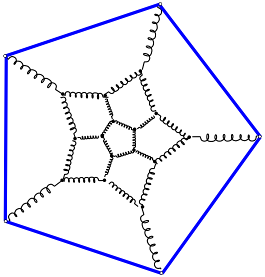
$$E(\rho) := \sum_{(i,j)\in E} |\rho(v_i) - \rho(v_j)|^2 = \sum_{(i,j)\in E} (x_i - x_j)^2 + (y_i - y_j)^2$$

ρ minimizes
$$\begin{cases} E(\rho) \\ \text{subject to } \rho(v_k) = p_k = (x_k, y_k) \text{ (for exterior vertices } v_k) \end{cases}$$

Lemma *The energy $E(\rho)$ is strictly convex*

$$\sum_{j\in N(i)} (\rho(v_i) - \rho(v_j)) = 0$$





$$\rho(v_i) = \frac{1}{d_i} \sum_{j\in N(i)} \rho(v_j)$$

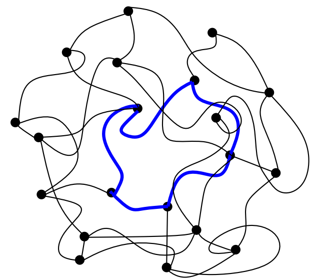
Tutte's theorem: main steps

- chose a cycle F (the outer face of G) in the right way

a cycle such that $G \setminus F$ is connected

(deletion of vertices and edges)

Tutte's theorem: main steps



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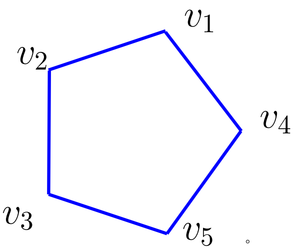
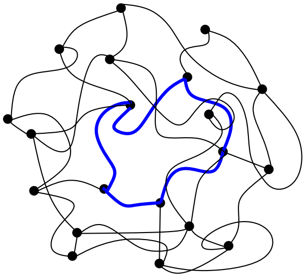


- choose a convex polygon P of size $k = |F|$

such that $\rho(F) = P$

Tutte's theorem: main steps



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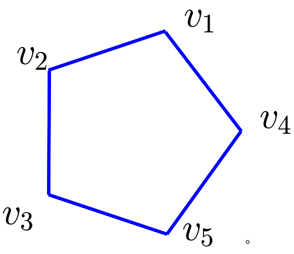
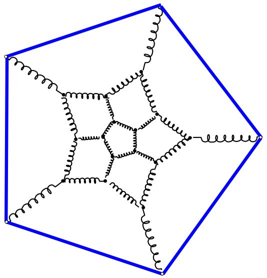


- choose a convex polygon P of size $k = |F|$

  such that $\rho(F) = P$

- solve equations for images of inner vertices $\rho(v_i)$:

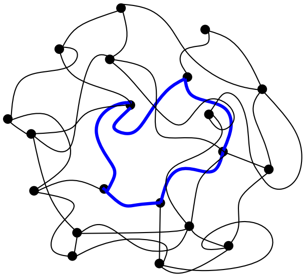


$$\rho(v_i) = \sum_{j \in N(i)} w_{ij}\rho(v_j)$$

$$\rho(v_i) - \sum_{j \in N(i)} w_{ij}\rho(v_j) = 0$$

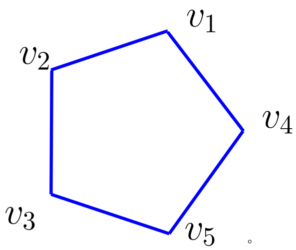
according to Tutte: $w_{ij} = \frac{1}{deg(v_i)}$

Tutte's theorem: main steps



- chose a cycle F (the outer face of G) in the right way

a cycle such that $G \setminus F$ is connected
(deletion of vertices and edges)



- choose a convex polygon P of size $k = |F|$

such that $\rho(F) = P$

- solve two linear systems:

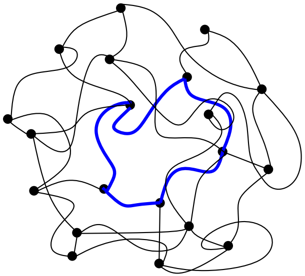
$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases} \longleftrightarrow \begin{cases} \rho_x(v_i) - \sum_{j \in N(i)} w_{ij} \rho_x(v_j) = 0 \\ \rho_y(v_i) - \sum_{j \in N(i)} w_{ij} \rho_y(v_j) = 0 \end{cases}$$

Tutte's theorem: main steps

- chose a cycle F (the outer face of G) in the right way

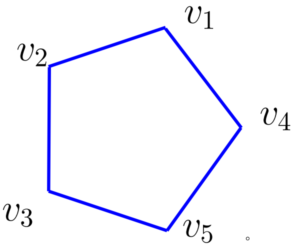
a cycle such that $G \setminus F$ is connected

(deletion of vertices and edges)



- choose a convex polygon P of size $k = |F|$

such that $\rho(F) = P$



- solve a linear system:

Validity of Tutte's theorem: main results

- show that the linear system admit a (unique) solution:

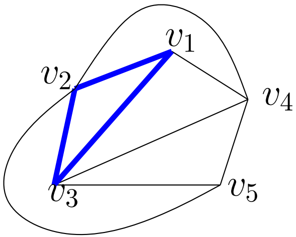
$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases}$$

matrix $(I - W)$ is inversible

- a barycentric drawing is planar: no edge crossing

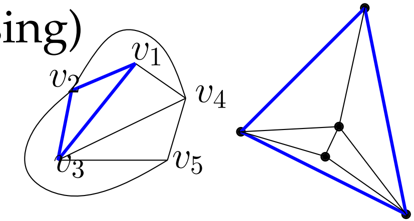
- a 3-connected planar graph G has a peripheral cycle

Claim (existence of peripheral cycles)
In a 3-connected planar graph peripheral cycles are exactly the faces (of the embedding)



Advantages of Tutte's drawing

- the drawing is guaranteed to be planar (no edge crossing)



- no need of the map structure

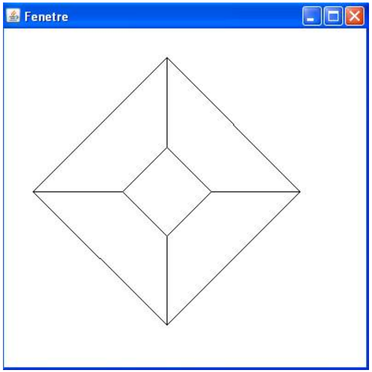
  graph structure + a peripheral cycle

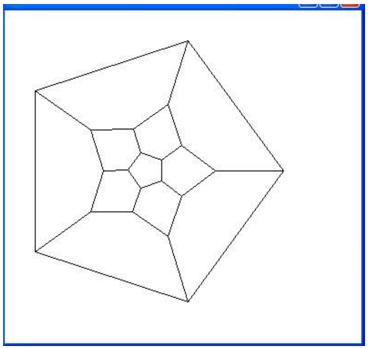
- very easy to implement: no need of sophisticated data structure or preprocessing

linear systems to solves

- nice drawings

  (detection of symmetries)





Drawbacks of Tutte's drawing

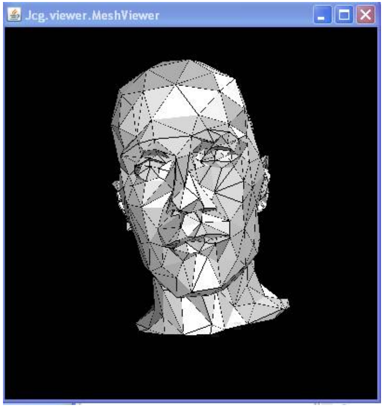
- requires to solve linear systems of equations (of size n)

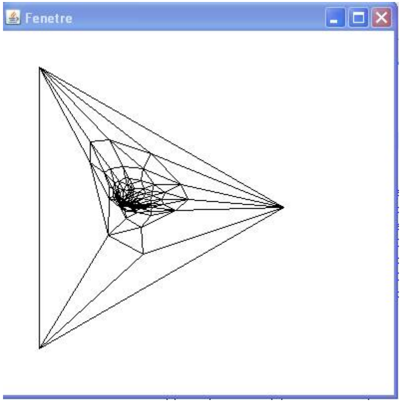
$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases}$$

complexity $O(n^3)$
or $O(n^{3/2})$ with methods more involved

- exponential size of the resulting vertex coordinates (with respect to n)

- drawings are not always "nice"





Tutte's spring embedder: iterative version

- choose an outer face F, and a convex polygon P
- put exterior vertices $v \in F$ on the polygon
- repeat (until convergence)

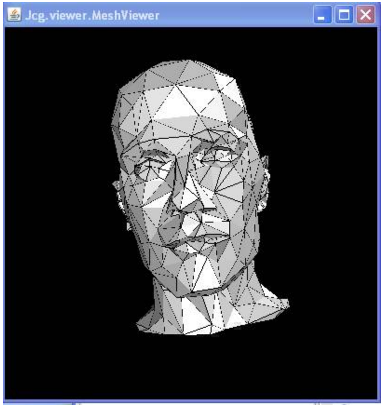
  for each inner vertex $v \in V_i$ compute

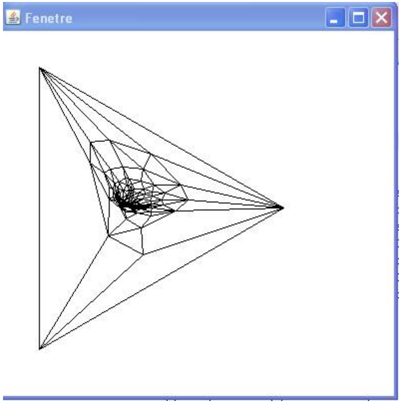
$$x_v = \frac{1}{deg(v)} \sum_{(u,v)\in E} x_u$$

$$y_v = \frac{1}{deg(v)} \sum_{(u,v)\in E} y_u$$

V_i inner vertices

(u, v) edge connecting v and u





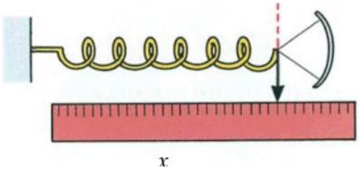
Graph drawing: force directed paradigm

- points are randmoly distributed in the plane
- repeat (until convergence)

  for each vertex $v \in V$ compute

  $$v = v + c_4 \cdot \mathbf{F}(v)$$

  $$\text{où } \mathbf{F}(v) := F_a(v) + F_r(v)$$



attractive force (between adjacent vertices)

$$\mathbf{F}_a(v) = c_1 \cdot \sum_{(u,v)\in E} \log(d_{uv})/c_2)$$

repulsive force (between non adjacent vertices)

$$\mathbf{F}_r(v) = c_3 \cdot \sum_{u\in V} \frac{1}{d_{uv}^2}$$

$$c_1 = 2 \; c_2 = 1 \; c_3 = 1 \; c_4 = 0.1$$

Part V
Tutte's theorem: the proof

First: existence and uniqueness of barycentric representations

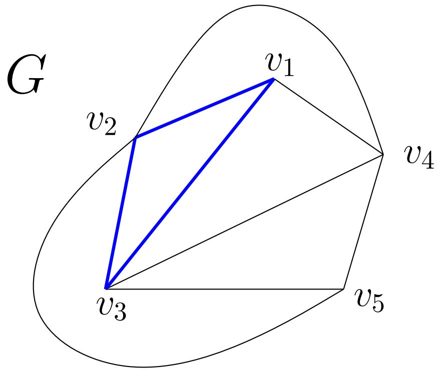
Second: the barycentric representation defines a planar drawing (no edge crossing)

Third: existence and computation of peripheral cycles

(Some notions of) Spectral graph theory

Laplacian matrix (simple graphs)

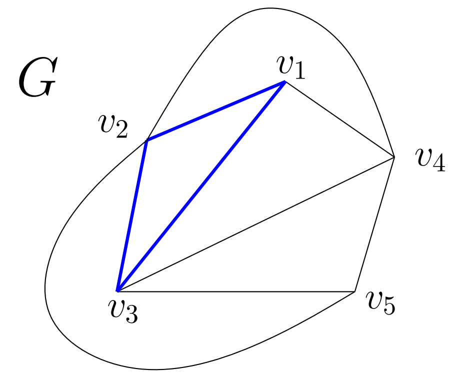
$$Q_G[i,k] = \begin{cases} deg(v_i) & \text{if } i = j \\ -A_G[i,j] & \text{otherwise} \end{cases}$$



$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

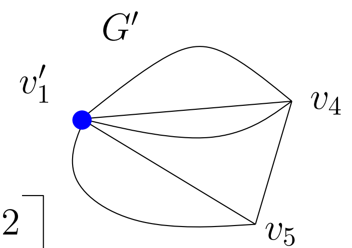
(Some notions of) Spectral graph theory

Laplacian matrix (counting multiple edges)



$$Q_G[i,k] = \begin{cases} deg(v_i) & \text{if } i = j \\ -|edges| \text{ from } v_i \text{ to } v_j & \text{otherwise} \end{cases}$$

$$Q_G[i_1, i_2, \ldots] = Q_G \backslash \begin{cases} \text{line } i_1, \text{ line } i_2, \cdots \\ \text{column } i_1, \text{ column } i_2, \cdots \end{cases}$$



$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

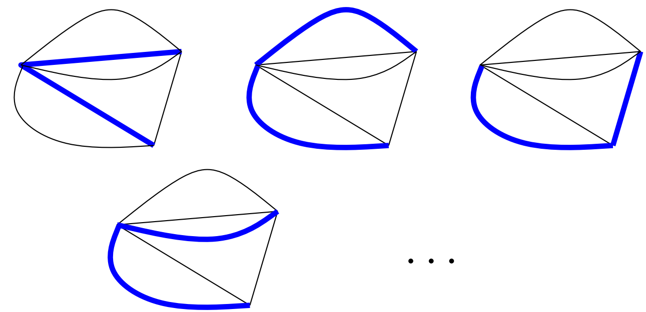
$$Q_{G'} = \begin{bmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

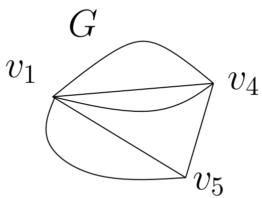
(Some notions of) Spectral graph theory

Lemma (Laplacian and the number of spanning trees)

Let Q be the laplacian of a graph G, with n vertices. Then the number of spannig trees of G is:

$$\tau(G) = det(Q[i]) \qquad (i \leq n)$$





$$Q_G \begin{bmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

$$Q_G[1] = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} = 11$$

First: existence and uniqueness of barycentric representations

First: existence and uniqueness of barycentric representations

Theorem

Let G be a 3-connected planar graph with n vertices, and F a peripheral cycle (such that $G \setminus F$ is connected). Let P be a convex polygon, such that $\rho(F) = P$. Then the barycentric representation ρ exists (and is unique)

Goal: *show that the two systems above admit a solution (unique)*

$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases} \longleftrightarrow \qquad \rho(v_i) = \sum_{1}^{n} w_{ij} \rho(v_j) \qquad i = 1, \ldots, (n-k)$$

(one equation for each inner vertex)

$\mathbf{x} = [x_1, x_2, \ldots, x_{n-k}]$

$\mathbf{y} = [y_1, y_2, \ldots, y_{n-k}]$

(coordinates of inner vertices)

$$\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j)$$

First: existence and uniqueness of barycentric representations

Proof

$$\rho(v_i) - \sum_{j \in N(i)} w_{ij}\rho(v_j) = 0$$

$$\Updownarrow$$

$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases}$$

First: existence and uniqueness of barycentric representations

Proof

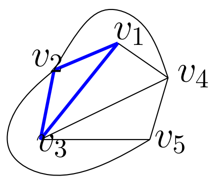
$$\rho(v_i) - \sum_{j \in N(i)} w_{ij}\rho(v_j) = 0$$

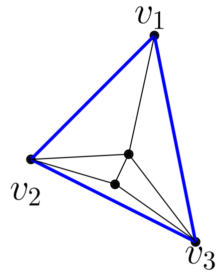
$$\Updownarrow$$

$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_{4x} \\ b_{5x} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} b_{4y} \\ b_{5y} \end{bmatrix}$$





$N(v_4) = \{v_1, v_2, v_3, v_5\}$

$$\rho(v_i) := (x_i, y_i)$$

$N(v_5) = \{v_2, v_3, v_4\}$

First: existence and uniqueness of barycentric representations

Proof

$$\rho(v_i) - \sum_{j \in N(i)} w_{ij} \rho(v_j) = 0 \qquad deg(v_i)\rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0$$

$$\Updownarrow$$

$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases} \qquad \begin{cases} M \cdot \underline{x} = \underline{a_x} \\ M \cdot \underline{y} = \underline{a_y} \end{cases}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_{4x} \\ b_{5x} \end{bmatrix} \qquad M = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} b_{4y} \\ b_{5y} \end{bmatrix}$$

$$\begin{cases} \rho(v_4) - \frac{1}{4}\rho(v_5) = \frac{1}{4}\rho(v_1) + \frac{1}{4}\rho(v_2) + \frac{1}{4}\rho(v_3) \\ -\frac{1}{3}\rho(v_4) + \rho(v_5) = \frac{1}{3}\rho(v_2) + \frac{1}{3}\rho(v_3) \end{cases}$$

First: existence and uniqueness of barycentric representations

$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$Q_G[1,2,3] := M = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

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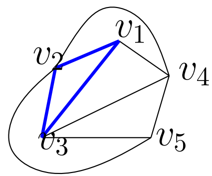
First: existence and uniqueness of barycentric representations

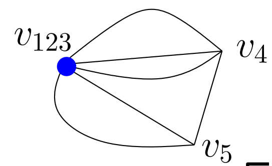
$$deg(v_i)\rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0$$

$$\begin{cases} M \cdot \underline{x} = \underline{a_x} \\ M \cdot \underline{y} = \underline{a_y} \end{cases}$$

$$G \longrightarrow G/F \xrightarrow[G/F \text{ is connected}]{} det(M) = \tau(Q_{G/F}) > 0$$



$$G/F = G/\{v_1, v_2, v_3\} \longrightarrow$$



$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$Q_G[1,2,3] := M = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

$$Q_{G/F} \begin{bmatrix} 5 & 3 & 2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

$$\begin{cases} \rho(v_4) - \frac{1}{4}\rho(v_5) = \frac{1}{4}\rho(v_1) + \frac{1}{4}\rho(v_2) + \frac{1}{4}\rho(v_3) \\ -\frac{1}{3}\rho(v_4) + \rho(v_5) = \frac{1}{3}\rho(v_2) + \frac{1}{3}\rho(v_3) \end{cases}$$

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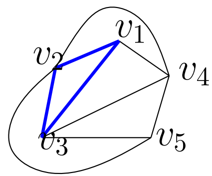
M admits inverse $\square$



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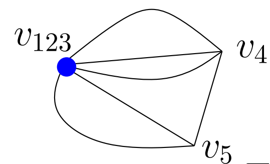

$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$$Q_G[1,2,3] := M = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

$$Q_{G/F} \begin{bmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

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Second: the barycentric representation defines a planar drawing

Theorem

Let G be a 3-connected planar graph with n vertices, and F a peripheral cycle (such that $G \setminus F$ is connected). Let P be a convex polygon, such that $\rho(F) = P$.

Then the barycentric representation defines a planar drawing (no edge crossing)

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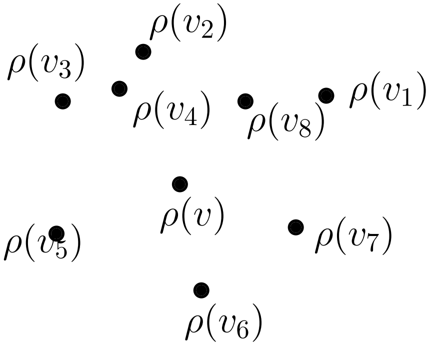
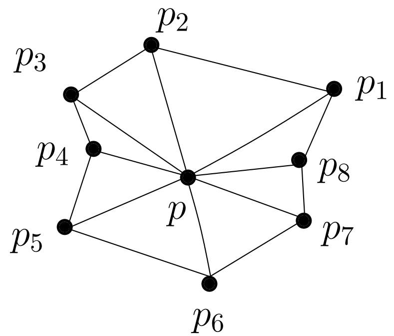


barycentric representation of G



planar drawing G

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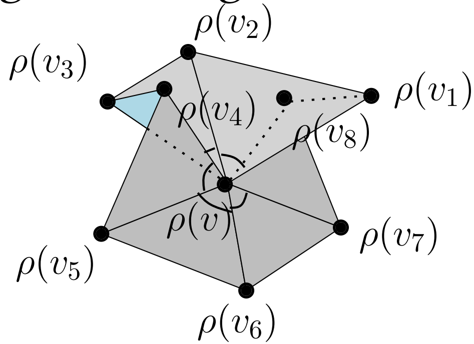
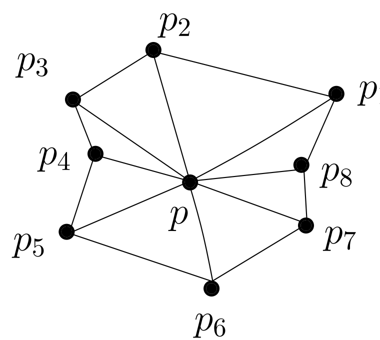




$$\sigma(v) \geq \alpha(v)$$
dessin non planaire de G

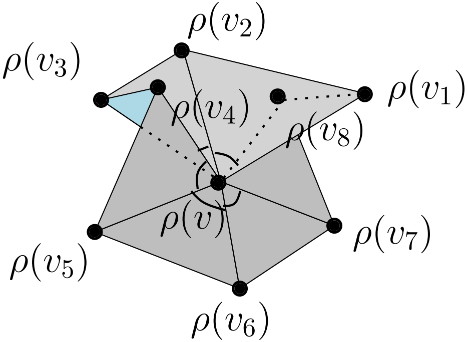
$$\boxed{\sigma(v) = \alpha(v)}$$

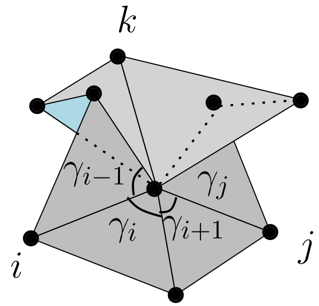
planar drawing of G
$$\alpha(v) = 2\pi$$

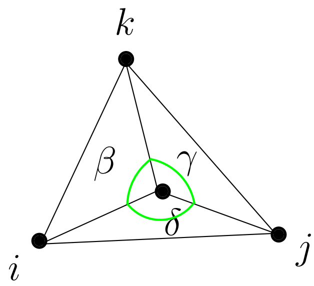
Second: the barycentric representation defines a planar drawing

Claim 1

$$\boxed{\sigma(v) \geq 2\pi = \alpha(v)}$$







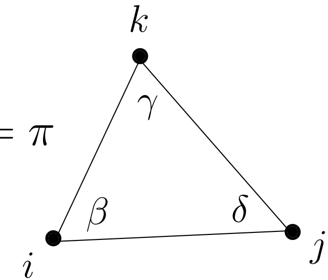
$$\sigma(v) := \sum_k \gamma_k \geq \quad \beta + \gamma + \delta = 2\pi$$

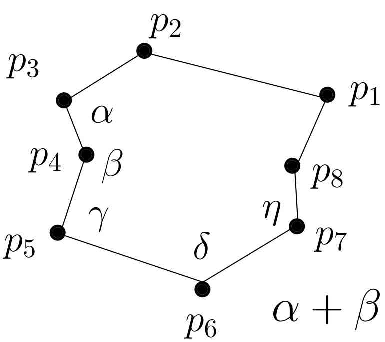
Second: the barycentric representation defines a planar drawing

Claim 2

$$\boxed{\sum_v \alpha(v) \leq \sum_v \sigma(v) = \pi f}$$

$$\beta + \gamma + \delta = \pi$$





$$\alpha + \beta + \gamma + \delta + \ldots = (|F| - 2)\pi$$

inner vertices outer vertices

$$\sum_v \alpha(v) = \sum_{v \in V \setminus F} \alpha(v) + \sum_{v \in F} \alpha(v) = 2\pi|V \setminus F| + (|F| - 2)\pi \leq \boxed{\sum_v \sigma(v) = \pi f}$$

sum over inner and outer vertices

sum of the angles of triangles
(3 angles per face)

Second: the barycentric representation defines a planar drawing

Conclusion

$$\boxed{\alpha(v) = \sigma(v)}$$

Euler formula

$$\left.\begin{array}{l} n - (\underset{\text{inner edges}}{e} + |F|) + f = (|V \setminus F| + |F|) - (e + |F|) + (\underset{\text{inner triangles}}{t} + 1) \\ \textbf{Counts the number of edges} \quad 3t = 2e + |F| \end{array}\right\} \longrightarrow 2\pi|V \setminus F| + (|F| - 2)\pi = \pi f$$

$$\left.\begin{array}{l} \sum_v \alpha(v) := 2\pi|V \setminus F| + (|F| - 2)\pi = \pi f \\ \textbf{Claim 2} \quad \sum_v \alpha(v) \leq \sum_v \sigma(v) = \pi f \end{array}\right\} \longrightarrow \sum_v \alpha(v) = \sum_v \sigma(v)$$

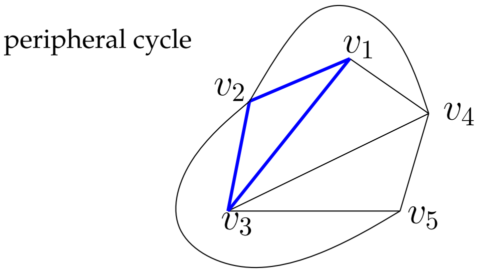
$$\left.\begin{array}{l} \sum_v \alpha(v) = \sum_v \sigma(v) \\ \textbf{Claim 1} \quad 2\pi = \alpha(v) \leq \sigma(v) \end{array}\right\} \quad \alpha(v) = \sigma(v) = 2\pi$$

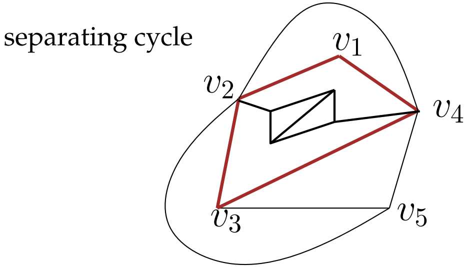
$\square$

Third: peripheral cycles (non separating cycles)

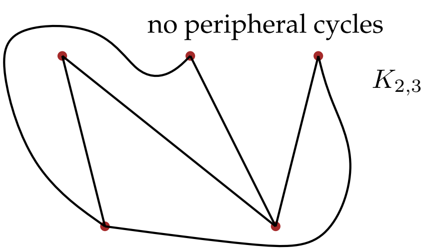
Definition

A *non-separating cycle* C of a graph G is an induced cycle in G such that $G \setminus C$ is connected (after the deletion of edges and vertices of C)





A *peripheral cycle* C is a simple cycle of a connected graph G such that for any two edges e_1 and e_2 in $G \setminus C$ there exists a path (starting at e_1 and ending at e_2) with no interior vertices on C



Third: existence of peripheral cycles

Lemma

In a 3-connected planar graph peripheral cycles are exactly the faces (of the embedding)

proof (exercise)

$\square$