INF563 Topological Data Analysis — Exercise Session
Stability of global topological signatures

Our goal here is to prove the following stability theorem for persistence diagrams of Rips filtrations:

Theorem 1. *For any compact metric spaces* (X, d_X) *and* (Y, d_Y)*, we have*

$$\mathrm{d}_{\mathrm{b}}^{\infty}(\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X),\ \mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y)) \leq 2\,\mathrm{d}_{\mathrm{GH}}(X, Y).$$

To simplify things a bit in the following, we will assume that X and Y are finite. Then, we can use the following well-known embedding result:

Lemma 1. *Any finite metric space* (Z, d_Z) *embeds isometrically into* $(\mathbb{R}^n, \ell^\infty)$*, where* n *denotes the cardinality of* Z*.*

Question 1. Prove Lemma 1.
Hint: letting $Z = \{z_1, \cdots, z_n\}$, for each point z_i consider the vector $(\mathrm{d}_Z(z_i, z_1), \mathrm{d}_Z(z_i, z_2), \cdots, \mathrm{d}_Z(z_i, z_n)) \in \mathbb{R}^n$, then show that the ℓ^∞-distances between the vectors are the same as the distances between the original points of Z.

Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $\varepsilon > \mathrm{d}_{\mathrm{GH}}(X, Y)$.

Question 2. Show that (X, d_X) and (Y, d_Y) can be jointly embedded isometrically into $(\mathbb{R}^d, \ell^\infty)$, for some $d > 0$, such that the Hausdorff distance between their images is at most ε.
Hint: look at the proof outline shown in Figure 1.

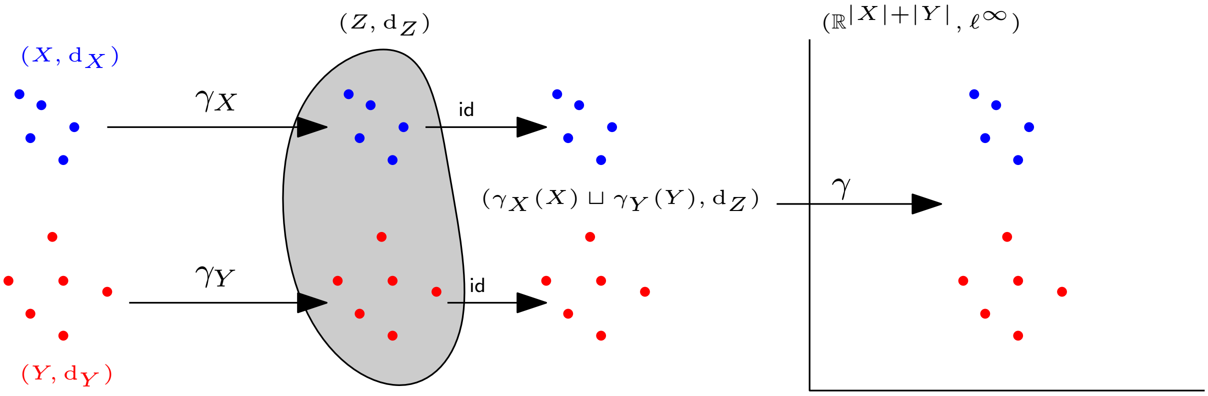

Figure 1: Outline of the proof of Theorem 1.

We call respectively X' and Y' the images of X and Y through the joint isometric embedding.

Question 3. Show that $\mathcal{R}(X', \ell^\infty)$ is isomorphic to $\mathcal{R}(X, \mathrm{d}_X)$ as a simplicial filtration.
Hint: this means that there is a bijection $X \to X'$ that induces a bijection between the simplices of the two filtrations, such that the times of appearance of the simplices are preserved.

Similarly, $\mathcal{R}(Y', \ell^\infty)$ is isomorphic to $\mathcal{R}(Y, \mathrm{d}_Y)$. Thus, we have:

$$\mathrm{d}_\mathrm{b}^\infty(\mathrm{Dg}\ \mathcal{R}(X, \mathrm{d}_X), \mathrm{Dg}\ \mathcal{R}(Y, \mathrm{d}_Y)) = \mathrm{d}_\mathrm{b}^\infty(\mathrm{Dg}\ \mathcal{R}(X', \ell^\infty), \mathrm{Dg}\ \mathcal{R}(Y', \ell^\infty)).$$

We call respectively $f_{X'}$ and $f_{Y'}$ the distance functions of X' and Y': $\forall p \in \mathbb{R}^d$,

$$f_{X'}(p) = \min_{x' \in X'} \|p - x'\|_\infty$$
$$f_{Y'}(p) = \min_{y' \in Y'} \|p - y'\|_\infty$$

Question 4. Show that $\|f_{X'} - f_{Y'}\|_\infty \leq \varepsilon$.
Hint: recall that $\mathrm{d}_\mathrm{H}(X', Y') \leq \varepsilon$.

Question 5. Deduce that $\mathrm{d}_\mathrm{b}^\infty(\mathrm{Dg}\ f_{X'}, \mathrm{Dg}\ f_{Y'}) \leq \varepsilon$, where $\mathrm{Dg}\ h$ denotes the persistence diagram of the filtration of the sublevel sets of h.

Question 6. Deduce now that $\mathrm{d}_\mathrm{b}^\infty(\mathrm{Dg}\ \mathcal{C}(X', \ell^\infty), \mathrm{Dg}\ \mathcal{C}(Y', \ell^\infty)) \leq \varepsilon$, where $\mathcal{C}(Z', \ell^\infty)$ denotes the Čech filtration of Z in the ℓ^∞-distance.
Hint: relate the sublevel sets of $f_{X'}$ to the unions of ℓ^∞-balls centered at the points of X', then apply the Nerve Theorem. Same for Y'.

Question 7. Deduce finally that $\mathrm{d}_\mathrm{b}^\infty(\mathrm{Dg}\ \mathcal{R}(X', \ell^\infty), \mathrm{Dg}\ \mathcal{R}(Y', \ell^\infty)) \leq 2\varepsilon$.
Hint: relate the Čech and Rips filtrations to each other in $(\mathbb{R}^d, \ell^\infty)$.

Question 8. Conclude.