INF 556: Topological Data Analysis	
Lecture 3: Solution to PC 3-4	
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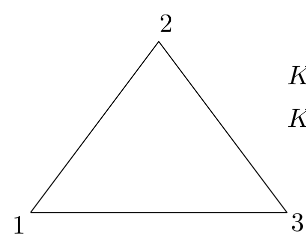
Disclaimer :
These notes have been written by Théo Lacombe. Some typos and errors may remain, please report them to the instructor if you find any.

Homology groups of some common spaces

All the following computations are done with the field of coefficient $\mathbb{Z}/2\mathbb{Z}$, which (among other things) means that we do not care about orientation. The notation $E \simeq F$ means that E and F are isomorphic as vector spaces.

1. The circle.



$K_0 = \{1\}, \{2\}, \{3\}$
$K_1 = \{1,2\}, \{1,3\}, \{2,3\}$

Figure 3.1: Triangulation of the circle

We detail in this simple example two approaches to compute homology. Let's begin with the "hand-craft" one:

To compute H_0, we need $Z_0 = \ker(\partial_0)$ and $B_0 = \text{im}(\partial_1)$. One has $\ker(\partial_0) = \text{span}\{\{1\}, \{2\}, \{3\}\}$, which are three linearly independent points in our complex (you cannot express $\{3\}$ as a linear combination of $\{1\}$ and $\{2\}$ and so on). Therefore:

$$\ker(\partial_0) \simeq \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^3$$

On the other hand, $\text{im}(\partial_1) = \text{span}\{\{2\} - \{1\}, \{3\} - \{2\}, \{1\} - \{3\}\}$. (Reminder: since we are working with $\mathbb{Z}/2\mathbb{Z}$, $+1 = -1$ and thus signs do not matter). However, one can observe that $\{1\} - \{3\} = \{2\} - \{1\} - (\{3\} - \{2\})$, so we actually have $\text{im}(\partial_1) = \text{span}\{\{2\} - \{1\}, \{3\} - \{2\}\}$ (both vectors are independent) and thus:

$$\text{im}(\partial_1) \simeq \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^2$$

and finally:

$$H_0\left(\mathcal{S}^1; \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \simeq \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right) \tag{3.1}$$

In particular, $\beta_0 = 1$, which can be interpreted as *the circle has one connected component.*

In order to compute H_1, we need to find the 1-cycles of our complex. One can easily observe that:

$$\begin{aligned}\partial_1(\{1,2\}+\{2,3\}+\{3,1\}) &= \{2\}-\{1\}+\{3\}-\{2\}+\{1\}-\{3\} \\ &= 0\end{aligned}$$

and we do note have any other 1-cycle, so:

$$\ker(\partial_1) \simeq \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)$$

Aside, we do not have any 2-simplex and so $\text{im}(\partial_2) = \{0\}$, which finally implies that:

$$H_1\left(\mathcal{S}^1; \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \simeq \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right) \tag{3.2}$$

In particular, $\beta_1 = 1$, which can be interpreted as *there is one non-trivial loop in the circle (which is not a boundary).*

For $k \geqslant 2$, $C_k\left(\mathcal{S}^1, \frac{\mathbb{Z}}{2\mathbb{Z}}\right)$ is trivial and thus so is H_k.

Other approach: The idea is the following one: we generally only care about the dimension of H_k (i.e. the Betti number β_k), and we have:

$$\beta_k = \dim(\ker(\partial_k)) - \text{rk}(\partial_{k+1})$$

We also remind the following fundamental result of linear algebra (in finite dimensional vector spaces):

$$\dim(E) = \text{rk}(u) + \dim(\ker(u))$$

for E a finite dimensional vector space and u a linear application from E to some other vector space.

Therefore, we can turn this into the problem of finding the rank of ∂_k (which will also give us the dimension of its kernel), which can be easily computed by writing the matrix of ∂_k:

$$\begin{array}{c} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} \begin{array}{c} \{1,2\} \quad \{1,3\} \quad \{2,3\} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 3.2: Matrix of ∂_1: the starting space is $C_1\left(K, \frac{\mathbb{Z}}{2\mathbb{Z}}\right)$, its base is given by the three vectors $\{1,2\}, \{2,3\}, \{3,1\}$, and we write the coordinates of ∂_1 in the basis $\{1\}, \{2\}, \{3\}$ of $C_0\left(K, \frac{\mathbb{Z}}{2\mathbb{Z}}\right)$.

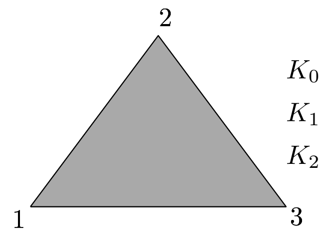
Standard computations show that this matrix (whose coefficients are in $\mathbb{Z}/2\mathbb{Z}$) has rank 2 (see *Gaussian elimination*):

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{c_3 \leftarrow c_3 - c_2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{c_3 \leftarrow c_3 - c_1} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Figure 3.3: Sketch of Gaussian elimination to compute the rank of ∂_1 for the circle, leading to a rank 2 matrix.

The interest of this algorithm is that it can be easily implemented (and is useful while dealing with more complicated simplices).

2. The disk $\mathbb{B}^2$.



$K_0 = \{1\}, \{2\}, \{3\}$
$K_1 = \{1,2\}, \{1,3\}, \{2,3\}$
$K_2 = \{1,2,3\}$

Figure 3.4: Triangulation of $\mathbb{B}^2$.

For H_0, computations are **exactly** the same as the circle (see above).

For H_1, we have the same result for $\ker(\partial_1)$ (one 1-cycle). However, in this case, $\operatorname{im}(\partial_2)$ is not empty (we have a 2-simplex), leading to:

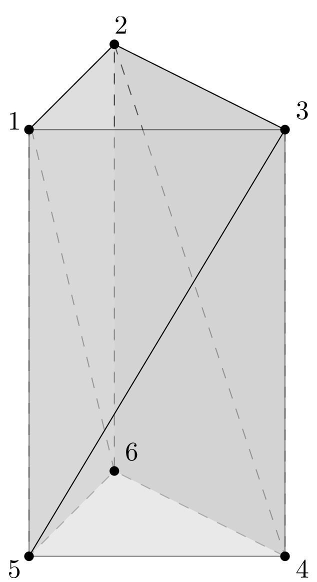
$$H_1\left(\mathbb{B}^2; \frac{\mathbb{Z}}{2\mathbb{Z}}\right) = \{0\} \tag{3.3}$$

For H_2, despite having a 2-simplex, we do not have any 2-cycle ($\partial_2\{1,2,3\} = \{1,2\}+\{2,3\}+\{3,1\} \neq 0$). Furthermore, we do not have any 3-simplex in this complex, and thus:

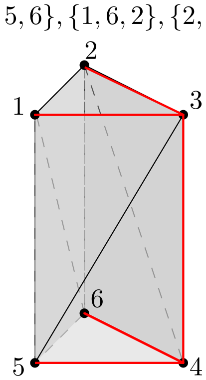
$$H_2\left(\mathbb{B}^2; \frac{\mathbb{Z}}{2\mathbb{Z}}\right) = \{0\} \tag{3.4}$$

Finally, $\mathbb{B}^2$ has the same homology groups as a single point, which is actually not a surprise since it is homotopy equivalent to a point!

3. The cylinder $c = \mathcal{S}^1 \times [0,1]$



$K_0 = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$
$K_1 = \{1,2\}, \{1,3\}, \{2,3\}, \{4,5\}, \{4,6\}, \{5,6\}$...
...$\{1,5\}, \{1,6\}, \{2,6\}, \{2,4\}, \{3,4\}, \{3,5\}$
$K_2 = \{1,5,6\}, \{1,6,2\}, \{2,4,6\}, \{2,3,4\}, \{1,5,3\}, \{3,4,5\}$



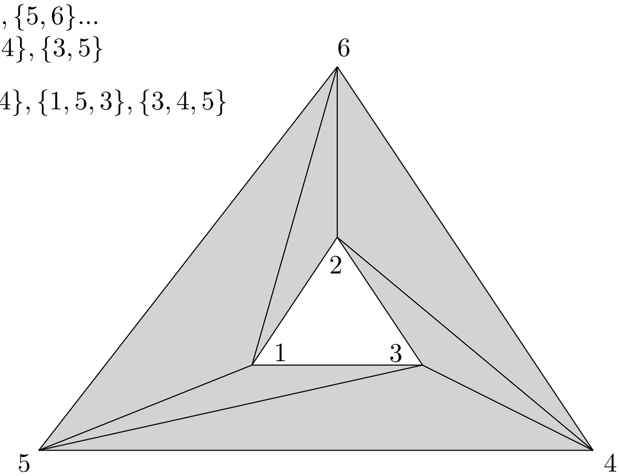


Figure 3.5: Triangulation of the cylinder. The 2-faces $\{1,2,3\}$ and $\{4,5,6\}$ **do not** belong to the complex. On the smaller graph, in red, edges such that $\partial_1(edges)$ gives the generators of B_0. On the right, a representation of the triangulation "from the top", which can help for computations.

Since we have 6 points in this triangulation, $\dim(Z_0) = 6$. On the other hand, computations show that $\operatorname{im}(\partial_1)$ has 5 (independent) generators (see Fig 3.5). The idea is that the boundary of any other

1-simplex (edge) in the complex can be obtained by going through these edges. For example, $\{1,6\}$ has $\{6\}-\{1\}$ has a boundary, which can be obtained by taking the boundary of $\{1,3\}+\{3,4\}+\{4,6\}$. Therefore,

$$H_0\left(c;\frac{\mathbb{Z}}{2\mathbb{Z}}\right)\simeq\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right) \tag{3.5}$$

In order to compute H_1, we have to find the 1-cycles and the 1-boundary. There are many 1-cycles in this complex...! For example, any element of the form $\{a,b\}+\{b,c\}+\{c,a\}$ (with $\{a,b\},\{b,c\},\{c,a\}$ in the complex) is a 1-cycle. However, there are 7 1-cycles (you will find 8 1-cycles, but one of them can be written as a linear combination of the others), showing that $dim(Z_1)=7$. On the other hand, we have six 2-faces (triangles). We finally have:

$$H_1\left(c;\frac{\mathbb{Z}}{2\mathbb{Z}}\right)\simeq\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right) \tag{3.6}$$

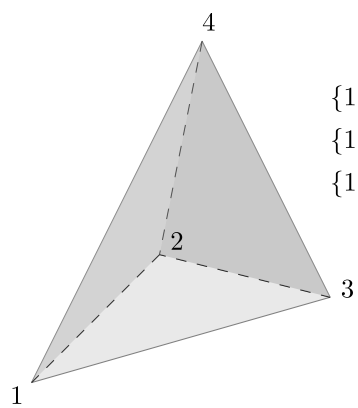
For H_2, we observe that we do not have any 2-cycle, and no 3-simplex, leading to:

$$H_2\left(c;\frac{\mathbb{Z}}{2\mathbb{Z}}\right)=\{0\} \tag{3.7}$$

Of course, higher dimensional homology groups are also trivial.

Remark: This is the same homology as the circle in question 1. This is not a surprise, since these two spaces are actually homotopy equivalent.

4. The sphere $\mathcal{S}^2$



$\{1\},\{2\},\{3\},\{4\}$
$\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{3,4\}$
$\{1,2,3\},\{1,2,4\},\{2,3,4\},\{1,3,4\}$

Figure 3.6: Triangulation of $\mathcal{S}^2$. Warning, the 3-face $\{1,2,3,4\}$ **does not** belong to the complex.

H_0 can be computed "by hand" or by computing the rank of the matrix:

$$\partial_1=\begin{bmatrix}1&1&0&1&0&0\\1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&0&1&1&1\end{bmatrix}$$

which are the coordinates of $\partial_1(x), x\in C_1$ written in the base $\{1\},\{2\},\{3\},\{4\}$. This matrix has rank 3, and thus $\beta_0=4-3=1$, then:

$$H_0\left(\mathcal{S}^2;\frac{\mathbb{Z}}{2\mathbb{Z}}\right)\simeq\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right) \tag{3.8}$$

Similarly, H_1 looks at the rank of (boundaries of 2-simplicies written on the base of 1-simplicies):

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

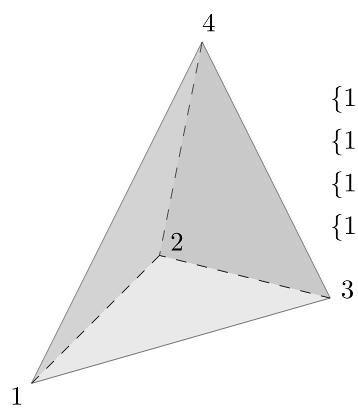
which is 3. Since we already know that the rank of ∂_1 is 3, we have $\dim(\ker(\partial_1)) = 6 - 3 = 3$. And thus, $\beta_1 = \dim(\ker(\partial_1)) - \text{rk}(\partial_2) = 3 - 3 = 0$. So:

$$H_1\left(\mathcal{S}^2; \frac{\mathbb{Z}}{2\mathbb{Z}}\right) = \{0\} \tag{3.9}$$

For H_2, we observe that we have one 2-cycle ($\{1,2,3\} + \{1,3,4\} + \{1,2,4\} + \{2,3,4\}$), and no 3-simplex. Thus:

$$H_2\left(\mathcal{S}^2; \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \simeq \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right) \tag{3.10}$$

5. The ball $\mathbb{R}^3$



$\{1\}, \{2\}, \{3\}, \{4\}$
$\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}$
$\{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\}$
$\{1,2,3,4\}$

Figure 3.7: Triangulation of $\mathbb{B}^3$.

Computation for H_0 and H_1 are exactly the same as above. For H_2, we still have one 2-cycle, but we also have an element in $\text{im}(\partial_3)$, leading to $H_2 = \{0\}$. Since there are no 3-cycles nor higher dimensional simplices, higher dimensional homology groups are trivial.

Remark: As for $\mathbb{B}^2$, the homology is the same as that of a single point.

6. The Torus

The first difficulty is to find a proper triangulation that is not too large, so that we can handle the calculations. Figure 3.8 (left) shows an example of a triangulation, which we will now use to compute homology.

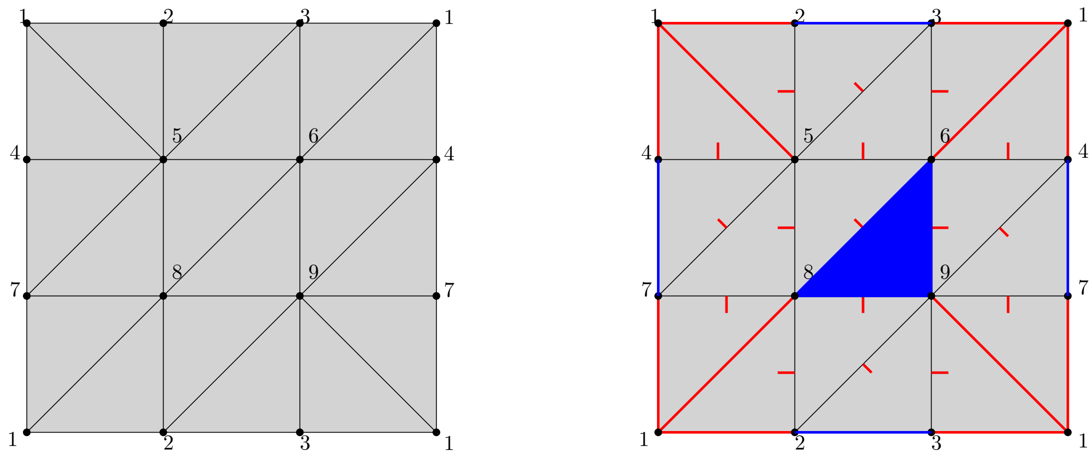


Figure 3.8: *(left)* Triangulation of the torus. *(right)* pairings for homology computation.

To compute β_0, we simply pair each vertex other than 1 with the edge connecting it to vertex 1 (see the red edges in Figure 3.8 (right)), so as to form a spanning tree. Then, inserting vertex 1 first increments β_0, and each further vertex insertion creates an independent 0-cycle that is immediately killed by the insertion of its paired edge. It follows that $\beta_0 = 1$.

Now, every remaining edge in the triangulation will create an independent 1-cycle at the time of its insertion. We therefore pair each edge other than $\{2,3\}$ and $\{4,7\}$ with an incident triangle (see the red ticks in Figure 3.8 (right)). Then, inserting $\{2,3\}$ then $\{4,7\}$ increases β_1 by 2, while each further edge insertion creates an independent 1-cycle that is immediately killed by the insertion of its paired triangle. It follows that $\beta_1 = 2$.

Finally, one triangle is left out by the pairing, namely $\{6,8,9\}$ (in blue in Figure 3.8 (right)). Inserting this triangle last gives $\partial_2\{6,8,9\} = 0$ because the triangle's boundary is already the boundary of the chain involving all the other triangles. As a consequence, the insertion of $\{6,8,9\}$ creates an independent 2-cycle and therefore increments β_2. In conclusion, $\beta_2 = 1$.

Note that $\beta_r = 0$ for all $r \geq 3$ since there are no r-simplices in the triangulation (hence the corresponding vector space of r-chains is trivial).

Homology groups of the sphere $\mathbb{S}^d$

We first observe that the sphere $\mathbb{S}^d$ is homeomorphic to the boundary of a $(d+1)$-simplex Δ embedded in $\mathbb{R}^d$. To see this, realign Δ so that its vertices lie on the sphere $\mathbb{S}^d$ and the origin O lies in its interior, then project its boundary radially onto $\mathbb{S}^d$. By convexity, the radial projection restricted to $\partial\Delta$ is bijective and bi-continuous, hence a homeomorphism. Thus, what we need to do now is compute the homology of the boundary $\partial\Delta$ of the $(d+1)$-simplex Δ.

Note that Δ itself is convex hence homotopy equivalent to a point. To see this, choose an arbitrary point p inside Δ, then consider the map $F : [0,1] \times \Delta \to \Delta$ defined by $F(t,x) = (1-t)\,x + t\,p$. This map is well-defined by convexity of Δ, and it is a homotopy between the identity map id_Δ and the projection π_p onto p. The homotopy equivalence is then given by π_p and by the inclusion $p \hookrightarrow \Delta$. Thus, we have $\beta_0(\Delta) = 1$ and $\beta_r(\Delta) = 0$ for all $r > 0$.

Now, let us apply the homology computation algorithm to Δ and to its boundary respectively. The only difference between the two executions is that, in the case of Δ, there is an extra column in the boundary matrix, corresponding to the insertion of the $(d+1)$-simplex itself. Since there are no other $(d+1)$-simplices, the column does not reduce to zero, hence the insertion of the $(d+1)$-simplex kills a d-cycle and thus decrements β_d. We conclude that

$$\begin{aligned}
\beta_d(\mathbb{S}^d) &= \beta_d(\partial\Delta) = \beta_d(\Delta) + 1 = 1\\
\beta_0(\mathbb{S}^d) &= \beta_0(\partial\Delta) = \beta_0(\Delta) = 1\\
\beta_r(\mathbb{S}^d) &= \beta_r(\partial\Delta) = \beta_r(\Delta) = 0 \quad \forall r \notin \{0,d\}.
\end{aligned}$$

Brouwer's fixed point theorem

1. The open half-line $]f(p), p)$ is always well-defined since there is no fixed point, and it evolves continuously with p as f is continuous. Finding its intersection $\phi(p)$ with the bounding circle of the unit disk boils down to solving for $\lambda > 0$ in the following equation:

   $$\|f(p) + \lambda\,(p - f(p))\|^2 = 1.$$

   The reduced discriminant of this degree-2 equation in λ is

   $$\langle f(p), p - f(p)\rangle^2 - (f(p)^2 - 1)(p - f(p))^2,$$

   which is always non-negative since $f(p)$ is located in the unit disk ($f(p)^2 \leq 1$). Moreover, the product of the two roots of the polynomial is non-positive, and when it is zero the sum is positive (since when the product is zero we have $f(p)^2 = 1$ and so $\langle f(p), p - f(p)\rangle < 0$ because p lies in the unit disk minus $f(p)$ and $f(p)$ lies on the disk's boundary). Therefore, there is always a unique positive root, and it evolves continuously with the parameters of the equation, hence with p. It follows that $\phi(p)$ is well-defined and continuous.

2. Note that $\phi \circ \iota = \mathrm{id}_{\mathbb{S}^1}$, therefore $\phi_* \circ \iota_*$ is an isomorphism and ϕ_* is surjective.

3. $\phi_* : H_*(\mathbb{B}^2; \mathbf{k}) \to H_*(\mathbb{S}^1; \mathbf{k})$ surjective implies that the dimension of $H_*(\mathbb{B}^2; \mathbf{k})$ is no smaller than that of $H_*(\mathbb{S}^1; \mathbf{k})$, which in the case $* = 1$ contradicts the fact that $H_1(\mathbb{S}^1; \mathbf{k}) = 1 > 0 = H_1(\mathbb{B}^2; \mathbf{k})$.

The chain of arguments used here is independent of the ambient dimension and of the field of coefficients.

The hairy ball theorem

1. We can define a homotopy Γ analytically via the formula:

$$(t,x) \mapsto (\cos \pi t)\, x + (\sin \pi t)\, V(x)/\|V(x)\|$$

   Note that the normalization of the vector $V(x)$ is possible because we assumed that $V(x) \neq 0$. To check that $\Gamma(t,x)$ lies on the unit sphere at any time t, we use that both vectors x and $V(x)/\|V(x)\|$ have norm 1, and that they are orthogonal to each other because x is on the sphere and $V(x)$ is a tangent vector at x — the rest is a simple calculation. Finally, the continuity of Γ in both parameters is immediate from the formula, as is the fact that $\Gamma(0,x) = x$ while $\Gamma(1,x) = -x$.

2. This is a direct consequence of $H_d(\mathbb{S}^d)$ being 1-dimensional (see Exercise on the homology of the sphere in all dimensions).

3. $\deg(\mathrm{id}(\mathbb{S}^d)) = 1$ because the morphism induced in homology by the identity map is itself the identity map. Now, since homotopy preserves the induced morphism, it also preserves the degree. As a consequence, we have $1 = (-1)^{d+1}$, which raises a contradiction when d is even.

The dunce hat

1. The homotopy f between $\mathrm{id}_{\mathbb{S}^1}$ and ϕ is illustrated in Figure 3.9, where the three copies of p (as well as the three copies of a) are matched after the transformation, as illustrated on the right-hand side of the figure.

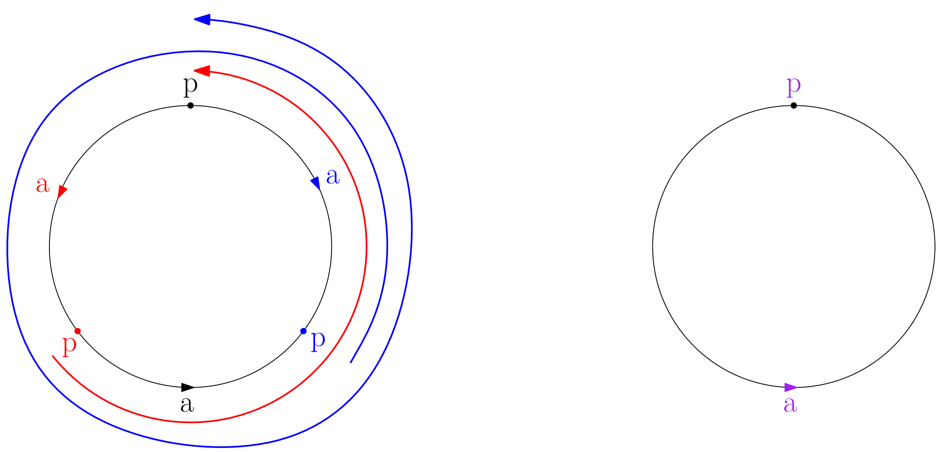


Figure 3.9: Homotopy between the identity and the gluing map.

   Formally, viewing $\mathbb{S}^1$ as the group of unit complex numbers, with the argument set to 0 at the top of the circle, f is defined analytically as follows:

$$f(t, e^{i\theta}) = \begin{cases} e^{3i\theta} & \text{if } \theta \in [0, 4\pi/3] \\ e^{-3i\theta} & \text{if } \theta \in [4\pi/3, 2\pi) \end{cases}$$

2. There is a homeomorphism h mapping $\mathbb{B}^2$ to a simplicial complex and $\mathbb{S}^1$ to a subcomplex. For instance, map $\mathbb{B}^2$ to a triangle and $\mathbb{S}^1$ to its boundary via a radial projection. Then, one can compose h with the homotopy for simplicial complexes given by the homotopy extension property, to obtain a homotopy for the continuous spaces.

3. We define maps between $\mathbb{B}^2$ and the dunce hat D as follows. For $f : \mathbb{B}^2 \to D$, we let $f(x) = \phi(x)$ if $x \in \mathbb{S}^1$ and $f(x) = x$ otherwise. For $g : D \to \mathbb{B}^2$, we let $g(x) = x$ if $x \in \mathbb{S}^1$ and $g(x) = F(1, x)$ otherwise, where F is the homotopy $[0, 1] \times \mathbb{B}^2 \to \mathbb{B}^2$ given by the extension of the homotopy between $\mathrm{id}_{\mathbb{S}^1}$ and ϕ. By construction, $g \circ f$ is homotopic to $\mathrm{id}_{\mathbb{B}^2}$ and $f \circ g$ is homotopic to id_D (run F backwards each time). Hence, $\mathbb{B}^2$ and D are homotopy equivalent, which means that the Dunce hat is contractible.