INF563 Topological Data Analysis — Exercise Session
Homology Groups and Homotopy

Exercise 1. Homology groups of some common spaces.
Propose triangulations of the following spaces, then compute their homology groups with coefficients in $\mathbb{Z}/2\mathbb{Z}$:

1. the circle (1-sphere) $\mathbb{S}^1$,
2. the disk $\mathbb{B}^2$,
3. the cylinder $\mathbb{S}^1 \times [0, 1]$
4. the 2-sphere $\mathbb{S}^2$,
5. the 3-ball $\mathbb{B}^3$,
6. the Torus $\mathbb{T}$, obtained from the unit square by identifying opposite edges as illustrated in Figure 1.

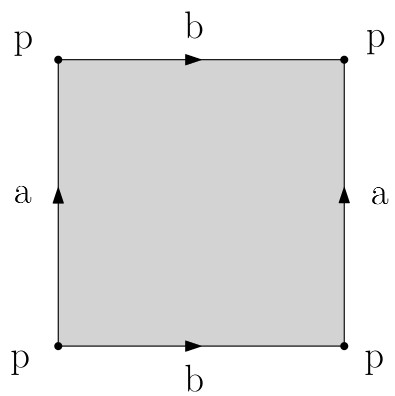

Figure 1: Edge identifications on the unit square to get the torus $\mathbb{T}$.

Exercise 2. Homology groups of the sphere $\mathbb{S}^d$.
Compute the homology groups of the d-dimensional sphere $\mathbb{S}^d$, for any $d \geq 1$.

Exercise 3. Brouwer's fixed point theorem.
Brouwer proved the following fixed point theorem: "*every continuous map* $f : \mathbb{B}^2 \to \mathbb{B}^2$ *has a fixed point, i.e. a point* $p \in \mathbb{B}^2$ *such that* $f(p) = p$." In particular, there is always a point at the surface of your coffee that is not moving. ☺

Here is a simple and elegant proof of this theorem, using homology theory. Let us assume that there exists a continuous map $f : \mathbb{B}^2 \to \mathbb{B}^2$ that has no fixed point, and let us look for a contradiction:

1. For any $p \in \mathbb{B}^2$, let $\phi(p)$ be the intersection of the open half-line $]f(p), p)$ with the circle $\mathbb{S}^1$. Show that $\phi : \mathbb{B}^2 \to \mathbb{S}^1$ is well-defined and continuous.

2. Show that ϕ induces a surjective homomorphism ϕ_* at the homology level. For this you can compose ϕ with the canonical inclusion $\iota : \mathbb{S}^1 \hookrightarrow \mathbb{B}^2$ and consider the induced homomorphism.

3. Conclude. Does the proof extend to higher dimensions?

Exercise 4. The hairy ball theorem.
The aim of this exercise is to prove the hairy ball theorem: "*for* d *even, there is no non-vanishing continuous tangent vector field* $V : \mathbb{S}^d \to \mathbb{R}^d$." In particular, one cannot comb a 2-sphere that has no baldness, and there is always a point on Earth where there is no wind. ☺

Here is again a simple and elegant proof using homology theory. Let us assume that there exists a non-vanishing continuous tangent vector field V over the sphere $\mathbb{S}^d$, for some arbitrary d, and let us look for a contradiction when d is even.

1. Use V to find a homotopy between $\mathrm{id}_{\mathbb{S}^d}$ and the antipodal map $x \mapsto -x$.

2. Show that every morphism $\phi_* : H_d(\mathbb{S}^d) \to H_d(\mathbb{S}^d)$ is of the form $x \mapsto \alpha_{\phi_*} x$ for some fixed scalar α_{ϕ_*}.

The *degree* of a continuous map $f : \mathbb{S}^d \to \mathbb{S}^d$ is defined as $\deg(f) = \alpha_{f_*}$. One can show that the degree of the antipodal map $x \mapsto -x$ is $(-1)^{d+1}$ (see Theorem 21.3 in "*Elements of Algebraic Topology*" by J. Munkres).

3. What is $\deg(\mathrm{id}_{\mathbb{S}^d})$? Conclude.

Exercise 5. The dunce hat.
The dunce hat is a classical example of a space that is contractible (homotopy equivalent to a point) but not collapsible (a stronger notion of contractibility, which plays no role in this exercise). It is obtained by identifying the edges of a triangle as illustrated in Figure 2 left.

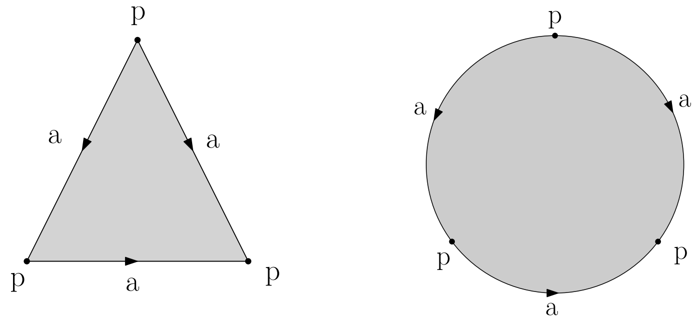

Figure 2: Left: from the triangle to the dunce hat. Right: homeomorphic representation where the triangle has been replaced by a disk.

The aim of this exercise is to show that the dunce hat is contractible. Let us turn the initial triangle into a disk, as shown in Figure 2 right, which is only a matter of applying some homeomorphism. Then, let $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ be the map induced by the edge identifications (whereby p is sent to the North pole and the rest of the circle is spanned by arc a).

1. Show that there is a homotopy between $\mathrm{id}_{\mathbb{S}^1}$ and ϕ, that is, a map $f : [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1$ such that $f(0,\cdot) = \mathrm{id}_{\mathbb{S}^1}$ and $f(1,\cdot) = \phi$.

Let $A \subseteq X$ be topological spaces. We say that the pair (X, A) has the *homotopy extension property* if, for any homotopy $f : [0,1] \times A \to Y$ and any map $F_0 : X \to Y$ such that $F_0|_A = f(0,\cdot)$, there is an extension of F_0 to a homotopy $F : [0,1] \times X \to Y$ such that $F(t,\cdot)|_A = f(t,\cdot)$ for all $t \in [0,1]$. It is known that any pair (X, A), where X is a simplicial complex and A is a subcomplex of X, has the homotopy extension property.

2. Show that the pair $(\mathbb{B}^2, \mathbb{S}^1)$ has the homotopy extension property. Deduce that there is a homotopy $F : [0,1] \times \mathbb{B}^2 \to \mathbb{B}^2$ such that $F(0,\cdot) = \mathrm{id}_{\mathbb{B}^2}$ and $F(t,\cdot)|_{\mathbb{S}^1} = f(t,\cdot)$ for all $t \in [0,1]$.

3. Show that the dunce hat and $\mathbb{B}^2$ are homotopy equivalent and conclude.