(0) Morse type pairs:

<u>Idea:</u> work out a set of axioms that guarantee that the Reeb graph has a 1-d stratification.

* topological changes occur at finitely many "critical" levels:

(Morse) (i) $\exists\, n \in \mathbb{N}$, $a_1 < \dots < a_n \in \mathbb{R}$ s.t.

$$X \setminus \bigcup_{i=1}^{n} f^{-1}(\{a_i\}) \underset{h}{\cong} \coprod_{i=1}^{n-1} Y_i \times (a_i, a_{i+1})$$

* gluing property:

(adjunction) (ii) h extends to $\bar{h} : \coprod_{i=1}^{n-1} Y_i \times [a_i, a_{i+1}]$ continuous

* critical levels quotient themselves to vertices, cylinders to edges:

(1-d) (iii) the $f^{-1}(\{x_i\})$ and Y_i are compact and locally connected

(finiteness) * finitely many vertices and edges:

(iv) $\forall t \in \mathbb{R}$, $H_*(f^{-1}(\{t\}))$ is finitely generated

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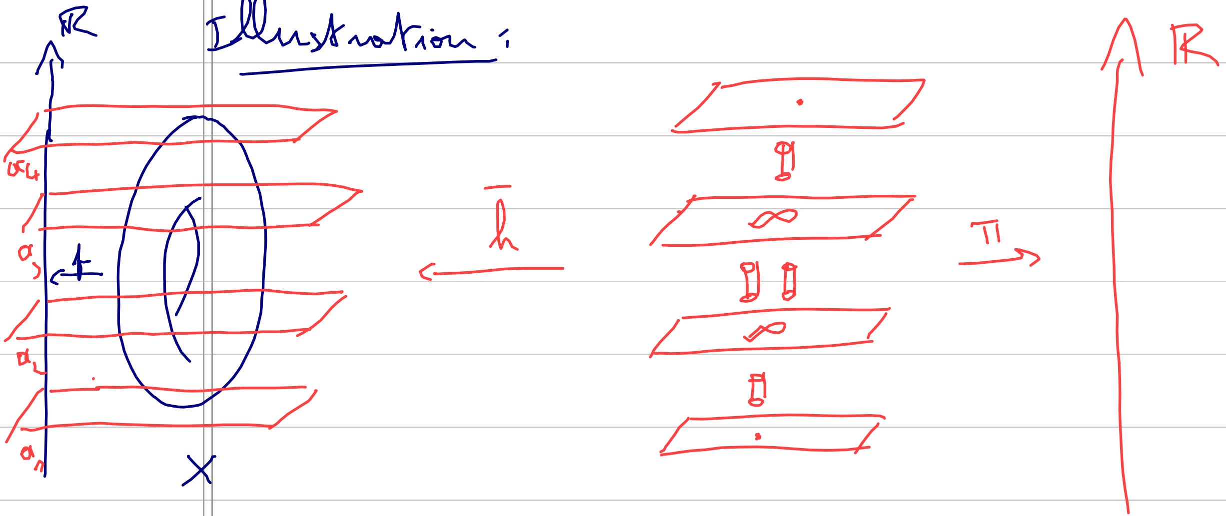
<u>Illustration:</u>



$\pi = f \circ \bar{h}$

Obs $\boxed{\text{Homology of a pair of spaces (a.k.a. relative homology).}}$

Let $A \subseteq X$ be a pair of (triangulable) spaces.

↳ define homology of X "modulo A":

let $i: A \to X$ inclusion map

$$\begin{tikzcd}
C_n(X;k) \arrow[r, "\partial_n"] & C_{n-1}(X;k) \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_1"] & C_0(X;k) \arrow[r, "\partial_0"] & 0 \\
C_n(A;k) \arrow[r, "\partial_n"] \arrow[u, "i_\#"] & C_{n-1}(A;k) \arrow[r, "\partial_{n-1}"] \arrow[u, "i_\#"] & \cdots \arrow[r, "\partial_1"] & C_0(A;k) \arrow[r, "\partial_0"] \arrow[u, "i_\#"] & 0
\end{tikzcd}$$

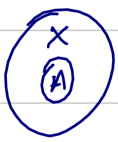
$i_\#$ chain map: $i_\# \circ \partial = \partial \circ i_\#$.

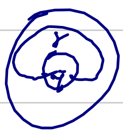
↳ take its <u>cokernel</u> (chain complex):

$$\operatorname{coker} i_\# = C_n(X;k)/C_n(A;k) \xrightarrow{\bar\partial_n} C_{n-1}(X;k)/C_{n-1}(X;k) \xrightarrow{\bar\partial_{n-1}} \cdots \xrightarrow{\bar\partial_0} 0$$

$\boxed{Q}$ What is this cokernel?

* in $C_n(X;k)$, $c \sim_A c'$ iff. $c - c' \in C_n(A;k)$





$c = \gamma + \eta$, $c' = \gamma$ $\mid$ $c - c' = \eta \sim_A 0$.

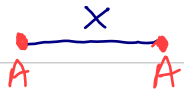
↳ in particular, $[c']$ is a cycle in $C_n(X;k)/C_n(A;k)$.

* $\bar\partial_n [c] = 0$ $\overset{\in C_{n-1}(X)/C_{n-1}(A)}{}$ $\iff$ $\partial c \in C_{n-1}(A)$. (cf. previous example)

* $\partial_{n-1} \circ \partial_n = 0 \implies \bar\partial_{n-1} \circ \bar\partial_n = 0$.

$\boxed{\text{Def:}}$ $$H_n(X, A; k) := \ker \bar\partial_n / \operatorname{Im} \bar\partial_{n+1}.$$

$\boxed{\text{Example:}}$



$$\begin{tikzcd}
\cdots\ 0 \arrow[r, "\partial_2"] & k \arrow[r, "\partial_1"] & k \arrow[r, "\partial_0"] & 0 \\
\cdots\ 0 \arrow[r, "\partial_2"] \arrow[u, "0"] & 0 \arrow[r, "\partial_1"] \arrow[u, "0"] & k^2 \arrow[r, "\partial_0"] \arrow[u, "{[1\ 1]}"] & 0 \arrow[u, "0"]
\end{tikzcd}$$

$\operatorname{coker} i_\# = \cdots 0 \xrightarrow{\bar\partial_2} k \xrightarrow{\bar\partial_1} 0 \xrightarrow{\bar\partial_0} 0 \implies \left(\begin{array}{l} H_1(X, A; k) \simeq k \\ H_n(X, A; k) = 0 \ \forall n \neq 1 \end{array} \right)$

① $\boxed{\text{Extended persistence:}}$

* indexing poset: $\xrightarrow{\mathbb{R}} +\infty \xleftarrow{\mathbb{R}^{op}}$

$T = \mathbb{R} \cup \{+\infty\} \cup \mathbb{R}^{op}$ equipped with the extended total order $\precsim$:

$$\begin{cases} \text{for } x,y \in \mathbb{R}: & x \precsim y \iff x \leq y \\ \text{for } x,y \in \mathbb{R}^{op}: & x \precsim y \iff x \geq y \\ \text{for } x \in \mathbb{R}, y \in \mathbb{R}^{op}: & x \precsim +\infty \precsim y \end{cases}$$

* Representations:

a) $F: (T, \precsim) \longrightarrow \mathrm{Top}$

on objects:

$$F(t) := \begin{cases} f^{-1}((-\infty, t]) & \text{for } t \in \mathbb{R} \\ \mathbb{R}f \cong (\mathbb{R}f, \emptyset) & \text{for } t = +\infty \\ (\mathbb{R}f, f^{-1}([t, +\infty))) & \text{for } t \in \mathbb{R}^{op} \end{cases}$$

on morphisms: $\forall t \precsim t' \in T$, $F(t \precsim t') = F(t) \subseteq F(t')$

indeed:
$$\begin{array}{l} t \leq t' \in \mathbb{R} \Rightarrow f^{-1}((-\infty, t]) \subseteq f^{-1}((-\infty, t']) \\ t \geq t' \in \mathbb{R}^{op} \Rightarrow (\mathbb{R}f, f^{-1}([t, +\infty))) \subseteq (\mathbb{R}f, f^{-1}([t', +\infty))) \end{array}$$

homology for space or pair of spaces → ← fixed field

b) $(T, \precsim) \xrightarrow{F} \mathrm{Top} \xrightarrow{H(-;k)} \mathrm{vect}_k$ ← finite-dim k-vector spaces

Extended persistence module: $FH \in \mathrm{vect}_k^{(T,\precsim)}$.

⚠ $\boxed{\text{Note:}}$ $(T,\preccurlyeq) \underset{\text{(poset)}}{\cong} (\mathbb{R},\leq)$ ← send $\mathbb{R}$ to $(-\infty,0)$, $+\infty$ to 0, and $\mathbb{R}^{op}$ to $(+\infty, 0)$

$\Rightarrow$ $\boxed{\mathrm{vect}_k^{(T,\preccurlyeq)} \cong \mathrm{vect}_k^{(\mathbb{R},\leq)}}$

↳ same as ordinary persistence modules.

* Decomposition:

$(T,\preccurlyeq)\cong(\mathbb{R},\leq)$ is a small category and vect_k is an abelian category

$\Rightarrow$ $\mathrm{vect}_k^{(T,\preccurlyeq)}$ is abelian. $\Rightarrow$ can talk about decompositions

$\boxed{\text{Thm:}}$ (cf. slide)

"Every persistence module decomposes uniquely as a direct sum of interval modules".

I on $\mathbb{R}$:

$$\cdots \to 0 \to k \xrightarrow{\mathrm{id}} k \to 0 \to \cdots$$

↳ the result applies verbatim to extended persistence modules.

* classification of interval summands:

$T := \mathbb{R} \to +\infty \leftarrow \mathbb{R}^{op}$

I: ordinary

I: relative

I: extended

Note: barcode $\to$ diagram: $I = a,b \mapsto (a,b) \in \mathbb{R}^2$

② Metrics between Reeb graphs / persistence diagrams:

* Metrized Reeb graph : $R_f \xrightarrow{f} \mathbb{R}$

(f is identified with the induced map)

$$d_f : \begin{cases} R_f \times R_f \longrightarrow \mathbb{R} \\ (x, y) \longmapsto \inf\limits_{\gamma : x \rightsquigarrow y} \max f\circ\gamma - \min f\circ\gamma . \end{cases}$$

$$\begin{cases} \gamma : [0,1] \longrightarrow R_f \quad C^0 \\ \gamma(0) = x \quad \gamma(1) = y \end{cases}$$

* Gromov – Hausdorff distance :

$$d_{GH}\big((R_f, d_f), (R_g, d_g)\big) := \inf_{(Z, d_Z)} d_H^Z\big(i_f(R_f), i_g(R_g)\big)$$

$i_f : R_f \to Z$ isometries
$i_g : R_g \to Z$

* Bottleneck distance : $d_B := \inf\limits_{D_g R_f \supseteq A \overset{m}{\longleftrightarrow} B \subseteq D_g R_g} \max\left\{ \sup\limits_{m(a)=b} \|a-b\|_\infty , \sup\limits_{p \notin A \cup B} \frac{|p_y - p_x|}{2} \right\}$

partial matching : bijection $D_g R_f \supseteq A \overset{m}{\longleftrightarrow} B \subseteq D_g R_g$

cost : $\begin{cases} m(a) = b : \quad \|a-b\|_\infty \\ p \notin A \cup B : \quad |p_y - p_x| / 2 \end{cases}$ (distance to diagonal Δ)

$$\text{cost}(m) = \max\left\{ \sup_{m(a)=b} \|a-b\|_\infty , \sup_{p \notin A\cup B} |p_y - p_x| / 2 \right\}$$ (bottleneck cost).

$$d_B(D_g R_f, D_g R_g) := \inf\left\{ \text{cost}(m) \mid D_g R_f \supseteq A \overset{m}{\longleftrightarrow} B \subseteq D_g R_g \right\}.$$

Alternate formulation using Wasserstein metrics (with a twist):
(μ_f depends on g)

Given R_f, R_g, let $\begin{cases} \mu_f := \sum_{p \in D_g R_f} \delta_p + \sum_{q \in D_g R_g} \delta_{\pi_\Delta(q)} \\ \mu_g := \sum_{q \in D_g R_g} \delta_q + \sum_{p \in D_g R_f} \delta_{\pi_\Delta(p)} \end{cases}$

where $\pi_\Delta : \mathbb{R}^2 \to \Delta$ is the ortho. proj. onto Δ.

Then: $d_B(D_g R_f, D_g R_g) = W_\infty(\mu_f, \mu_g)$

Note: here the optimal transport plan is not mass-splitting.

$\boxed{\text{Thm}}$ [Carrière, O. 2016] (work in progress)

$\forall R_f \; \exists \varepsilon > 0$ s.t. $\forall R_g$ with $d_{GH}(R_f, R_g) \leq \varepsilon$:

$$\frac{1}{4} d_{GH}(R_f, R_g) \leq d_B(D_g R_f, D_g R_g) \leq 6\, d_{GH}(R_f, R_g)$$

⚠ "locally" is understood in the Gromov-Hausdorff metric (otherwise the first inequality does not hold)