Stability of descriptors for metric spaces

Thm: [Chazal, de Silva, O. 2013]

For any compact metric spaces (X, d_X) and (Y, d_Y),
$$d_b(\mathrm{Dgm}\,\mathcal{R}(X,d_X), \mathrm{Dgm}\,\mathcal{R}(Y,d_Y)) \leq 2\, d_{GH}(X,Y).$$

Ingredients for the proof:

① Stability theorem:

$\forall f, g : X \to \mathbb{R}$ tame, $d_b(\mathrm{Dgm}\, f, \mathrm{Dgm}\, g) \leq \|f-g\|_\infty$.

② Nerve theorem in $(\mathbb{R}^d, \ell_\infty)$:

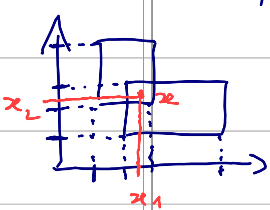
$\forall P \subset \mathbb{R}^d$ finite, $\forall t \geq 0$, the ℓ_∞-balls centered at pts of P are convex $\Rightarrow$ their intersections are either empty or homotopy equivalent to a point.

$$\Rightarrow C_t(P, \ell_\infty) \simeq \bigcup_{p \in P} B_\infty(p,t).$$

③ Equality Čech-Rips in $(\mathbb{R}^d, \ell_\infty)$:

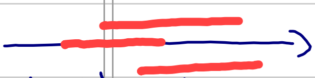
$\forall P \subset \mathbb{R}^d$ finite, $\forall t \geq 0$, $C_t(P, \ell_\infty) = R_{2t}(P, \ell_\infty)$.

proof: $\underset{\{p_0, \dots, p_r\}}{\sigma} \in C_t(P) \Leftrightarrow \bigcap_{i=0}^{r} B_\infty(p_i, t) \neq \emptyset \Leftrightarrow \bigcap_{i=0}^{r} \prod_{k=1}^{d} [p_i^k - t, p_i^k + t] \neq \emptyset$



$\Leftrightarrow \prod_{k=1}^{d} \bigcap_{i=0}^{r} [p_i^k - t, p_i^k + t] \neq \emptyset$ (product structure)

$\Leftrightarrow \forall k = 1 \dots d, \forall 0 \leq i < j \leq r, [p_i^k - t, p_i^k + t] \cap [p_j^k - t, p_j^k + t] \neq \emptyset$ (Helly's thm)



$\Leftrightarrow$ ________________ , $|p_i^k - p_j^k| \leq 2t$

pairwise intersections $\Leftrightarrow$ common intersection

$\Leftrightarrow \forall 0 \leq i < j \leq r, \forall k = 1, \dots, d, |p_i^k - p_j^k| \leq 2t \Leftrightarrow \sigma \in R_{2t}(P)$.

(4) Isometric embedding of finite metric spaces:

$\boxed{\text{Lemma}}$ Every finite metric space of size n embeds isometrically into $(\mathbb{R}^n, \ell_\infty)$.

proof: Let (P, d_P) be a finite metric space. Label the pts of P: $P = \{p_1, \dots, p_n\}$.

$$\gamma : \left| \begin{array}{l} (P, d_P) \longrightarrow (\mathbb{R}^d, \ell_\infty) \\ p_i \longmapsto \left(d_P(p_i, p_1), d_P(p_i, p_2), \dots, d_P(p_i, p_n)\right). \end{array} \right.$$

Then: $\forall i, j, k$, $|\gamma(p_i)^k - \gamma(p_j)^k| = |d_P(p_i, p_k) - d_P(p_j, p_k)| \le d_P(p_i, p_j)$. (triangle inequality)

Take $k = j \rightsquigarrow |\gamma(p_i)^j - \gamma(p_j)^j| = d_P(p_i, p_j)$.

$\Rightarrow \|\gamma(p_i) - \gamma(p_j)\|_\infty = d_P(p_i, p_j)$. ∎

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proof of the stability theorem: (cf. illustration slide)

$$\begin{tikzcd}
(X, d_X) \arrow[dr, "\gamma_X"] & & (Z, d_Z) \\
& \gamma_X(X) \sqcup \gamma_Y(Y) \arrow[r, "\gamma"] & \gamma\circ\gamma_X(X) \sqcup \gamma\circ\gamma_Y(Y) \\
(Y, d_Y) \arrow[ur, "\gamma_Y"'] & &
\end{tikzcd}$$

$(Z, d_Z) \supseteq \gamma_X(X) \sqcup \gamma_Y(Y)$ (def. of d_{GH}); $\quad \xrightarrow{\gamma}$ (by (4)); $\quad (\mathbb{R}^n, \ell_\infty) \supseteq \gamma\circ\gamma_X(X) \sqcup \gamma\circ\gamma_Y(Y)$ $\quad (n = \#X + \#Y)$

Then: $d_b(\mathrm{Dgm}\,\mathcal{R}(X, d_X), \mathrm{Dgm}\,\mathcal{R}(Y, d_Y)) = d_b(\mathrm{Dgm}\,\mathcal{R}(\gamma\circ\gamma_X(X), \ell_\infty), \mathrm{Dgm}(\gamma\circ\gamma_Y(Y), \ell_\infty))$

$= d_b(2\,\mathrm{Dgm}\,\check{C}(\gamma\circ\gamma_X(X), \ell_\infty), 2\,\mathrm{Dgm}\,\check{C}(\gamma\circ\gamma_Y(Y), \ell_\infty))$ (by (3))

coordinates of diagram pts are scaled by 2

$= 2\, d_b(\mathrm{Dgm}\,\check{C}(\gamma\circ\gamma_X(X), \ell_\infty), \mathrm{Dgm}\,\check{C}(\gamma\circ\gamma_Y(Y), \ell_\infty))$

$= 2\, d_b(\mathrm{Dgm}\, d^\infty_{\gamma\circ\gamma_X(X)}, \mathrm{Dgm}\, d^\infty_{\gamma\circ\gamma_Y(Y)})$ (by (2))

$\le 2\, d_H^\infty(\gamma\circ\gamma_X(X), \gamma\circ\gamma_Y(Y))$ (by (1))

$= 2\, d_{GH}((X, d_X), (Y, d_Y))$ (by def. of d_{GH} + (4)) ∎