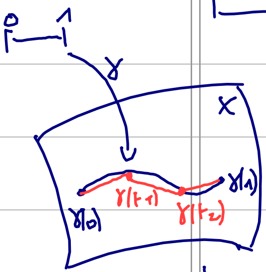
Topological descriptors for geometric data

Def: Given (X, d_X), (Y, d_Y) compact metric spaces, the Gromov-Hausdorff distance $d_{GH}(X,Y)$ is defined by:

$$d_{GH}(X,Y) := \inf_{\substack{(Z,d_Z) \text{ metric space} \\ \left.\begin{array}{l}\gamma_X : X \to Z \\ \gamma_Y : Y \to Z\end{array}\right) \text{isometries}}} d_H^Z\left(\gamma_X(X), \gamma_Y(Y)\right).$$

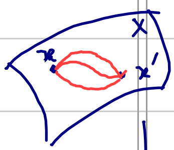
Hausdorff distance in (Z, d_Z)

Def: Given (X, d_X) compact and $\gamma : [0,1] \to X$ continuous, the length of γ is:



$$|\gamma| := \sup_{\substack{n \in \mathbb{N}_{>0} \\ 0 = t_0 < t_1 < \dots < t_n = 1}} \sum_{i=0}^{n-1} d_X\left(\gamma(t_i), \gamma(t_{i+1})\right).$$

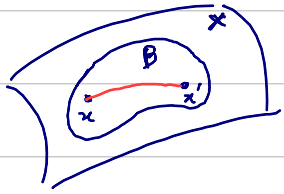
Def: (X, d_X) is an intrinsic metric space if the distance d_X coincides with the shortest-path distance, that is:



$$\forall x, x' \in X, \quad d_X(x,x') = \inf_{\substack{\gamma : [0,1] \to X \\ \gamma(0) = x \\ \gamma(1) = x'}} |\gamma|.$$

Note: When X is compact, the infimum above is a minimum (when it is not infinite), i.e. every pair of points x, x' in the same path-connected component of X are connected by a (possibly non-unique) shortest path.

Def: Given (X, d_X) compact intrinsic, a subset $B \subseteq X$ is <u>convex</u> if every pair of points $x, x' \in B$ is connected by a unique shortest path and that path is included in B.



Def: Given (X, d_X) compact intrinsic, the <u>convexity radius</u> at $x \in X$ is defined by:

$$\rho(X, x) := \sup \{ r > 0 \mid B_X(x, r') \text{ is convex for all } r' \leq r \}$$

ball of center x and radius r': $B(x, r') := \{ x' \in X \mid d_X(x, x') \leq r' \}$.

The <u>convexity radius</u> of X is:

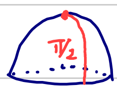
$$\rho(X) := \inf_{x \in X} \rho(X, x).$$

Examples:

$\rho((\mathbb{R}^d, \|.\|_2)) = +\infty$

$\forall X \subseteq \mathbb{R}^d$ convex, $\rho(X) = +\infty$

$\rho(S^d) = \pi/2$ (radius of hemisphere)



Def: Given X, Y compact sets in (Z, d_Z), and basepoints $x \in X$, $y \in Y$, the <u>Hausdorff distance</u> between pointed spaces is: $d_H^Z((X,x),(Y,y)) := \max\{ d_H^Z(X, Y), d_Z(x, y)\}$.

The <u>Gromov–Hausdorff distance</u> between pointed spaces is:

$$d_{GH}((X,x),(Y,y)) := \inf_{\substack{(Z, d_Z) \text{ metric space} \\ \gamma_X : X \to Z \\ \gamma_Y : Y \to Z \text{ isometries}}} d_H^Z((\gamma_X(X), \gamma_X(x)), (\gamma_Y(Y), \gamma_Y(y)))$$