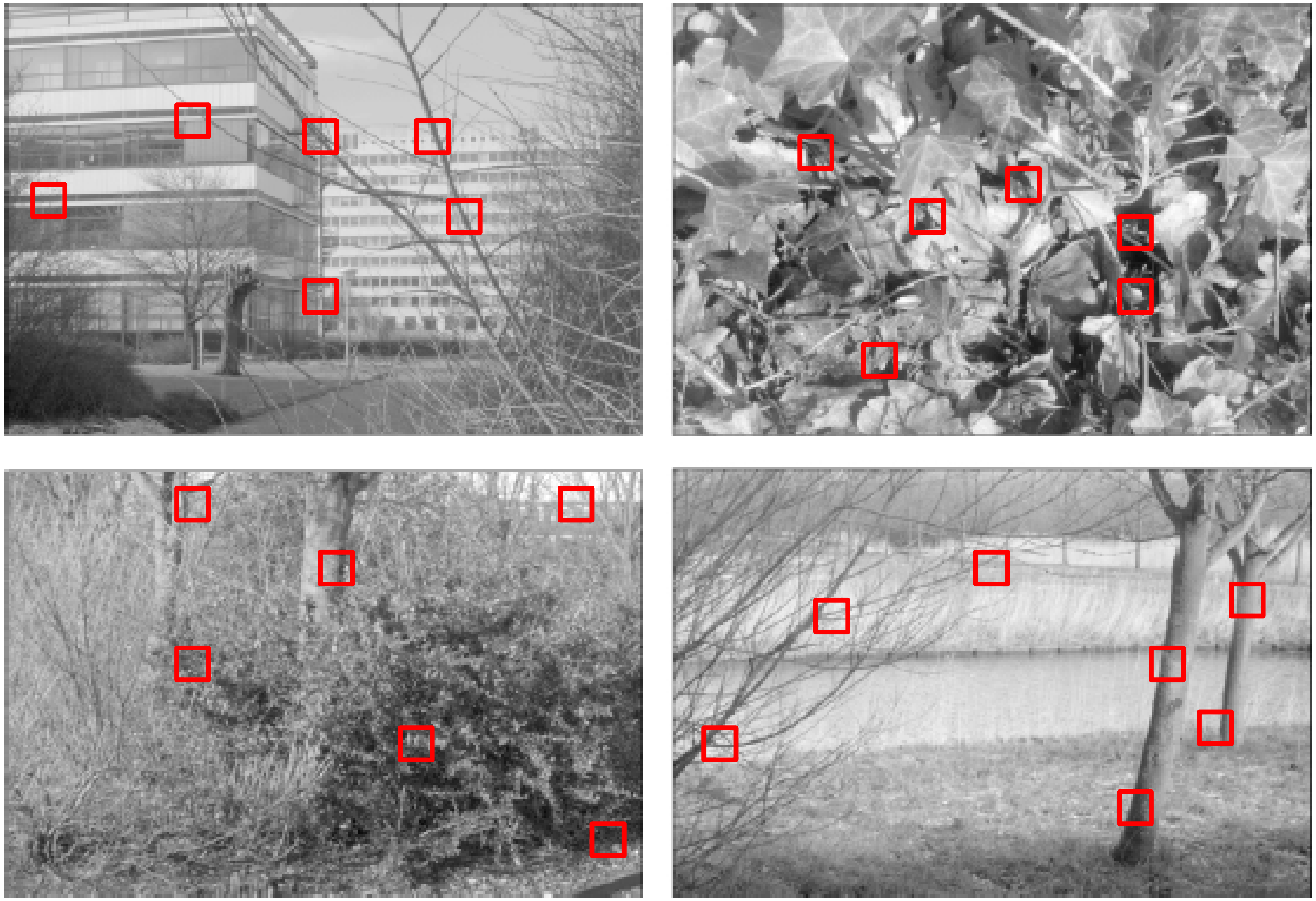
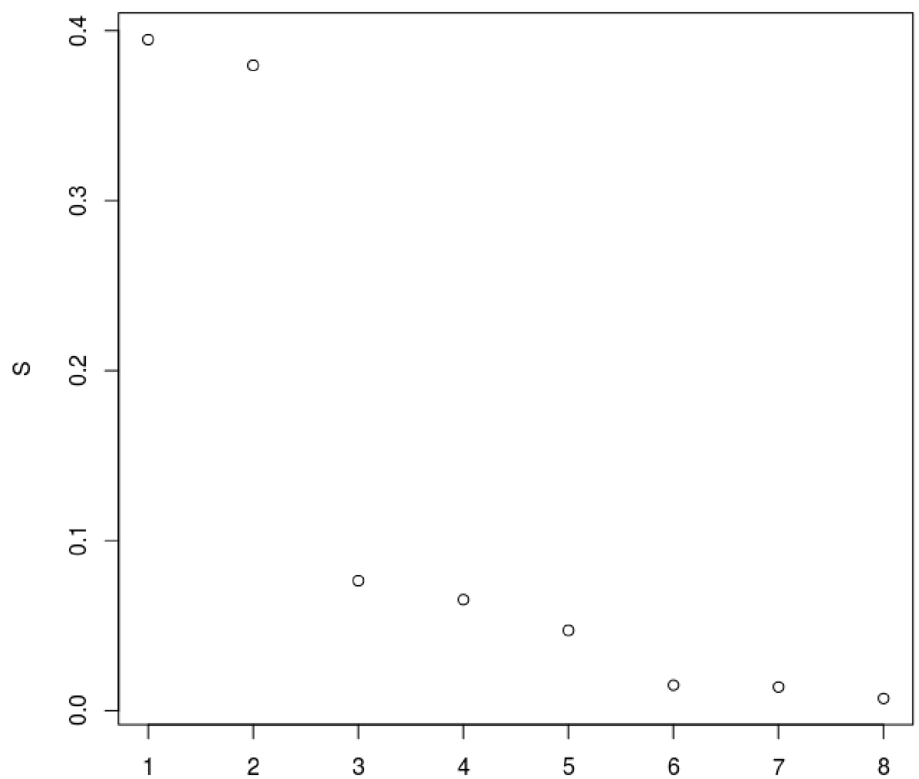
Example: Natural Images Data

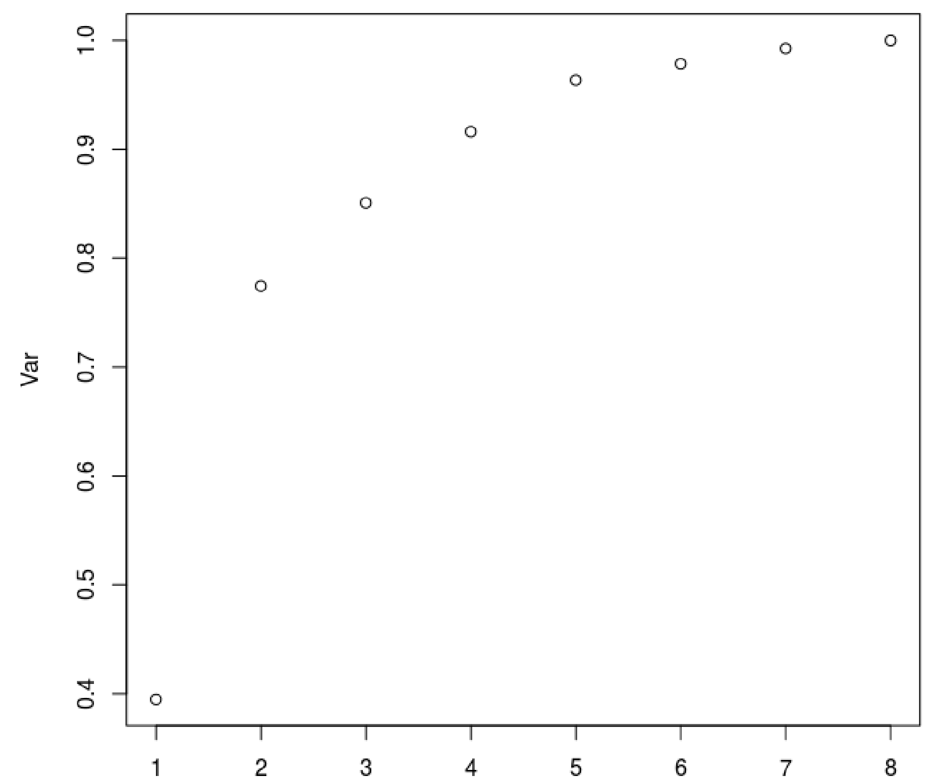
Input: 4 million data points on $\mathbb{S}^7$, coming from high-contrast 3×3 image patches

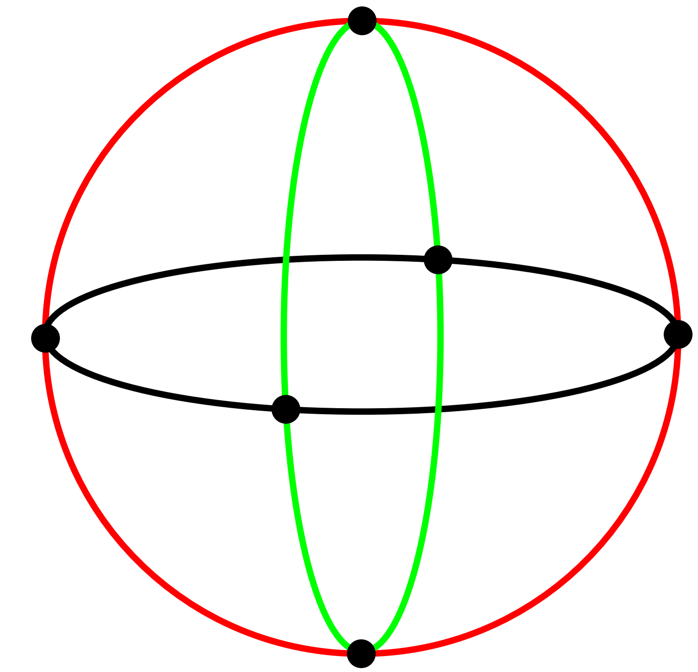


(source: [Lee, Pederson, Mumford 03])

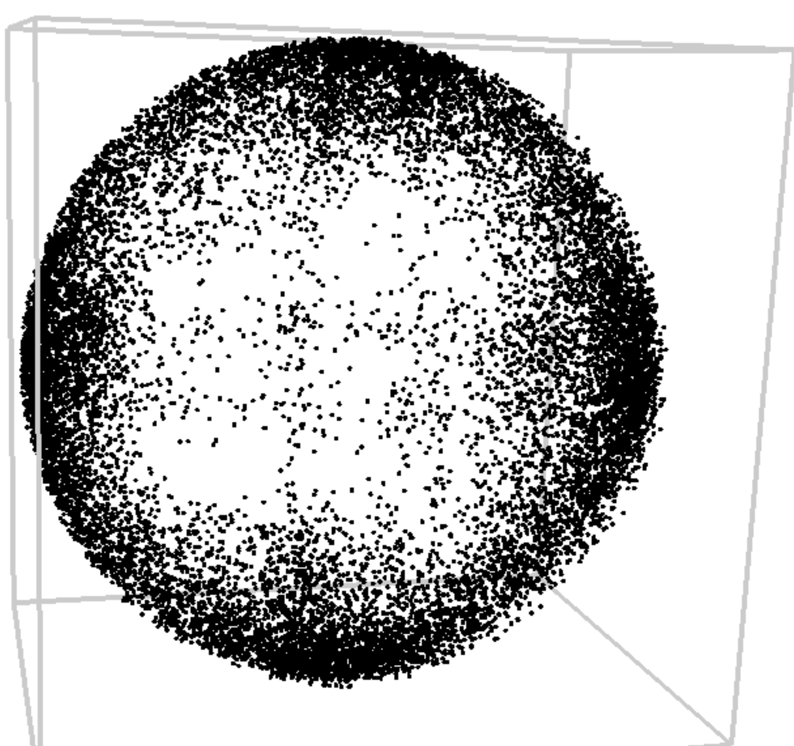
Example: Natural Images Data



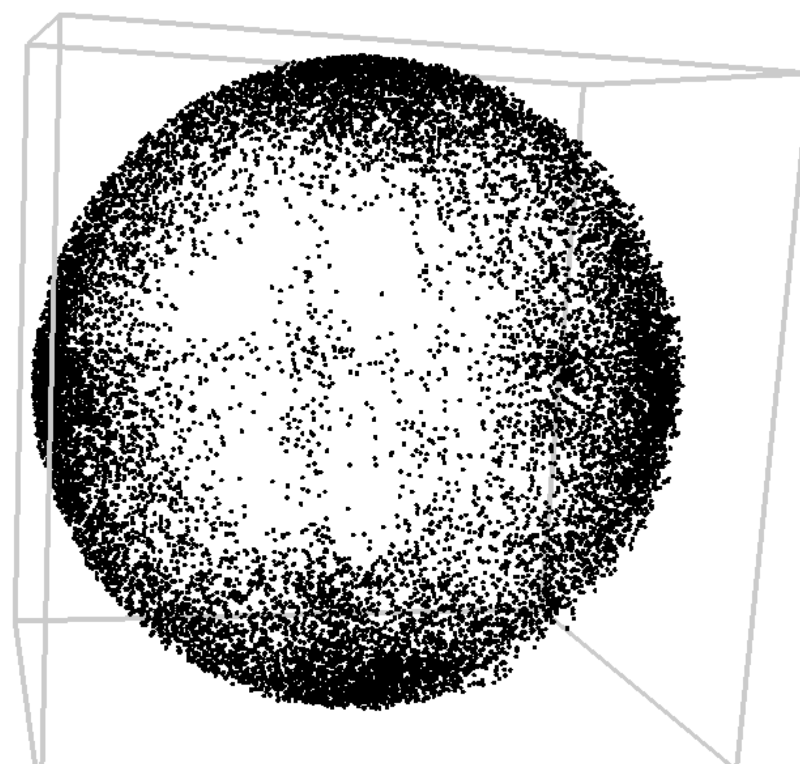




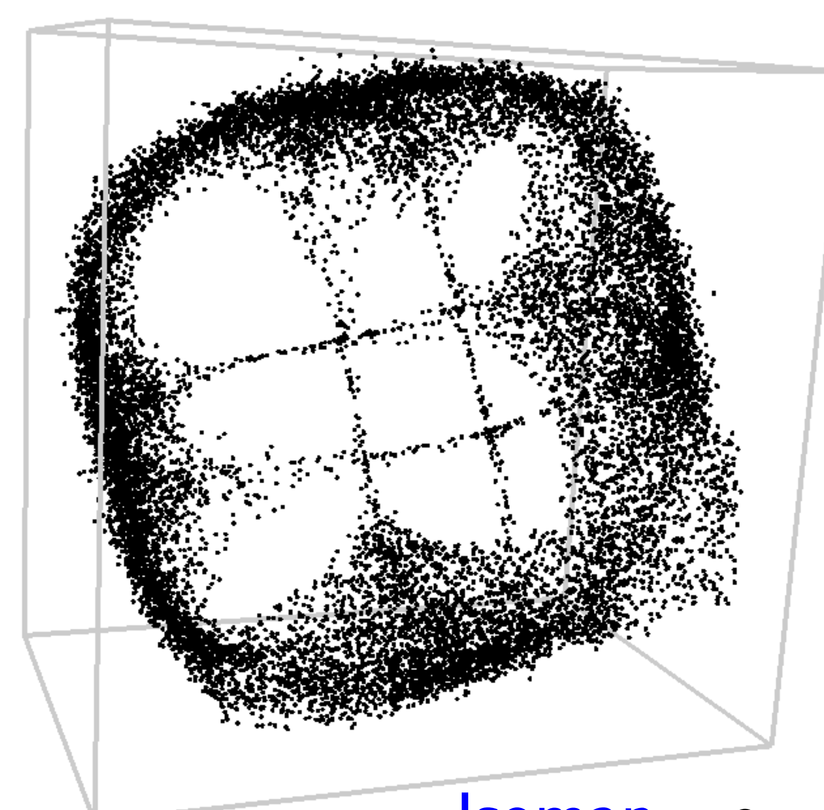
$(\beta_1 = 7)$



PCA

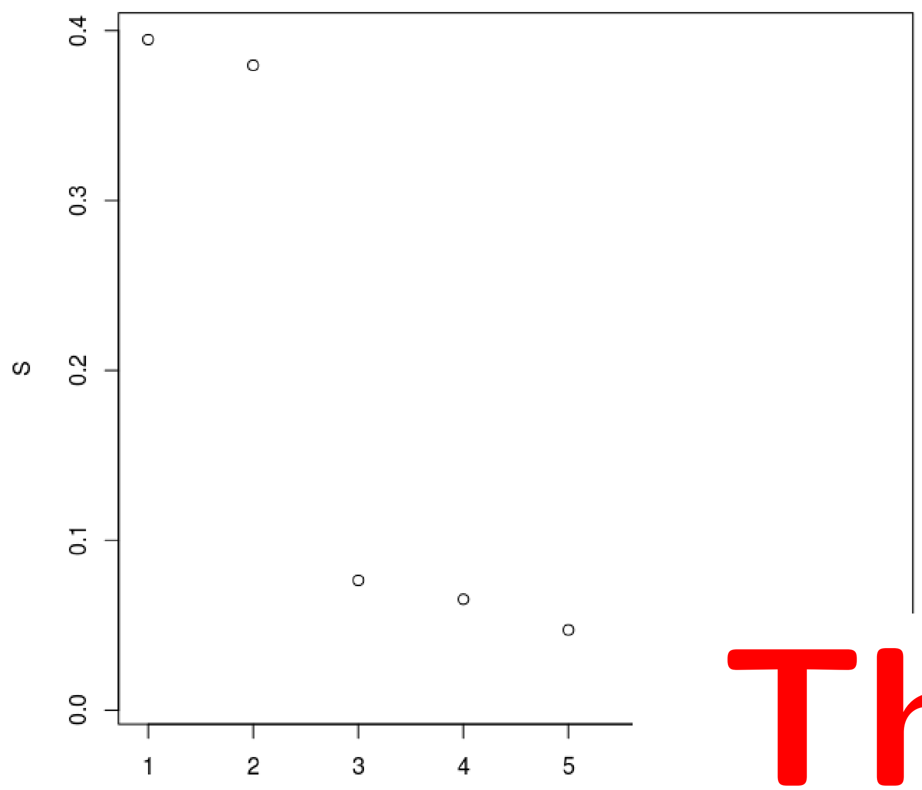


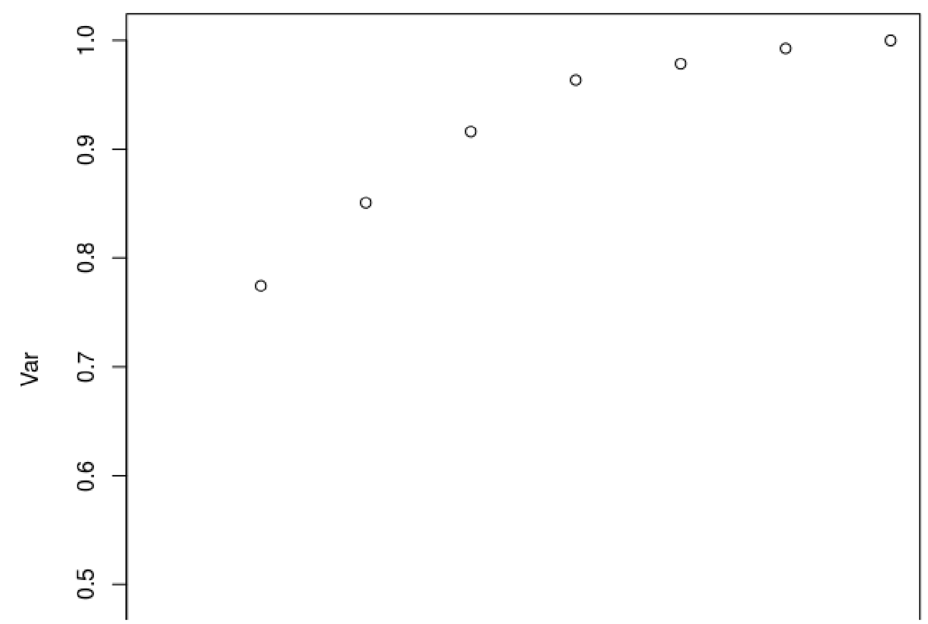
MDS

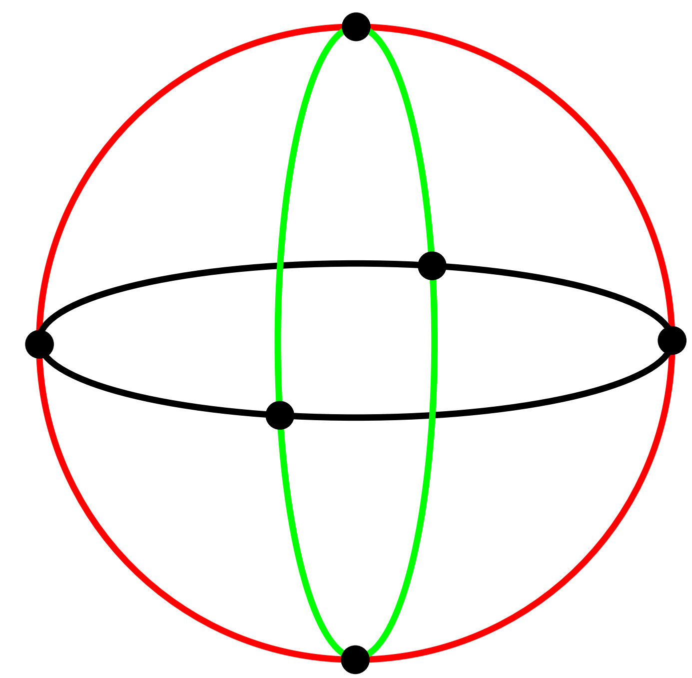


Isomap

Example: Natural Images Data

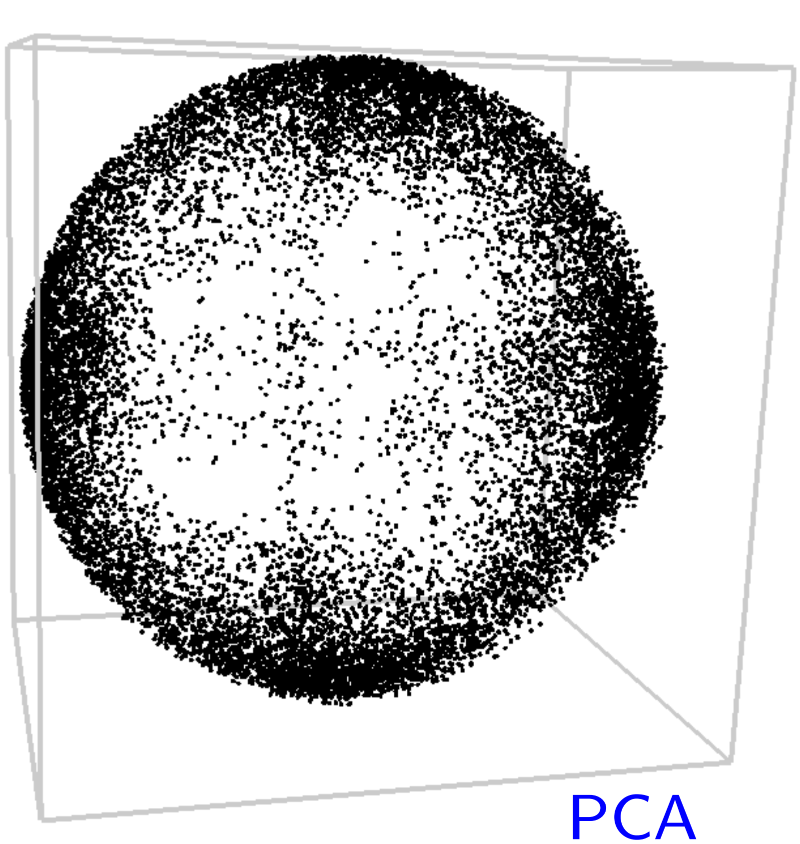




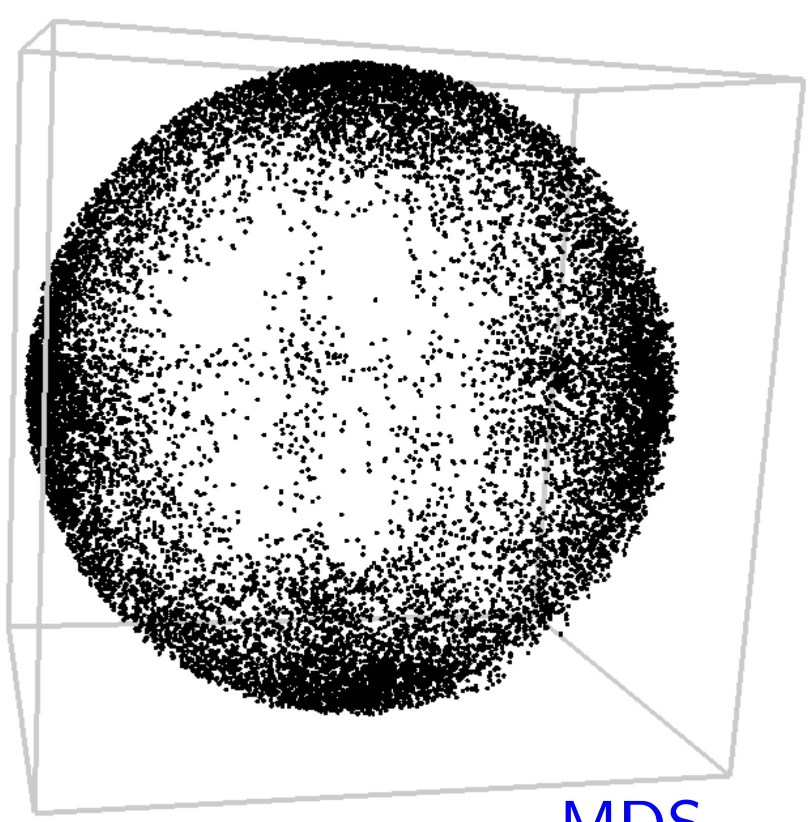


$(\beta_1 = 7)$

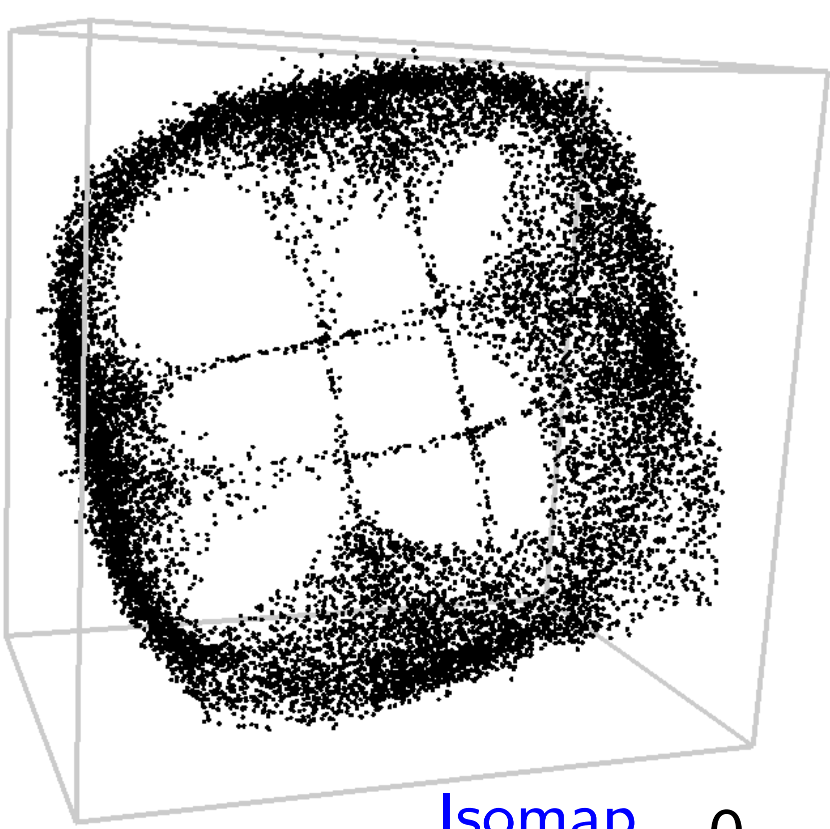
This is a lie!



PCA



MDS



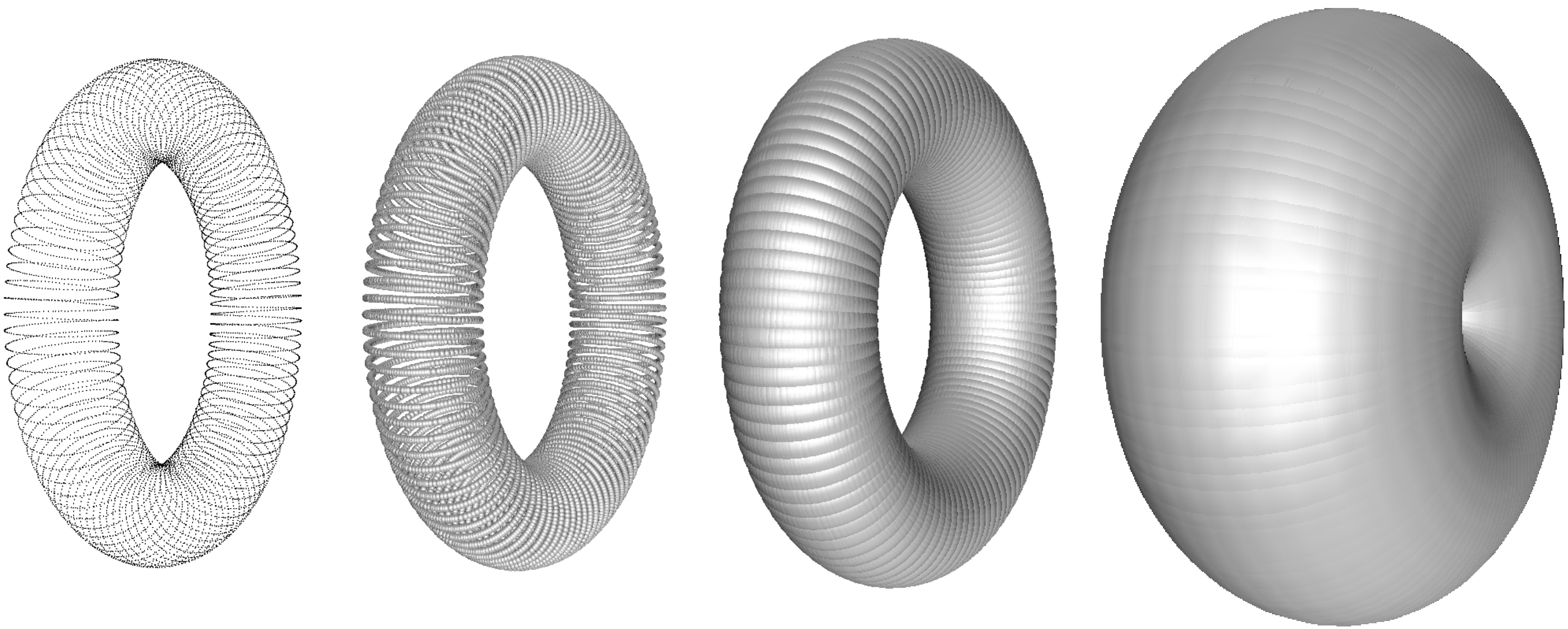
Isomap

Topology from Data

Input: point cloud $P \subset \mathbb{R}^d$

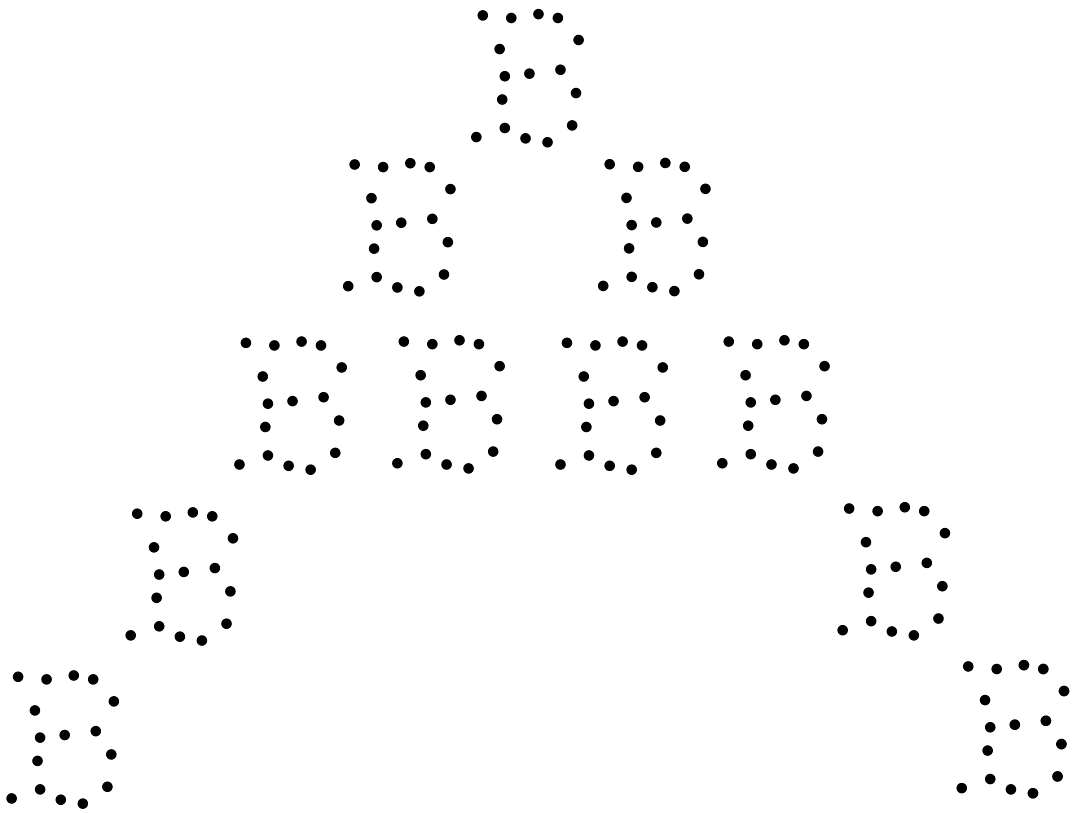
$\rightarrow$ uncover the topological structure of the space(s) underlying the data

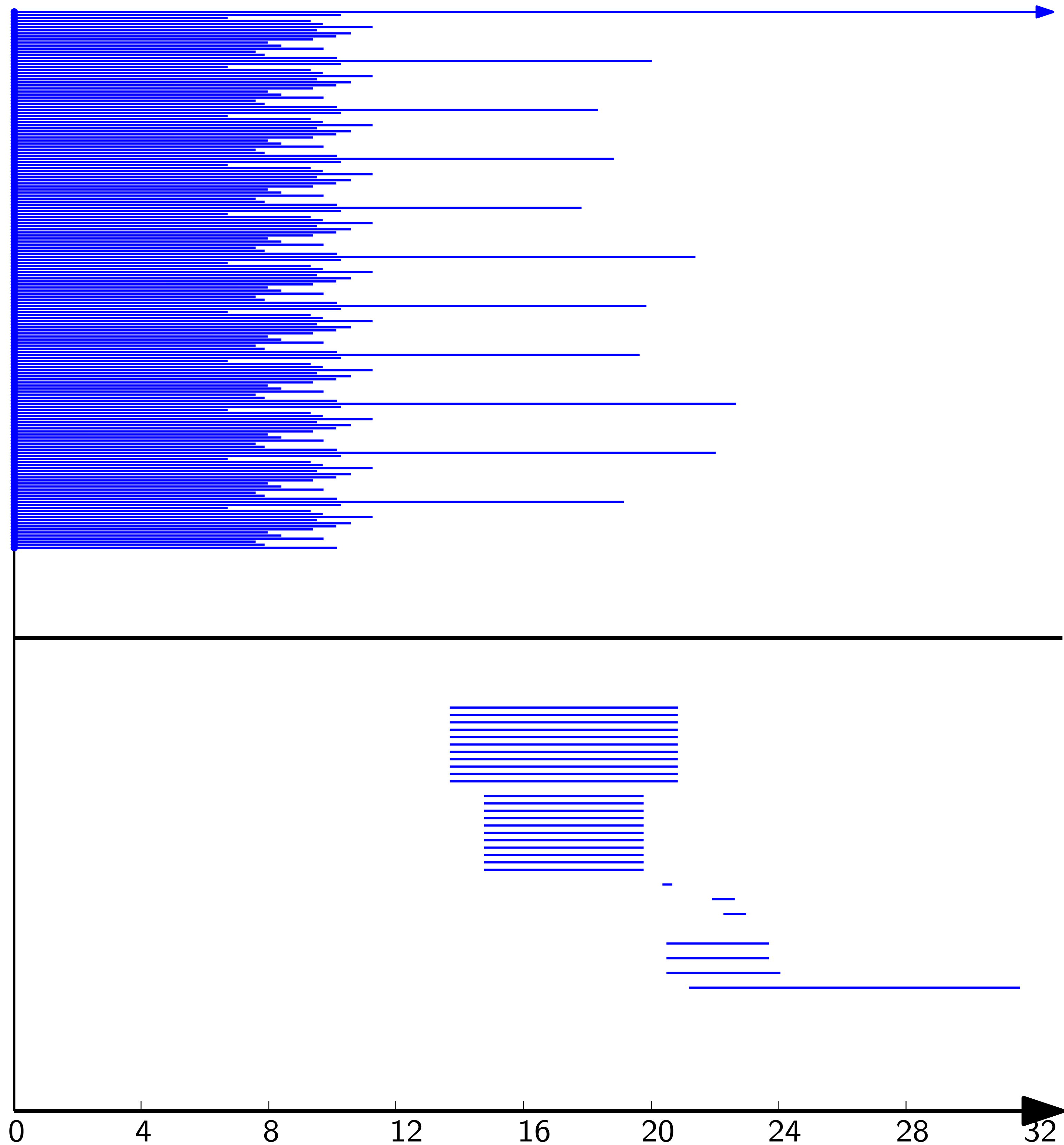
$\rightarrow$ inspect data at all scales and see what 'persists'



Approach: Compute persistence of distance function

$$\mathrm{d}_P : \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$





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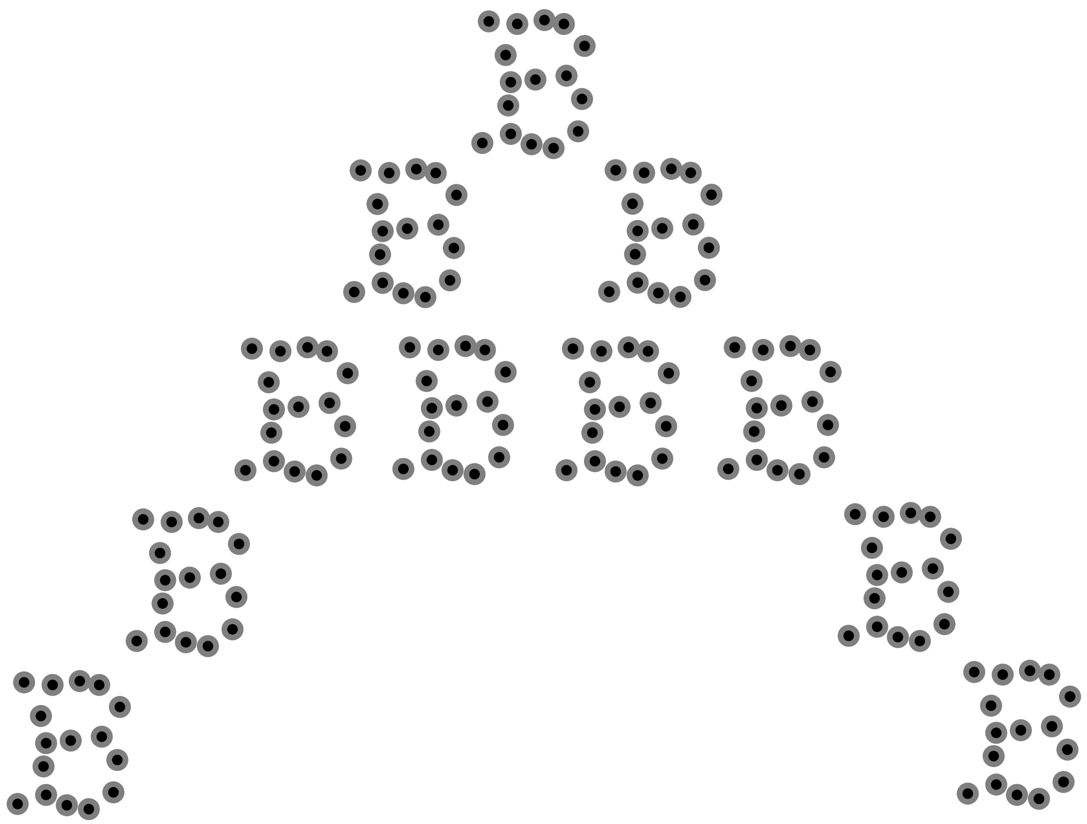

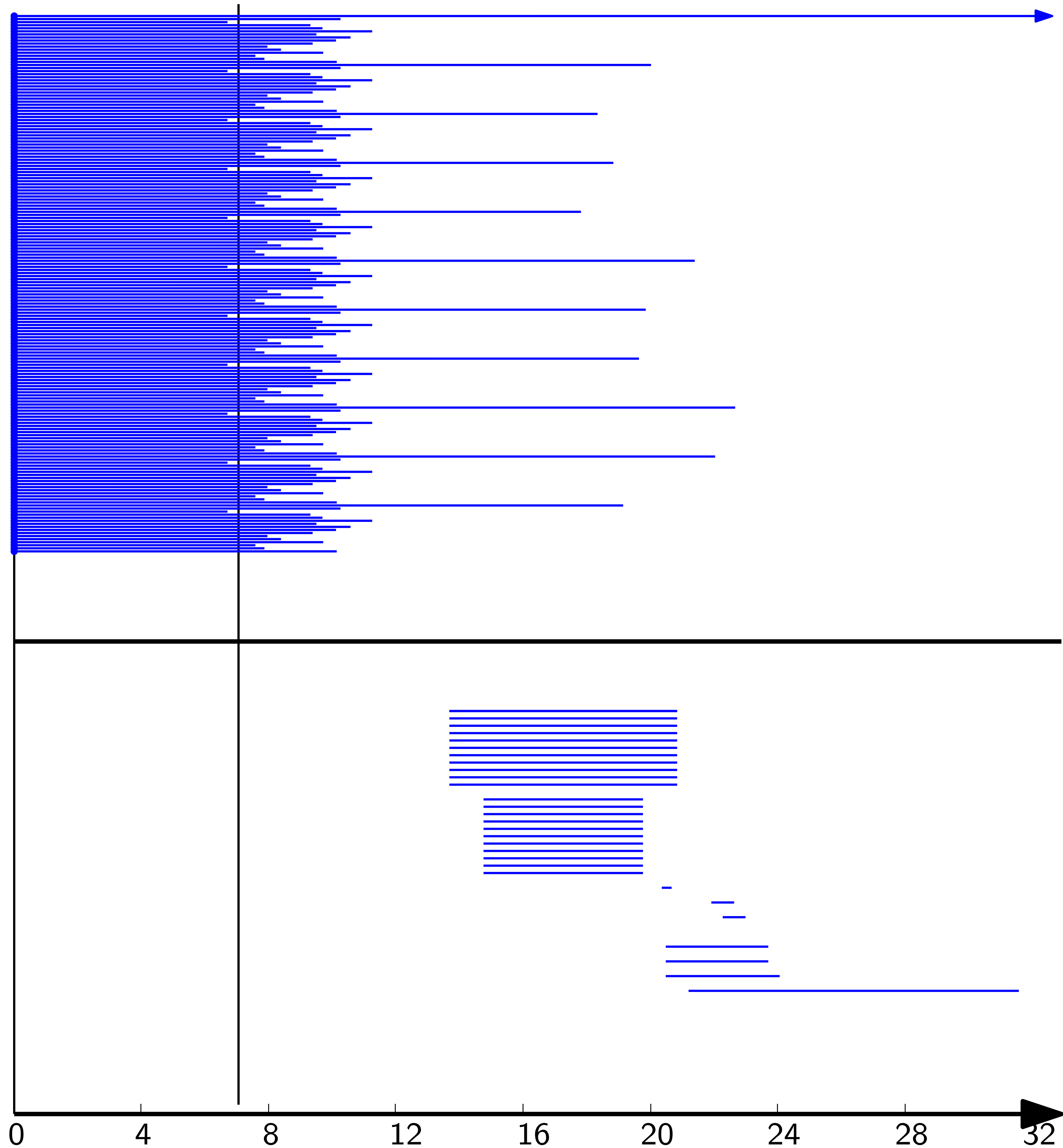

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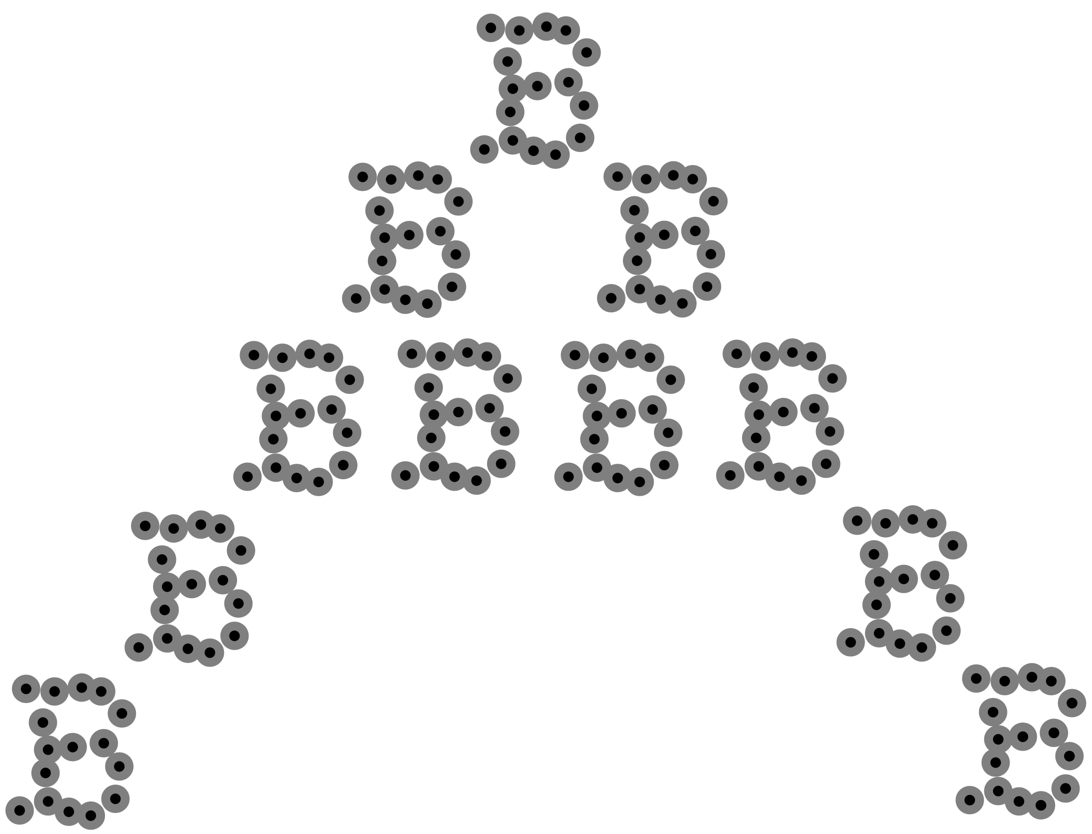

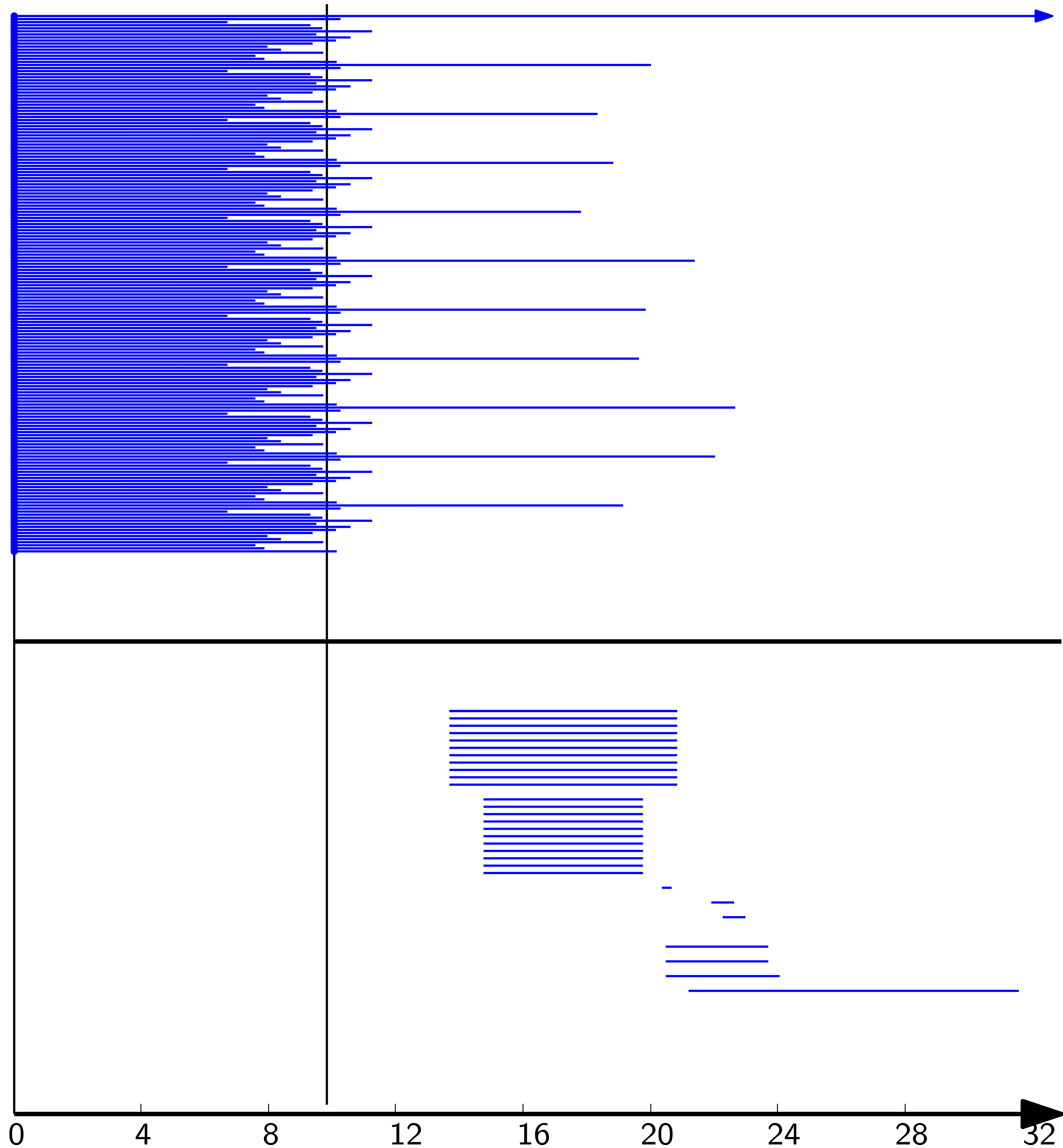

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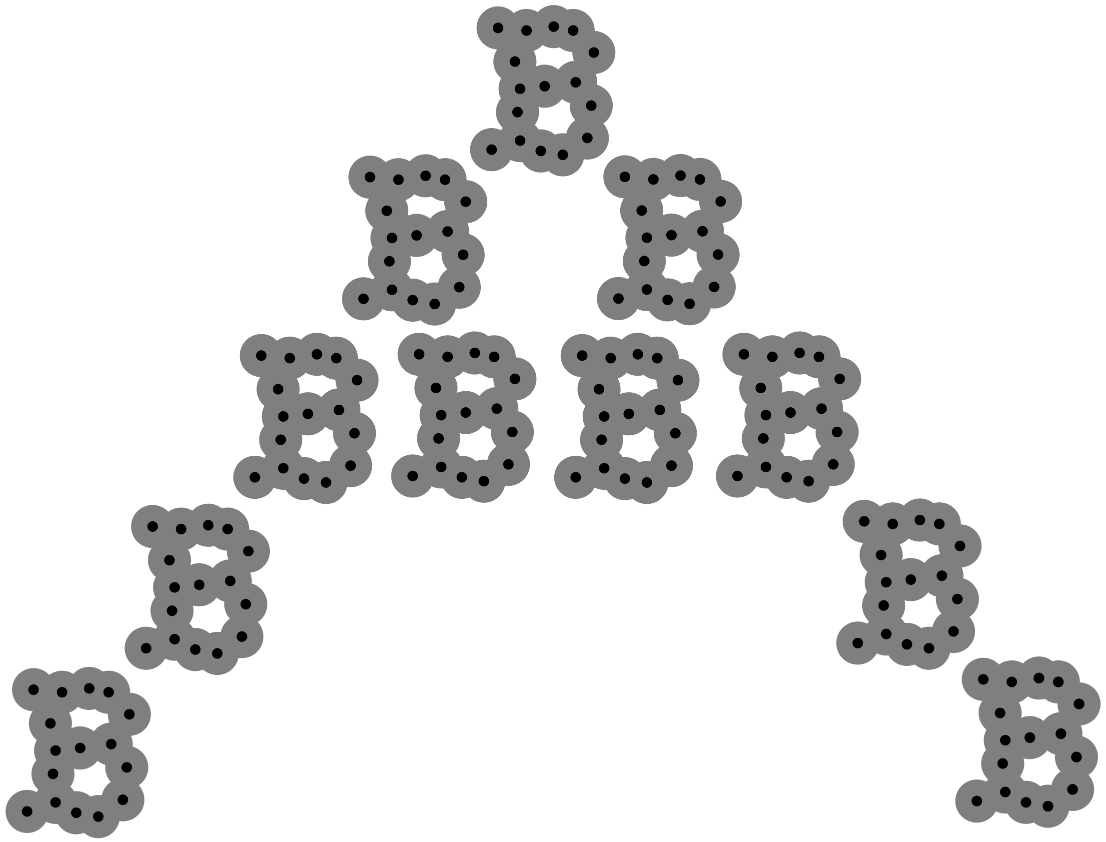

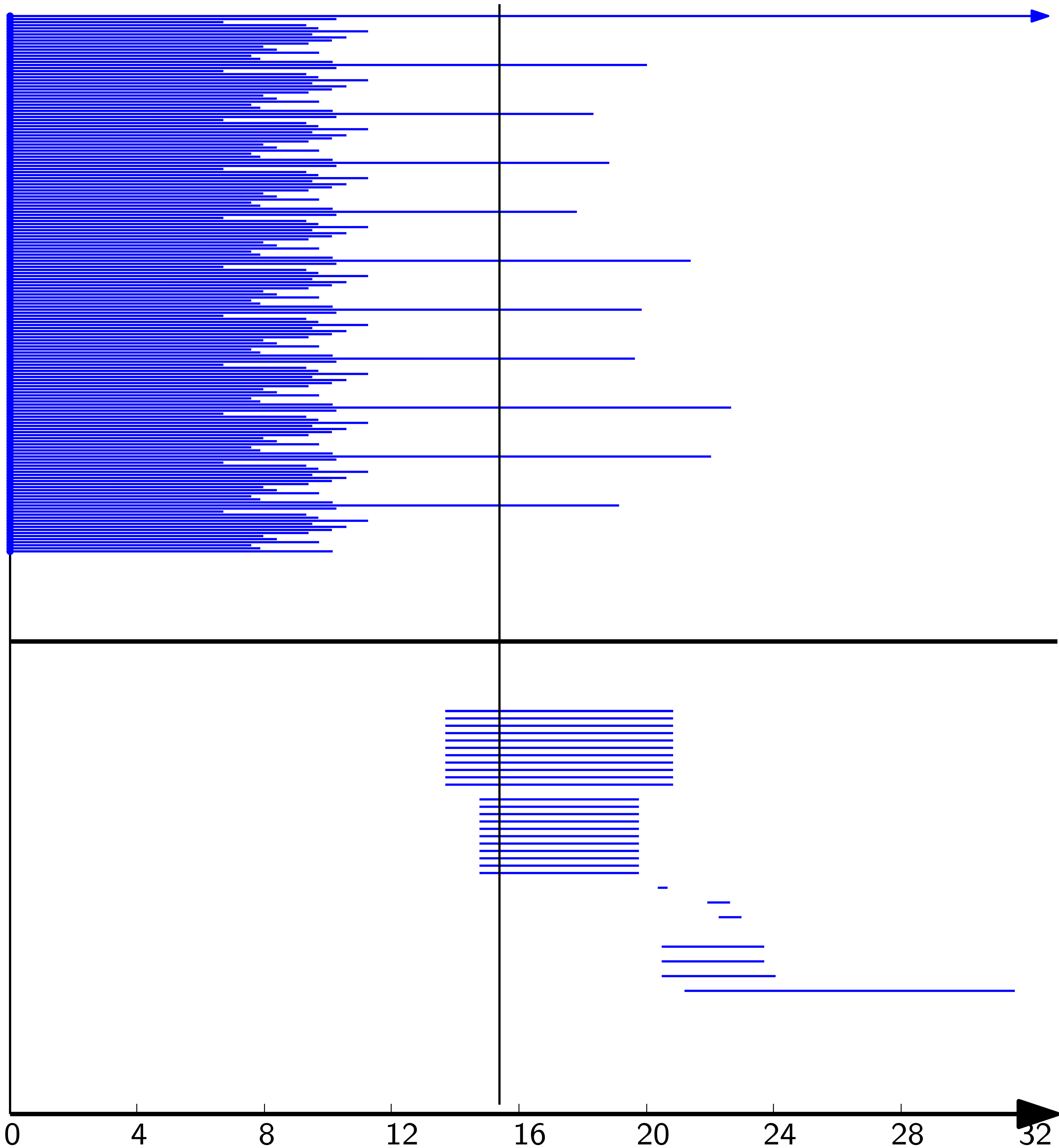

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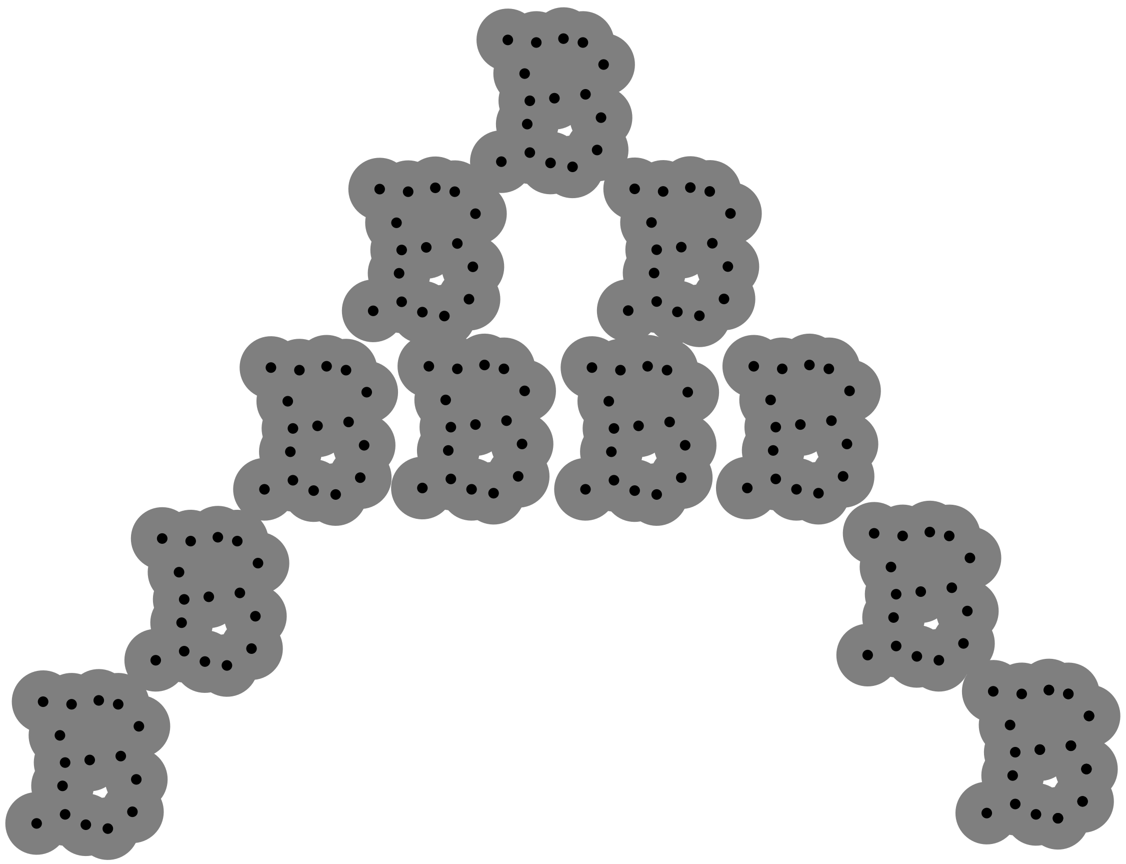
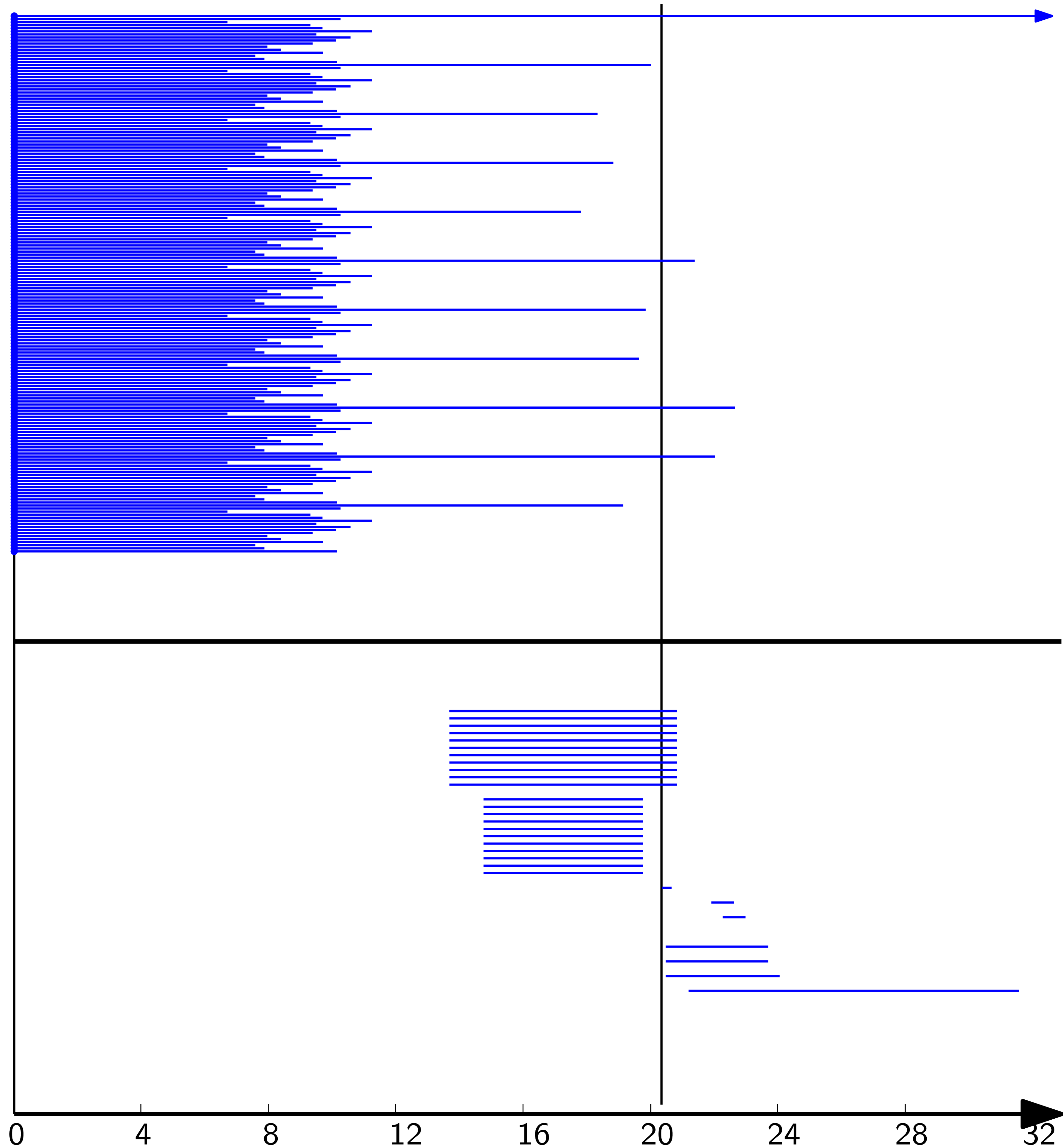
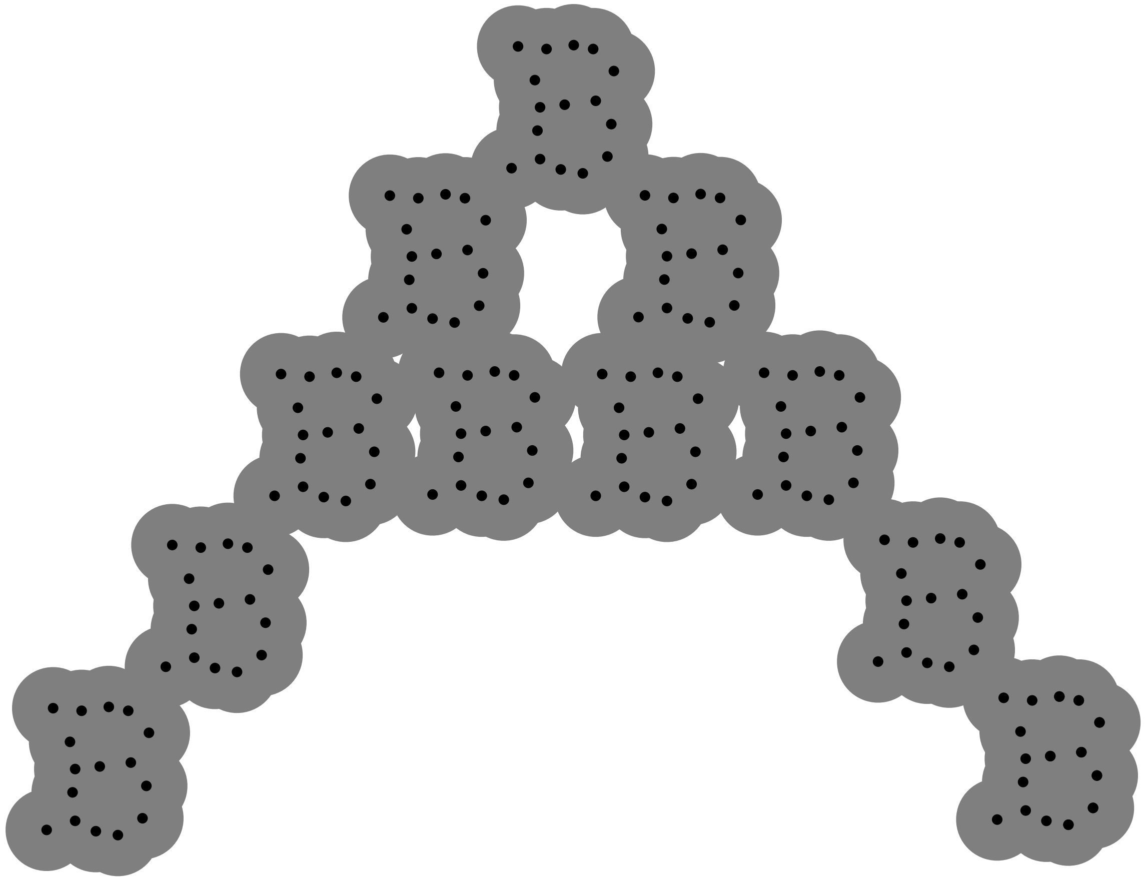
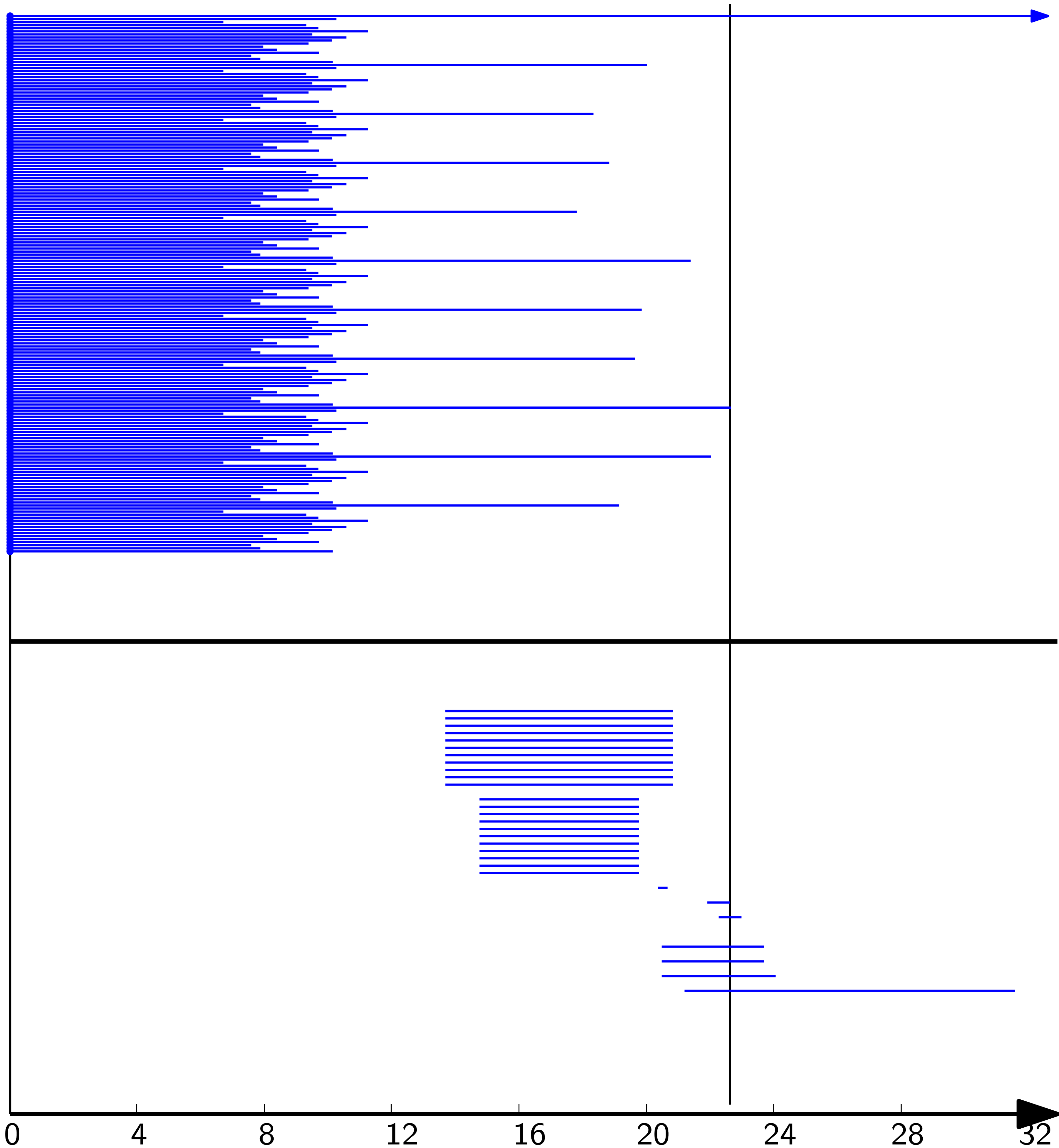




Approach: Compute persistence of distance function

$$\begin{aligned} \mathrm{d}_P : \quad & \mathbb{R}^2 \to \mathbb{R} \\ & x \mapsto \min_{p \in P} \|x - p\|_2 \end{aligned}$$





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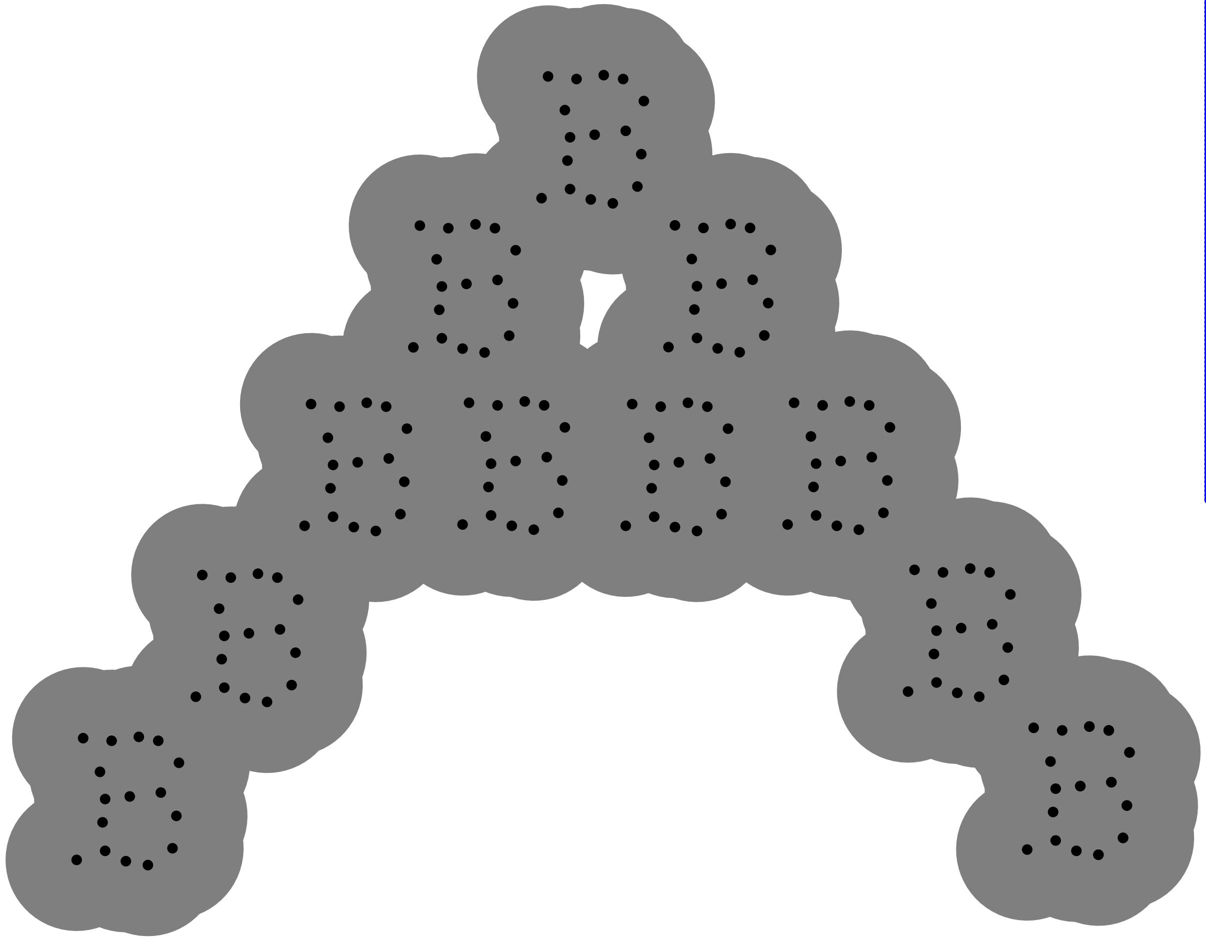

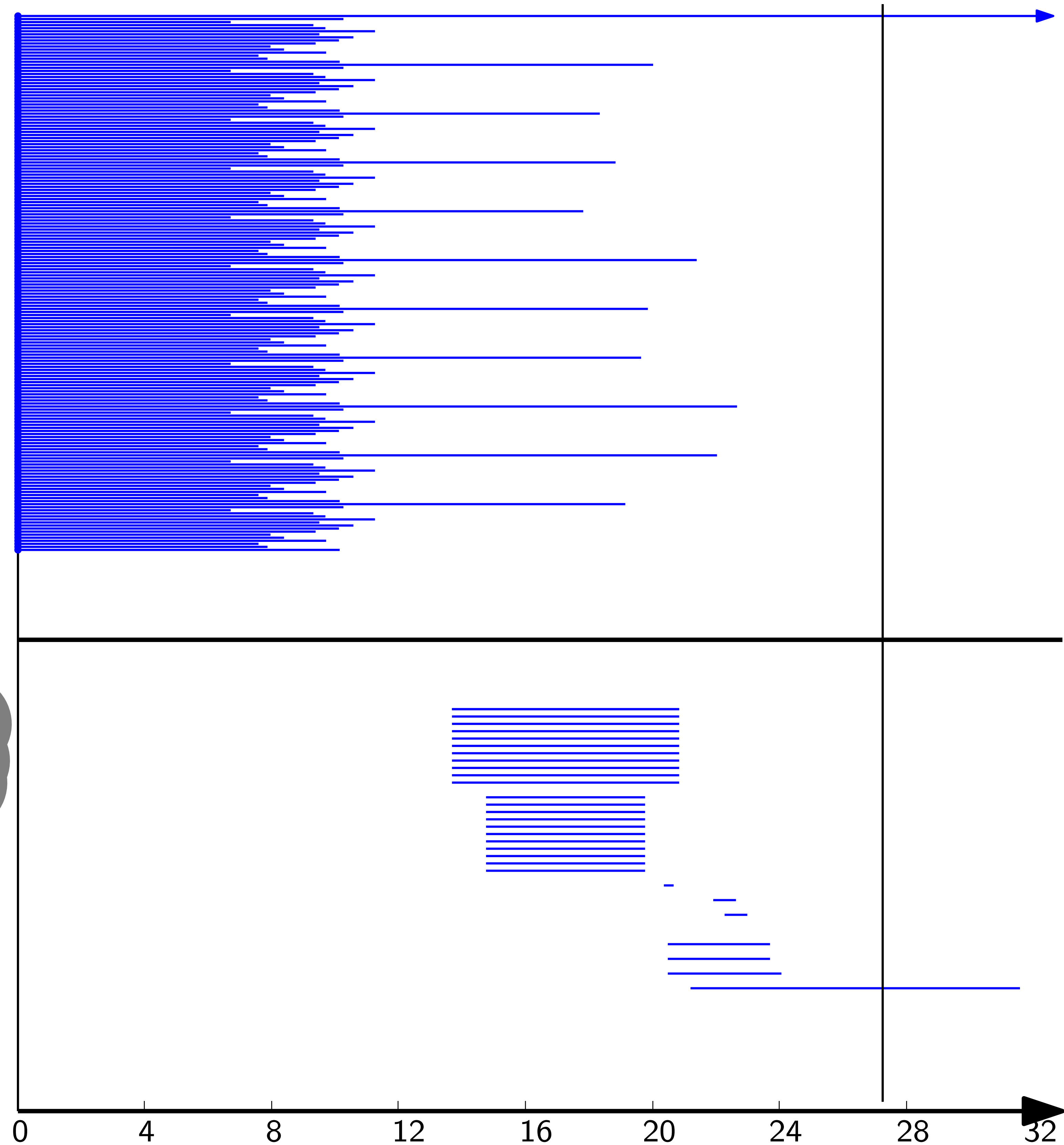

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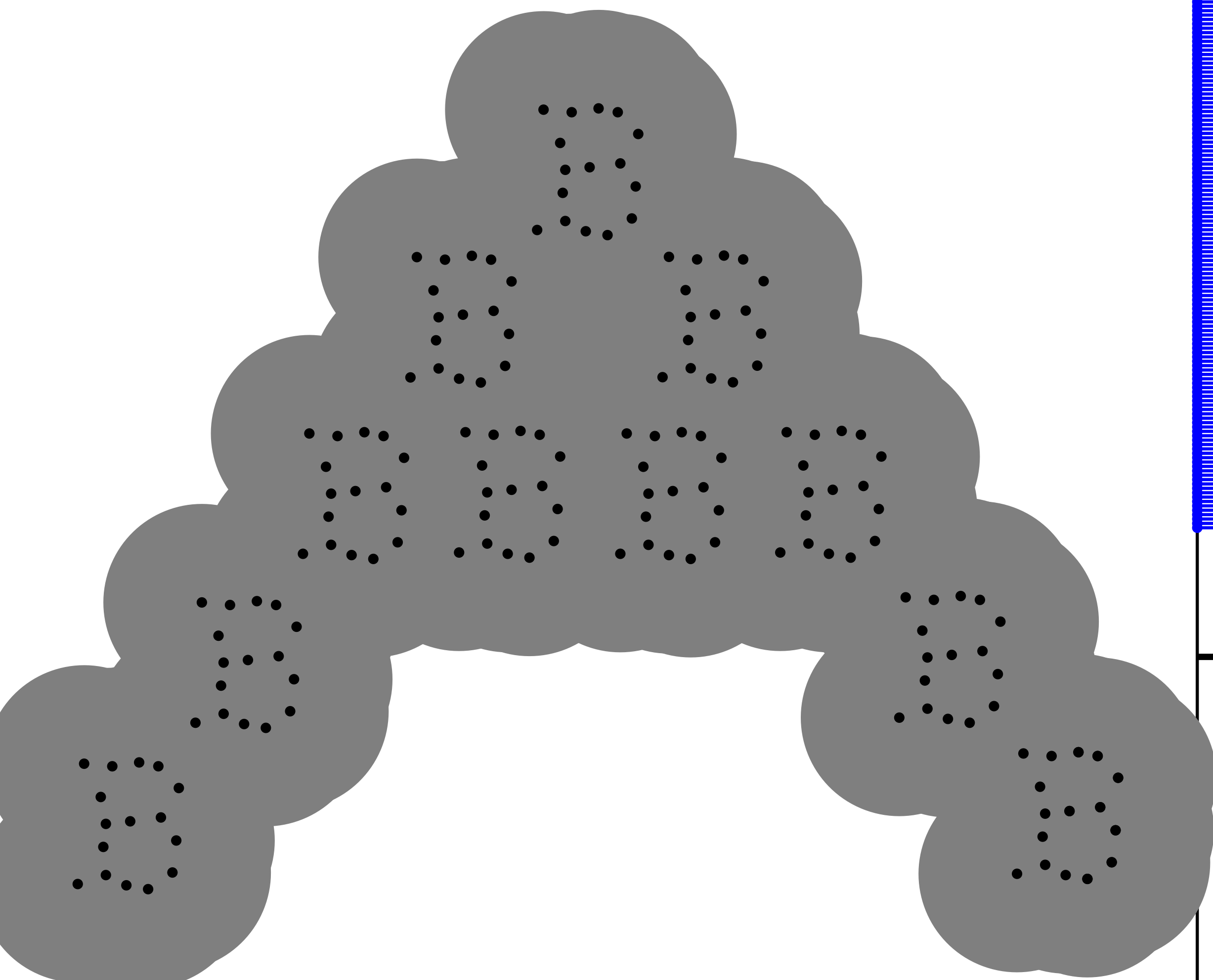

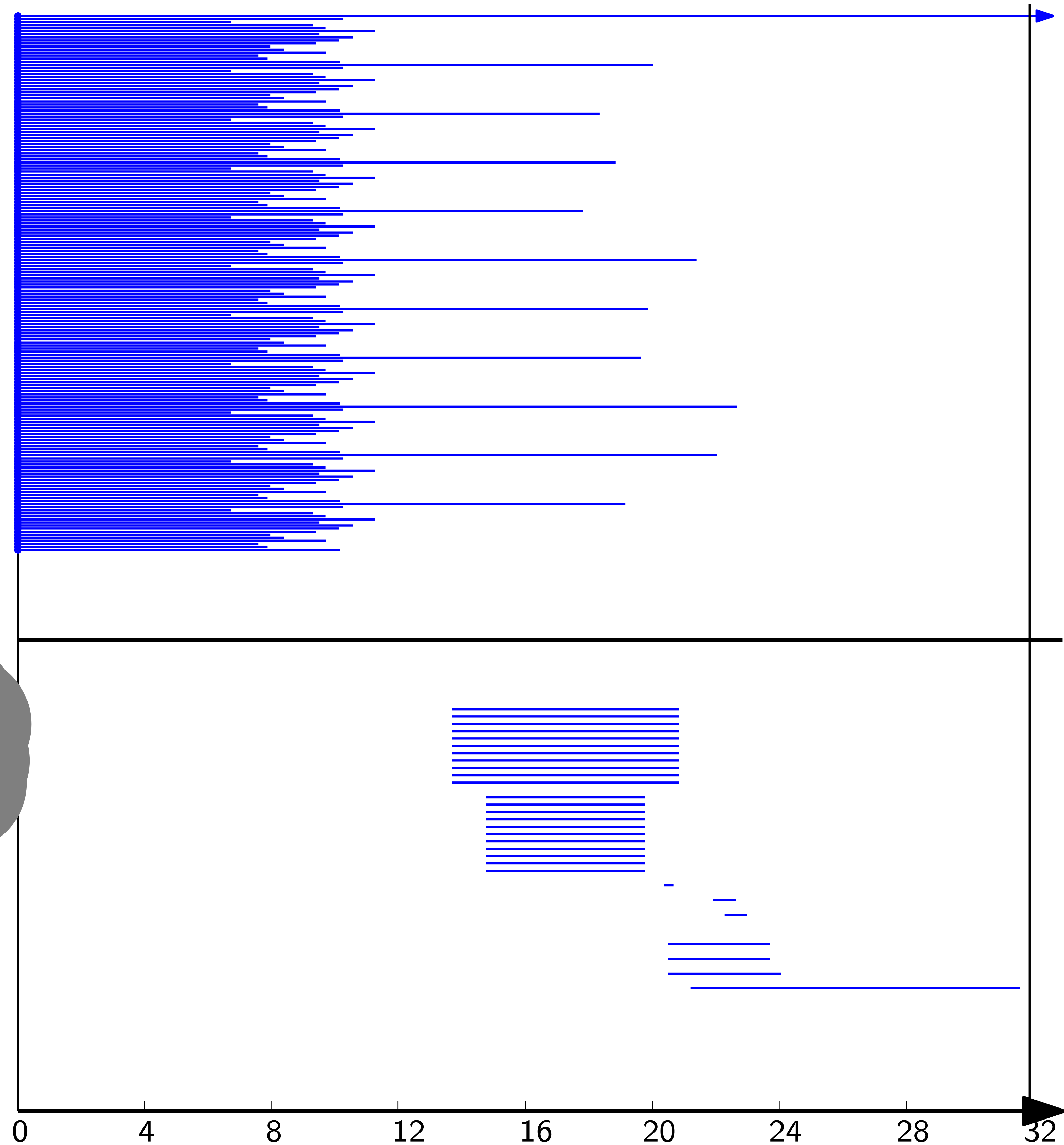

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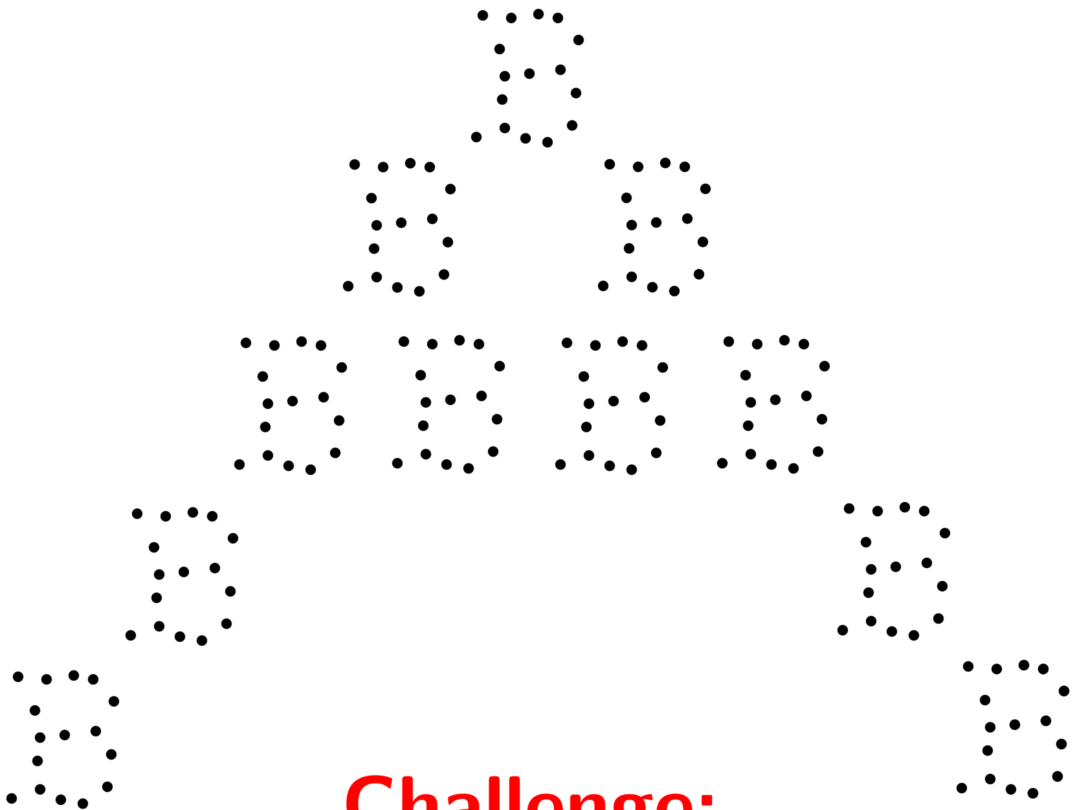
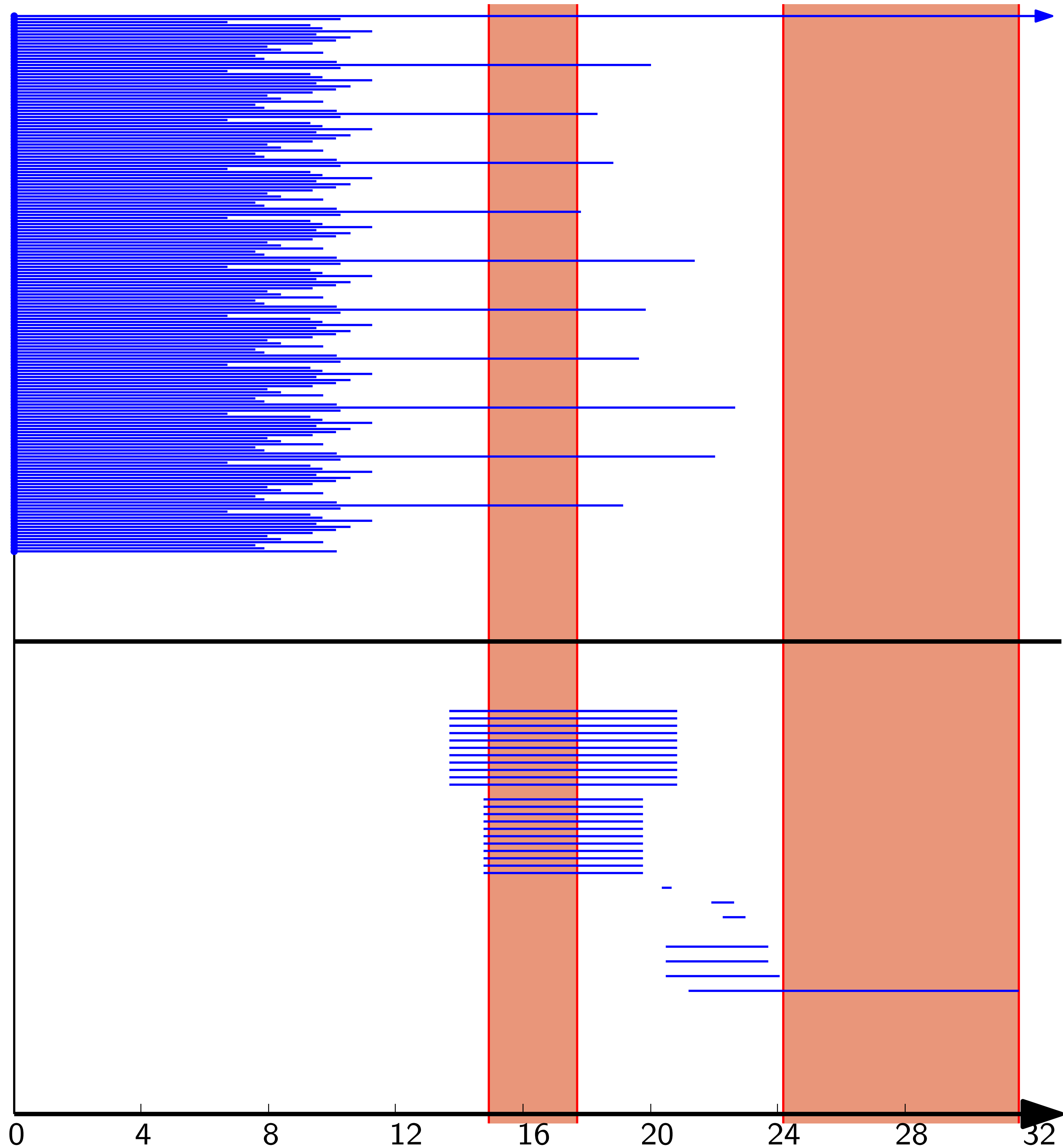
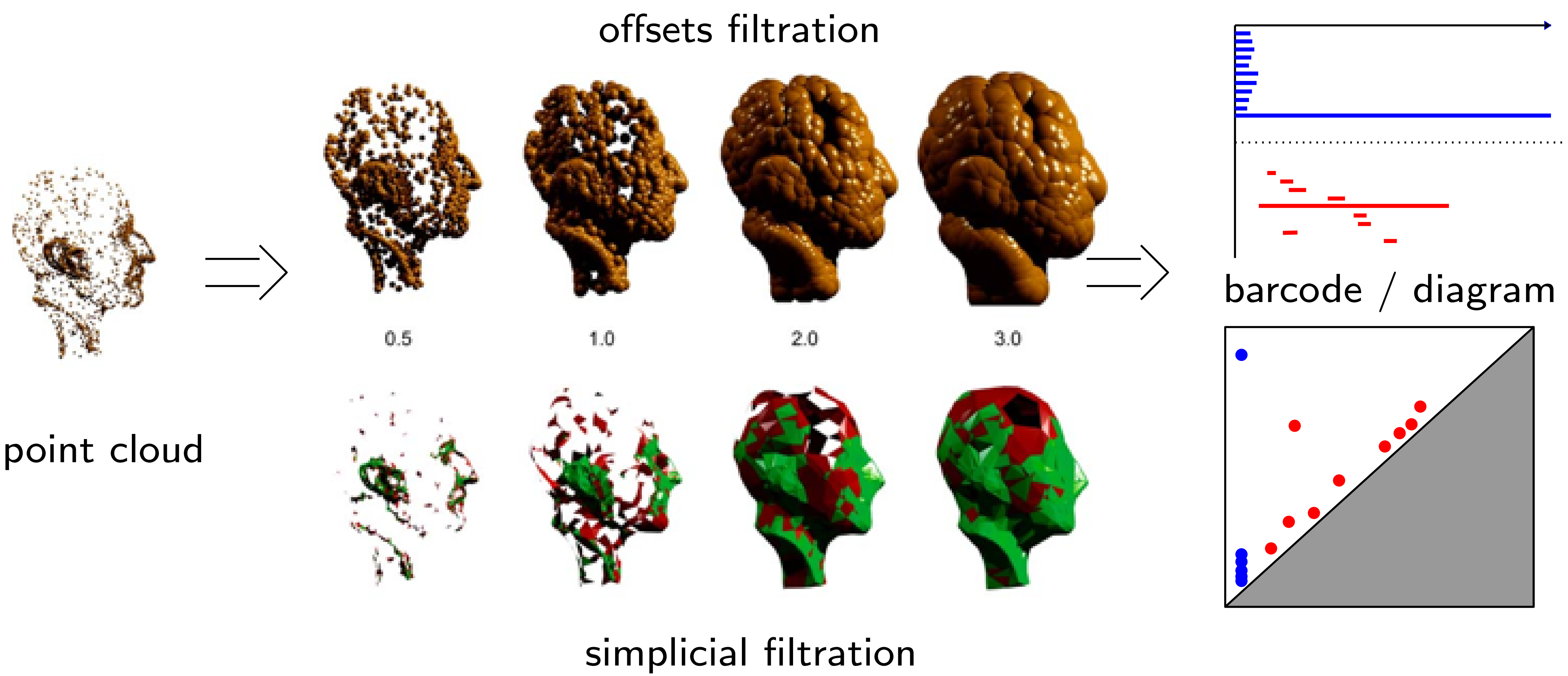




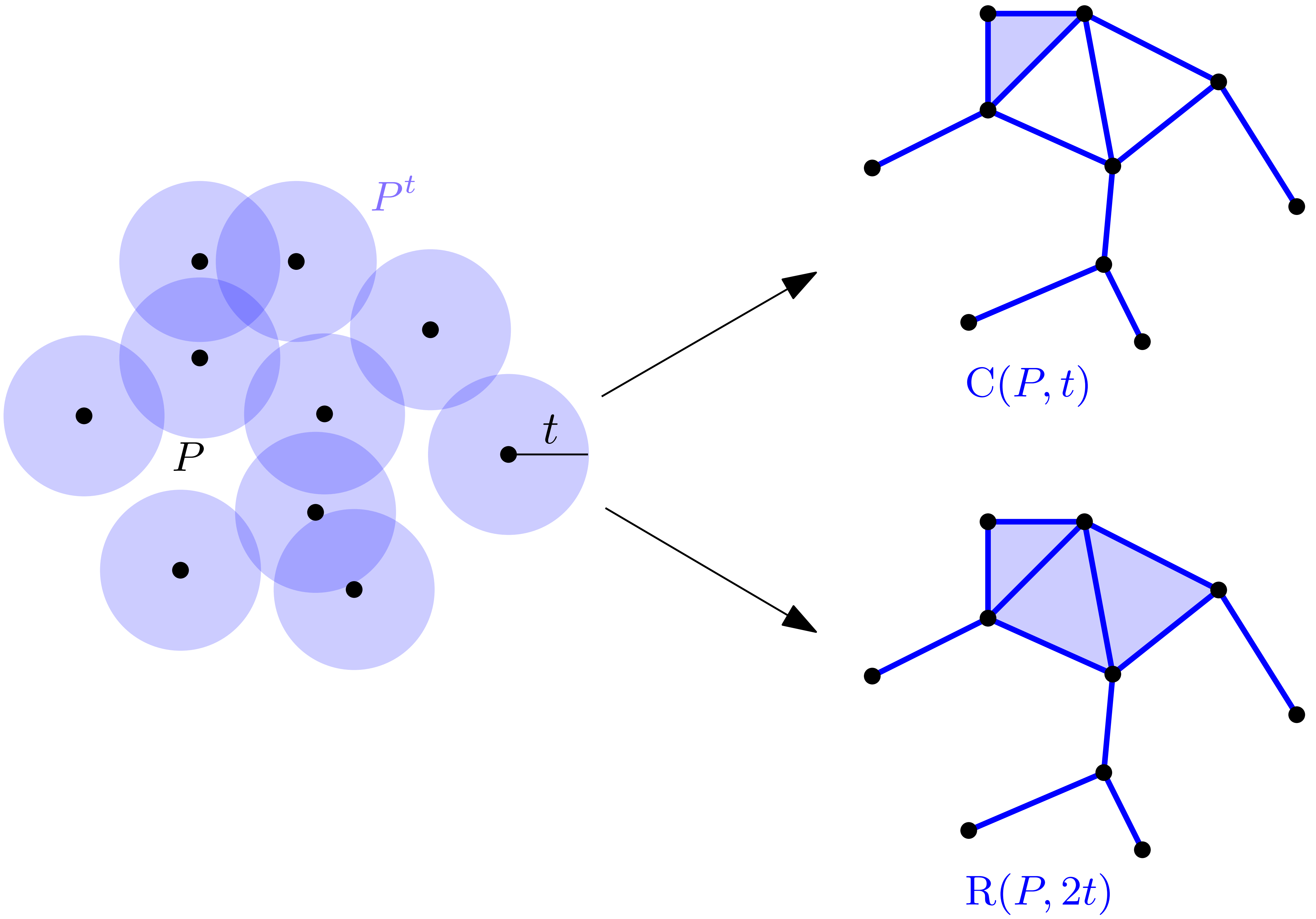
Challenge:
provide theoretical guarantees

(sufficient sampling conditions under which the barcode of d_P reveals the homology of the underlying space)

In practice: The inference pipeline



Čech and Rips filtrations

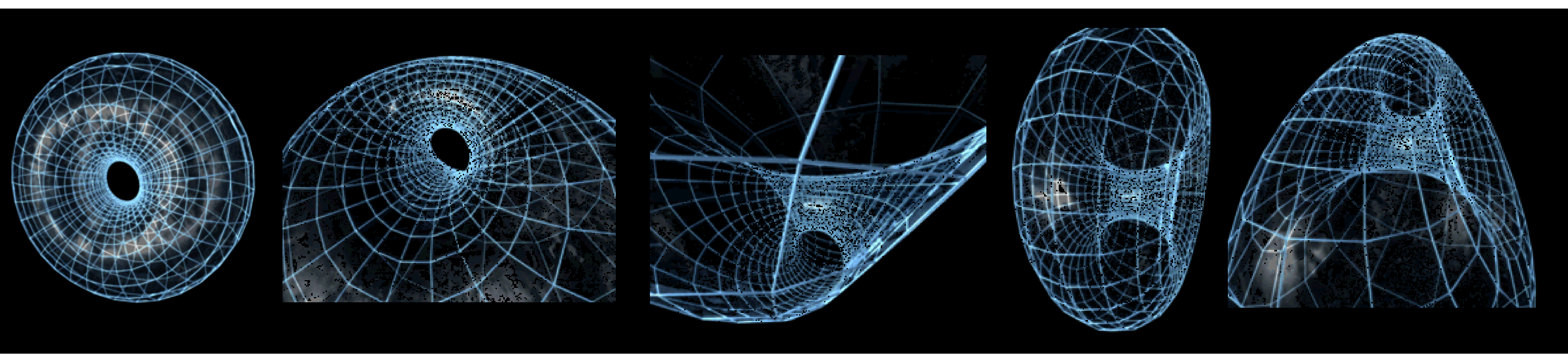


Example (manufactured data)

$(\mathbb{R} \mod \mathbb{Z})^2$

$$(u, v) \mapsto \frac{1}{\sqrt{2}}(\cos(2\pi u), \sin(2\pi u), \cos(2\pi v), \sin(2\pi v))$$

$\subset \mathbb{S}3 \subset \mathbb{R}^4$



source: `http://http://en.wikipedia.org/wiki/Clifford_torus`

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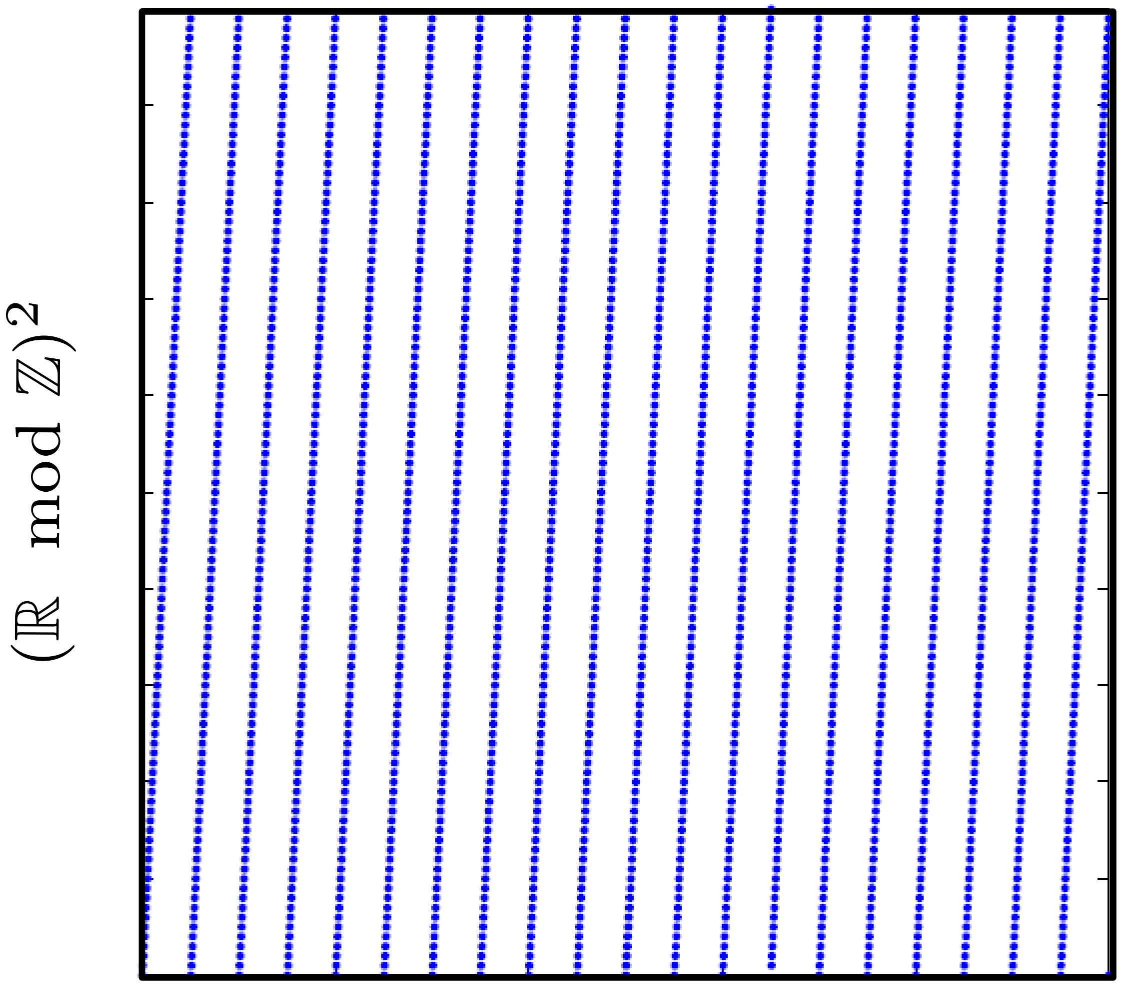
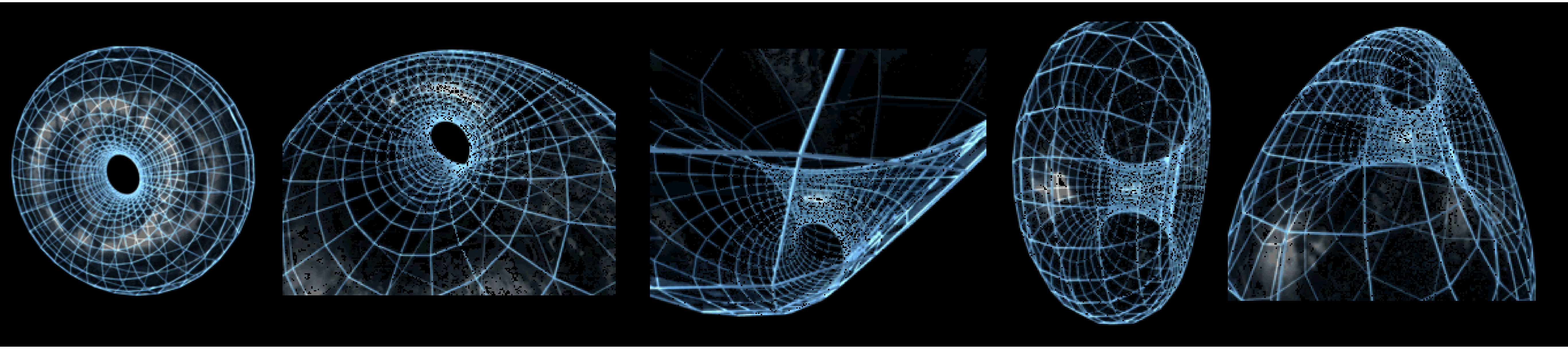


$$(u, v) \mapsto \frac{1}{\sqrt{2}}(\cos(2\pi u), \sin(2\pi u), \cos(2\pi v), \sin(2\pi v))$$

$n = 2000$ data points
ambient dimension $d = 4$
intrinsic dimension $k = 1, 2, 3$

$\subset \mathbb{S}3 \subset \mathbb{R}^4$



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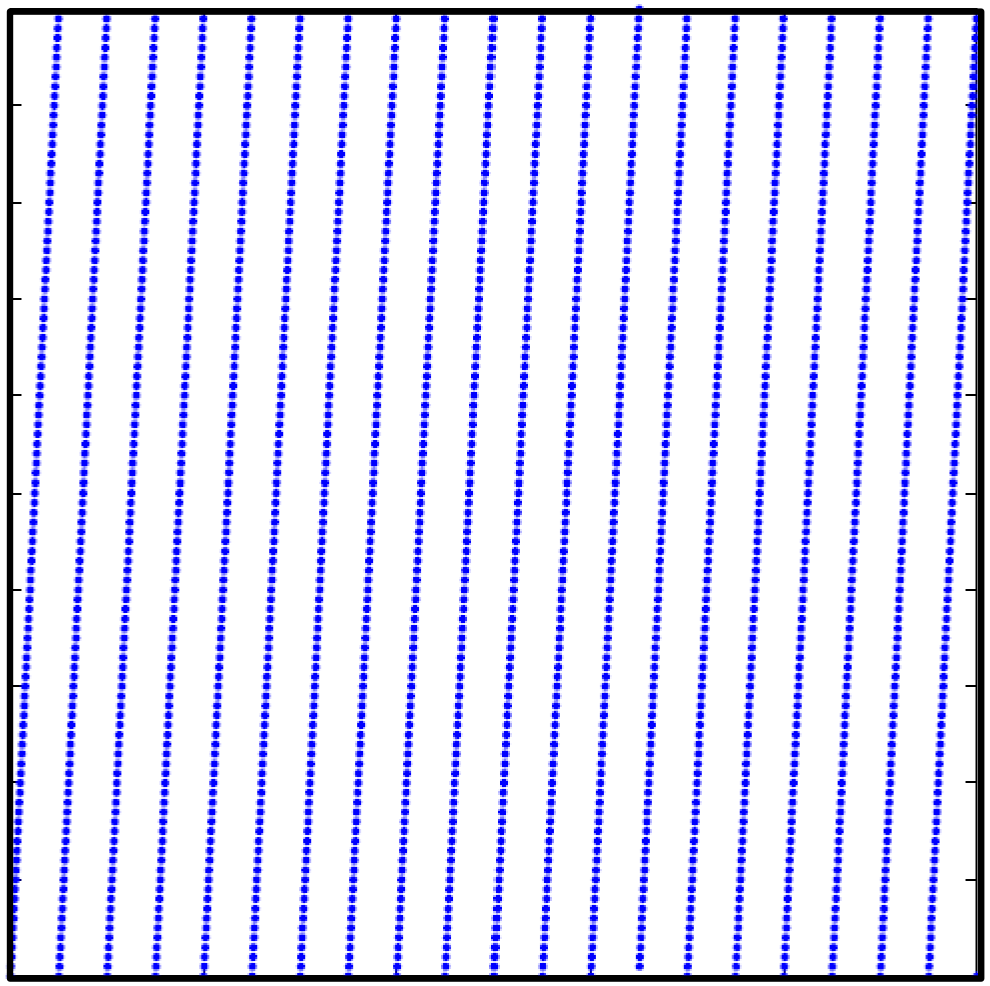

$n = 2000$ data points

ambient dimension $d = 4$

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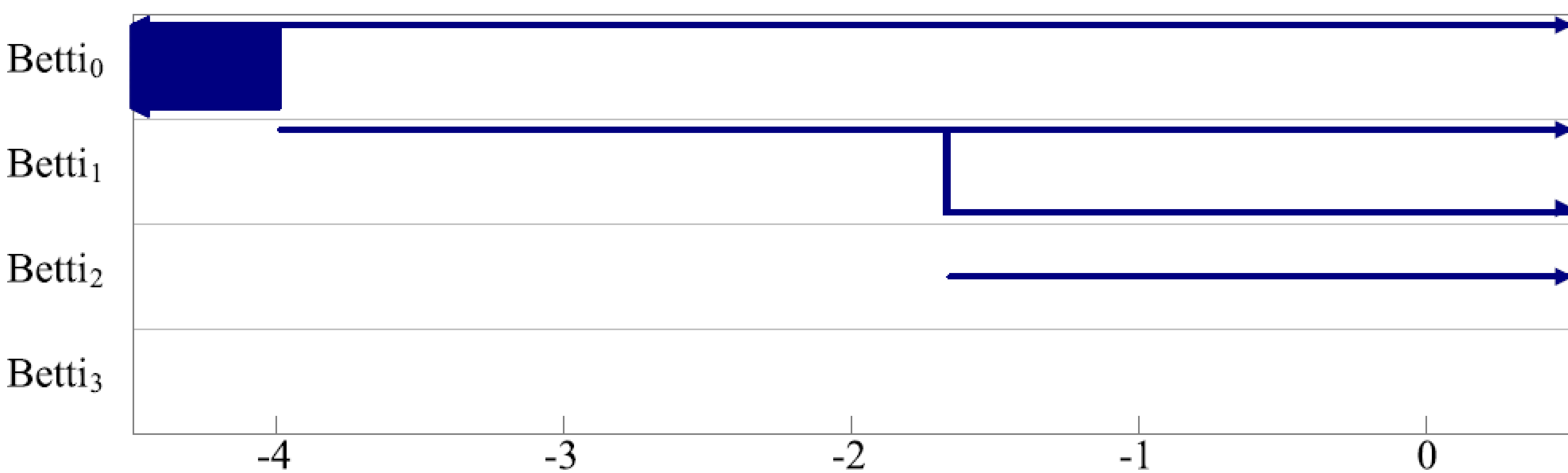
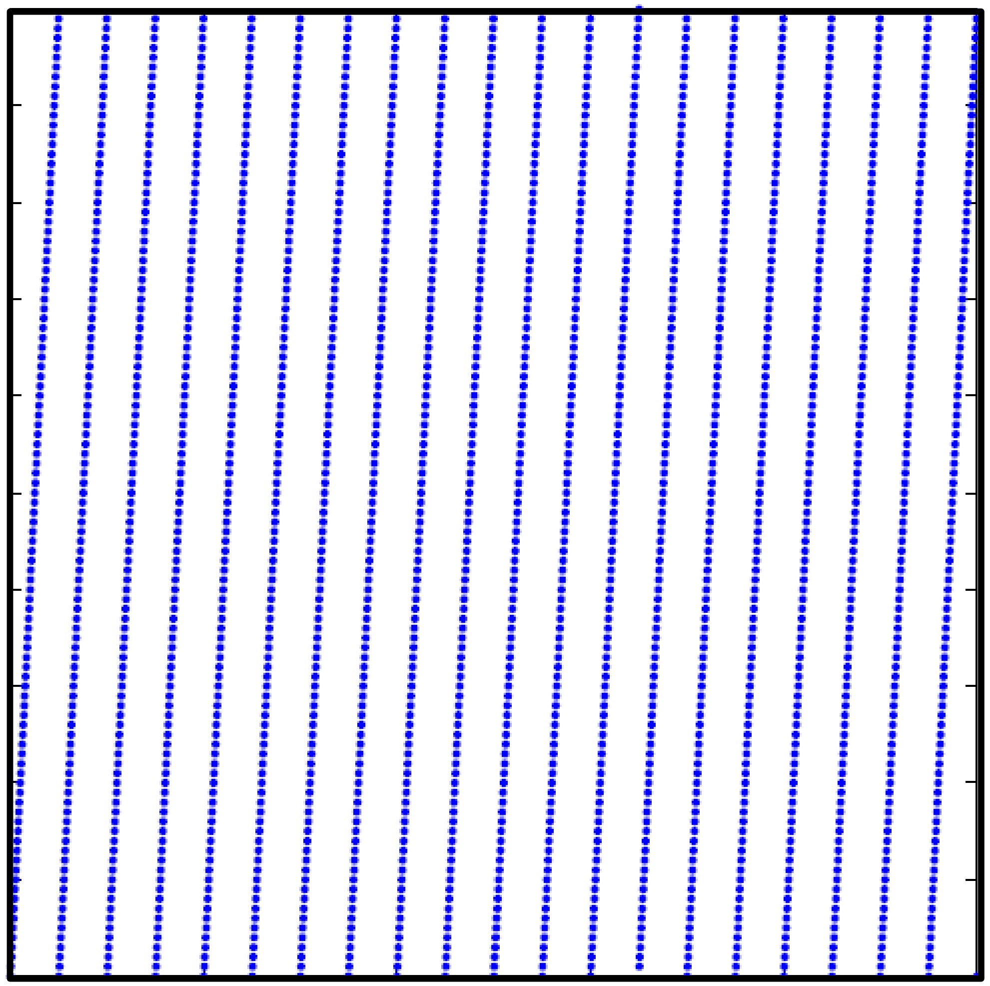


Vietoris-Rips filtration

size $\sim 2^n \mid n^{d+1}$

Example (manufactured data)



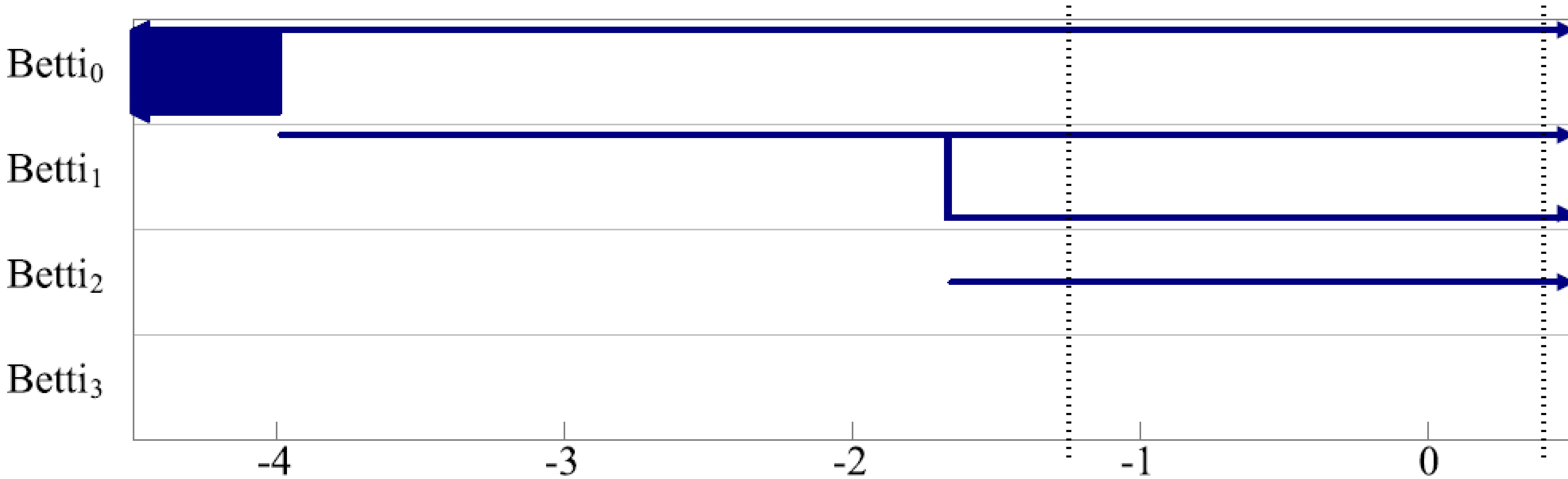
$n = 2000$ data points

ambient dimension $d = 4$

intrinsic dimension $k = 1, 2, 3$

computation limit $(500 \cdot 10^6$ simplices)

3-sphere $(37 \cdot 10^9$ simplices)



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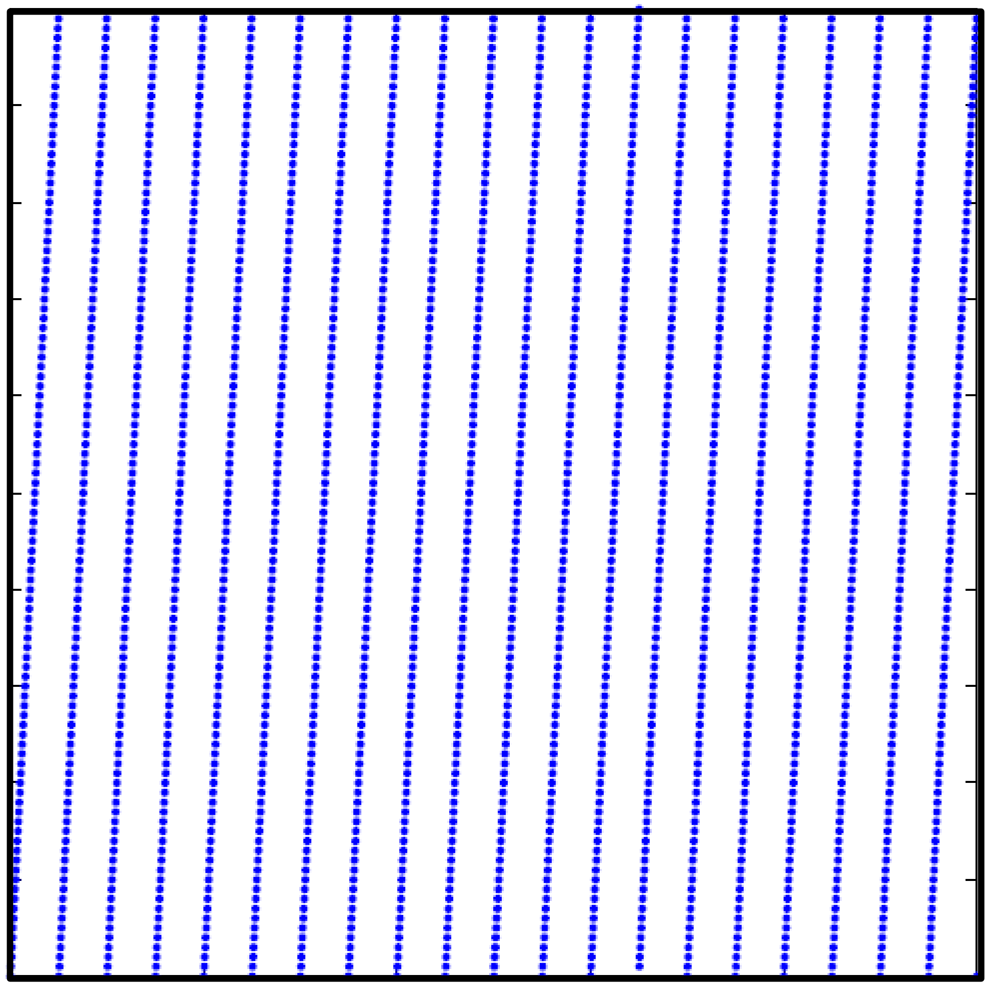
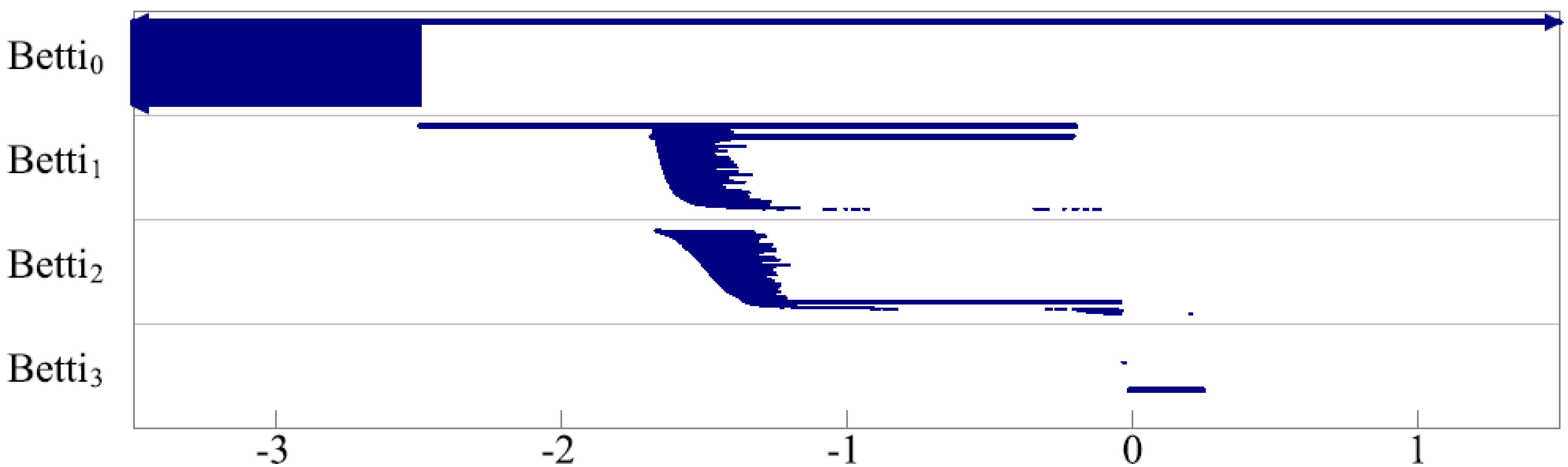


$n = 2000$ data points
ambient dimension $d = 4$
intrinsic dimension $k = 1, 2, 3$

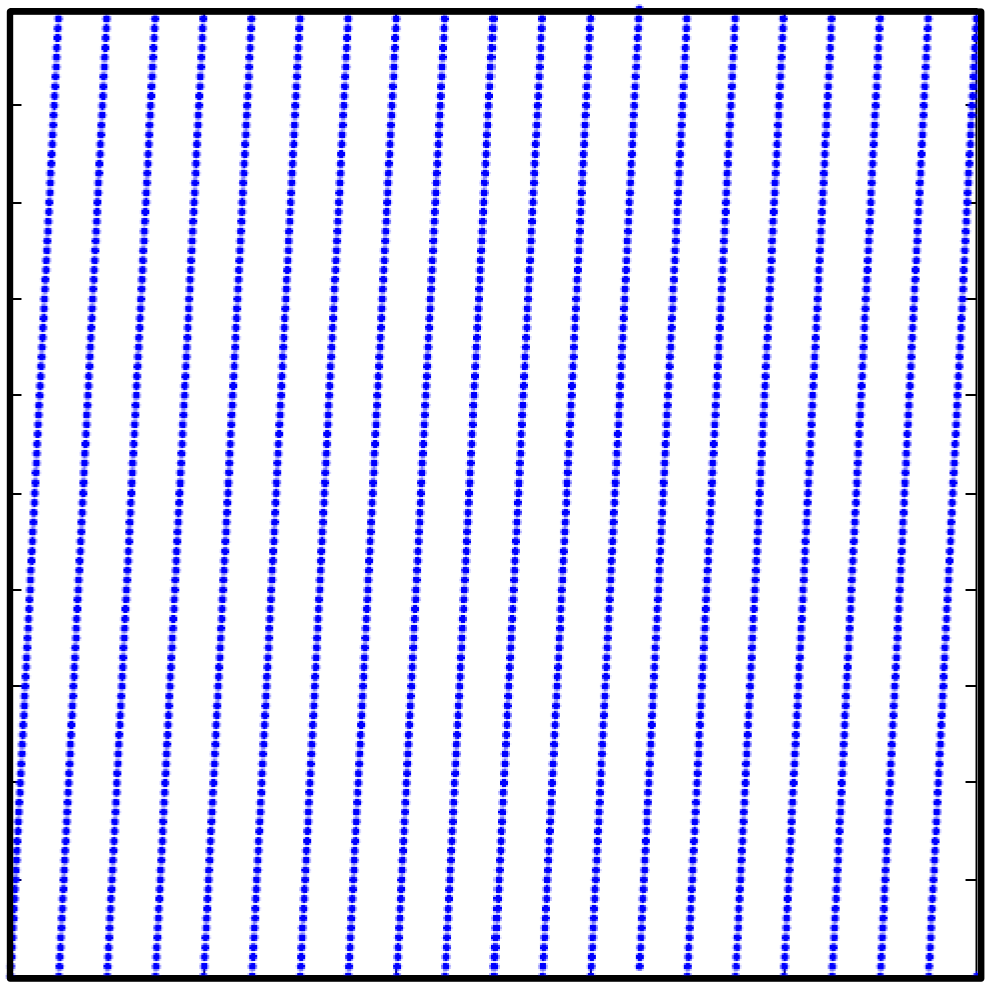
$(12 \cdot 10^6$ simplices)



mesh-based filtration

size $\sim 2^{d^2} n$

Example (manufactured data)

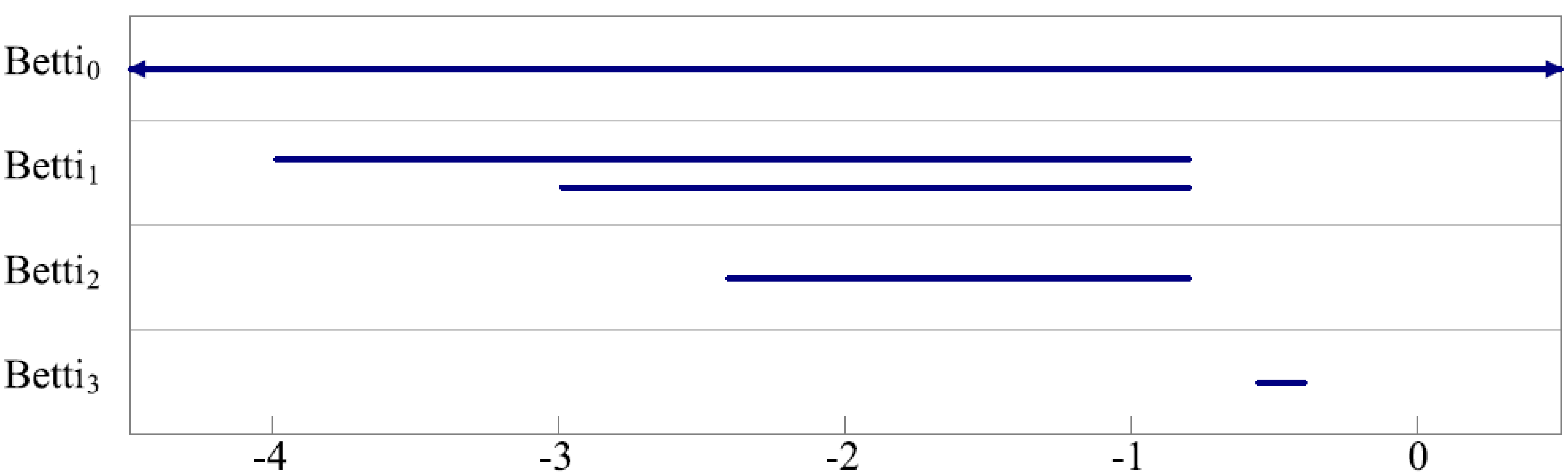


$n = 2000$ data points

ambient dimension $d = 4$

intrinsic dimension $k = 1, 2, 3$

($200 \cdot 10^3$ simplices)



sparse Rips filtration

size $\sim 2^{k^2} n$

Natural Images Data

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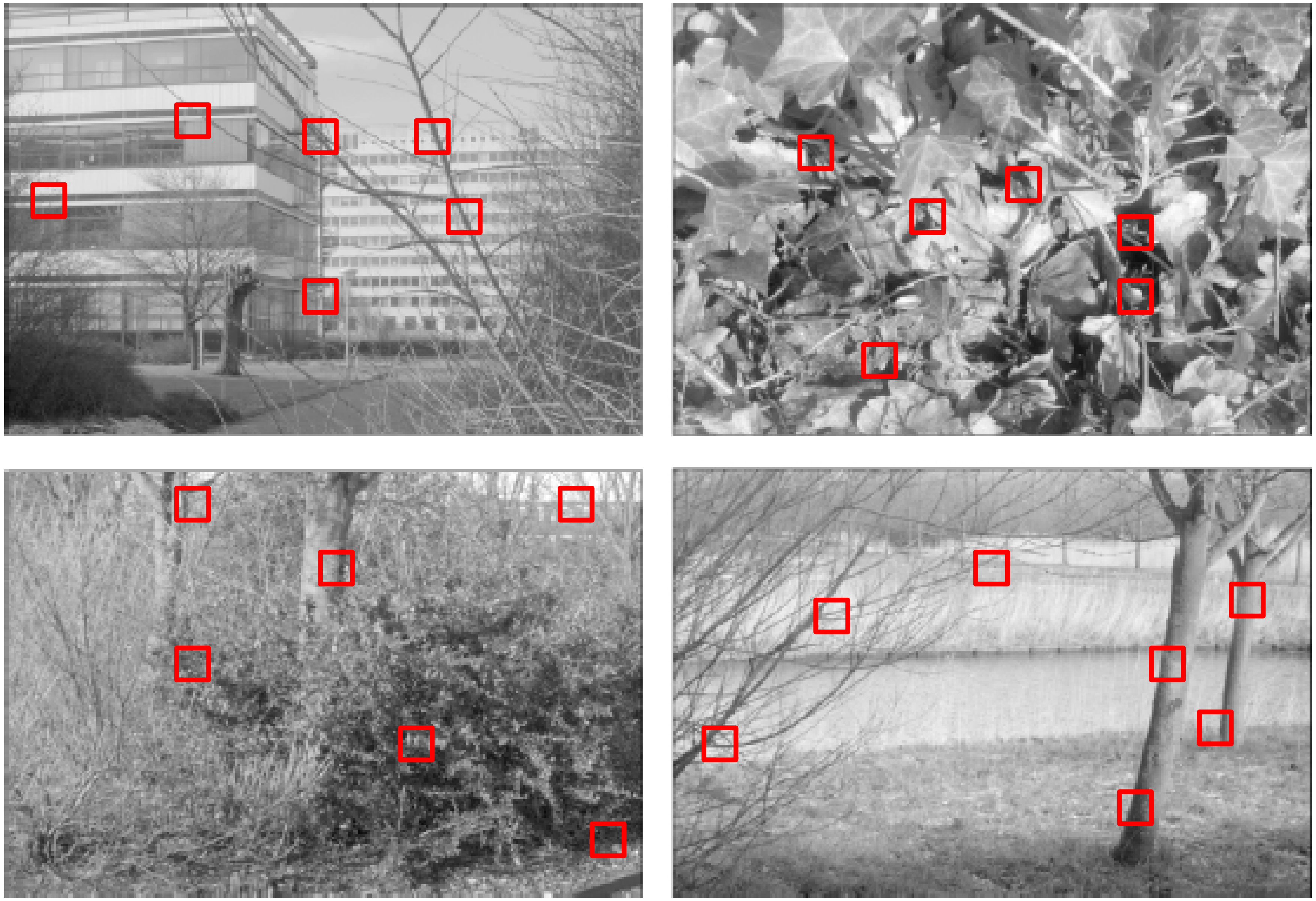
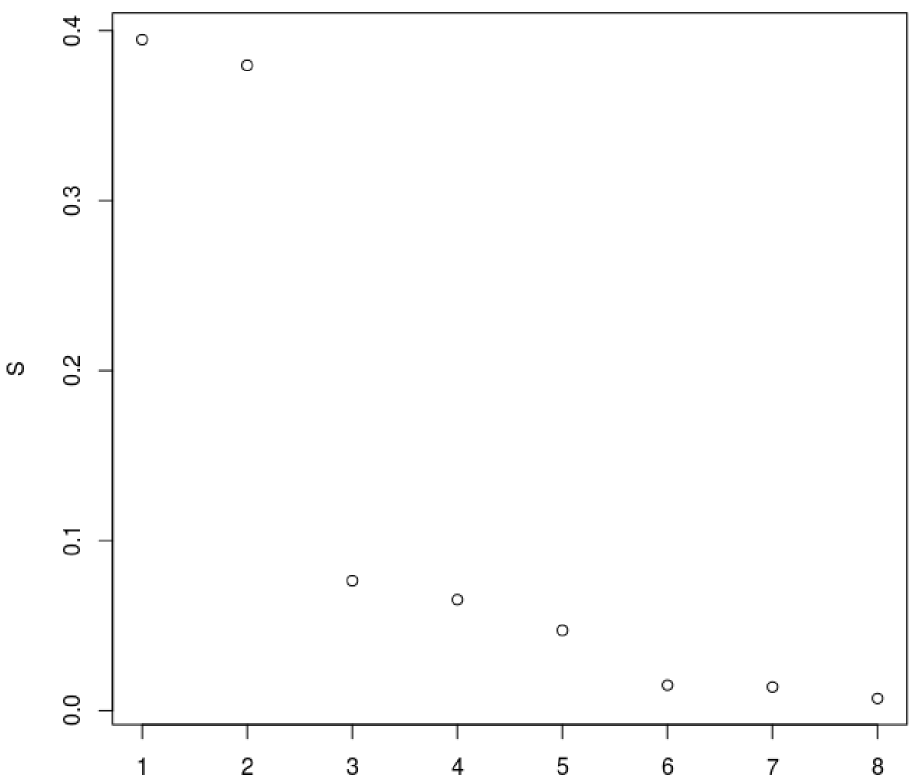
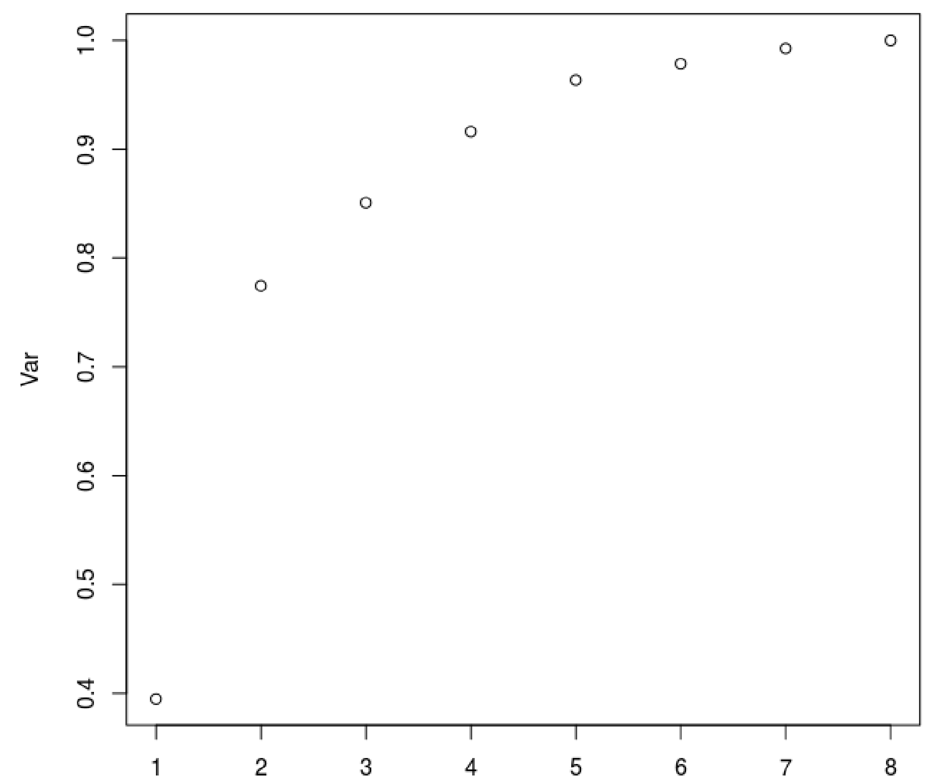
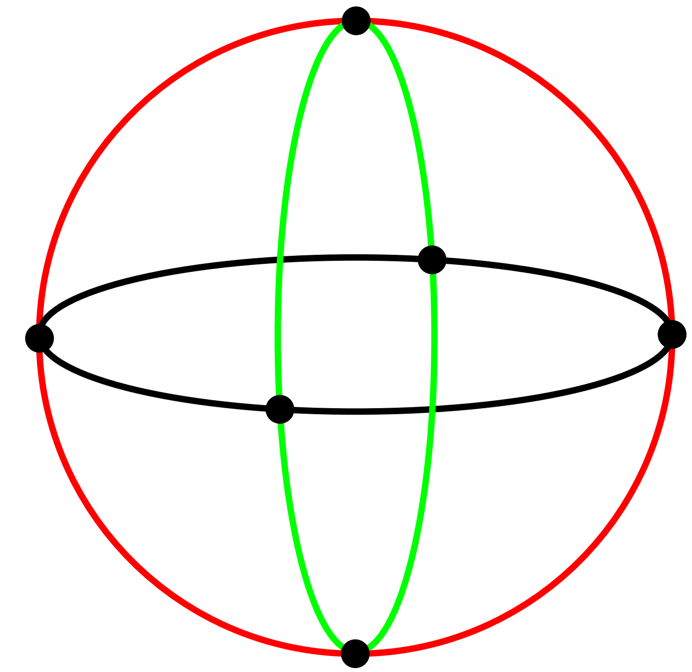


(source: [Lee, Pederson, Mumford 03])

Natural Images Data







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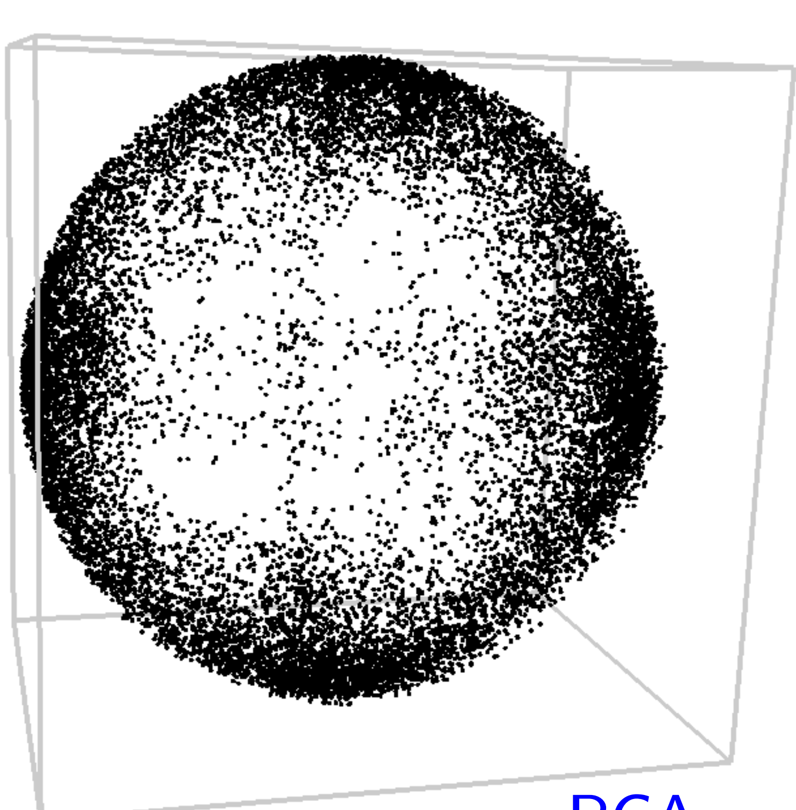
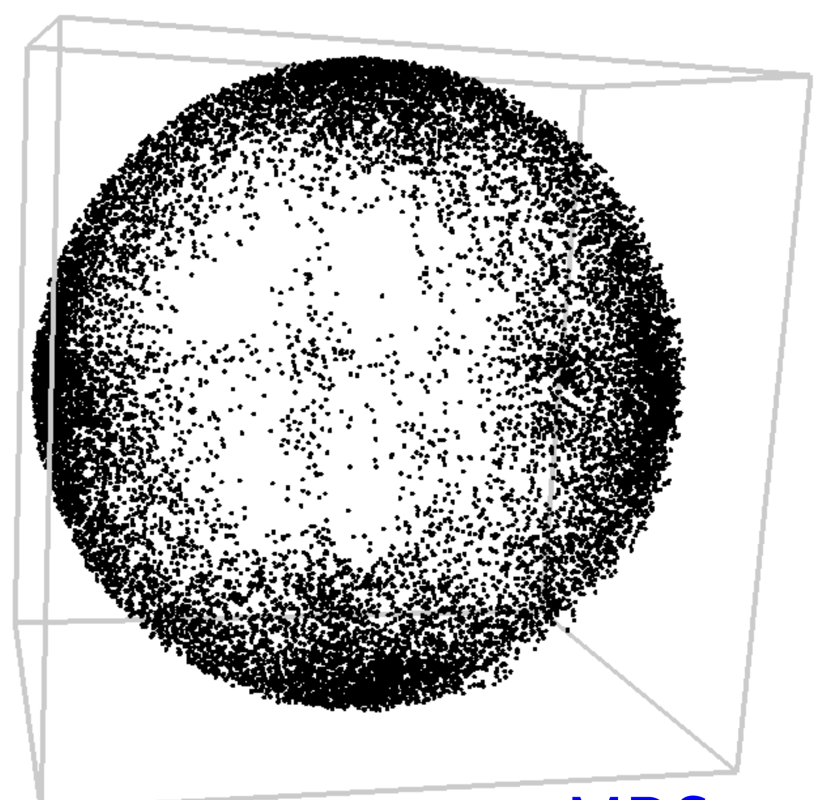
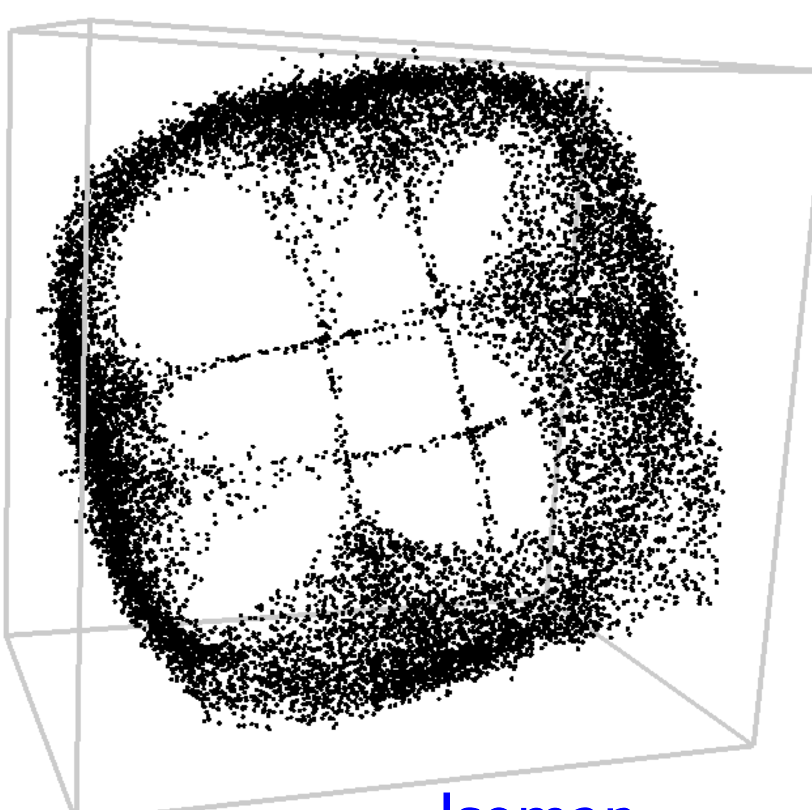


PCA



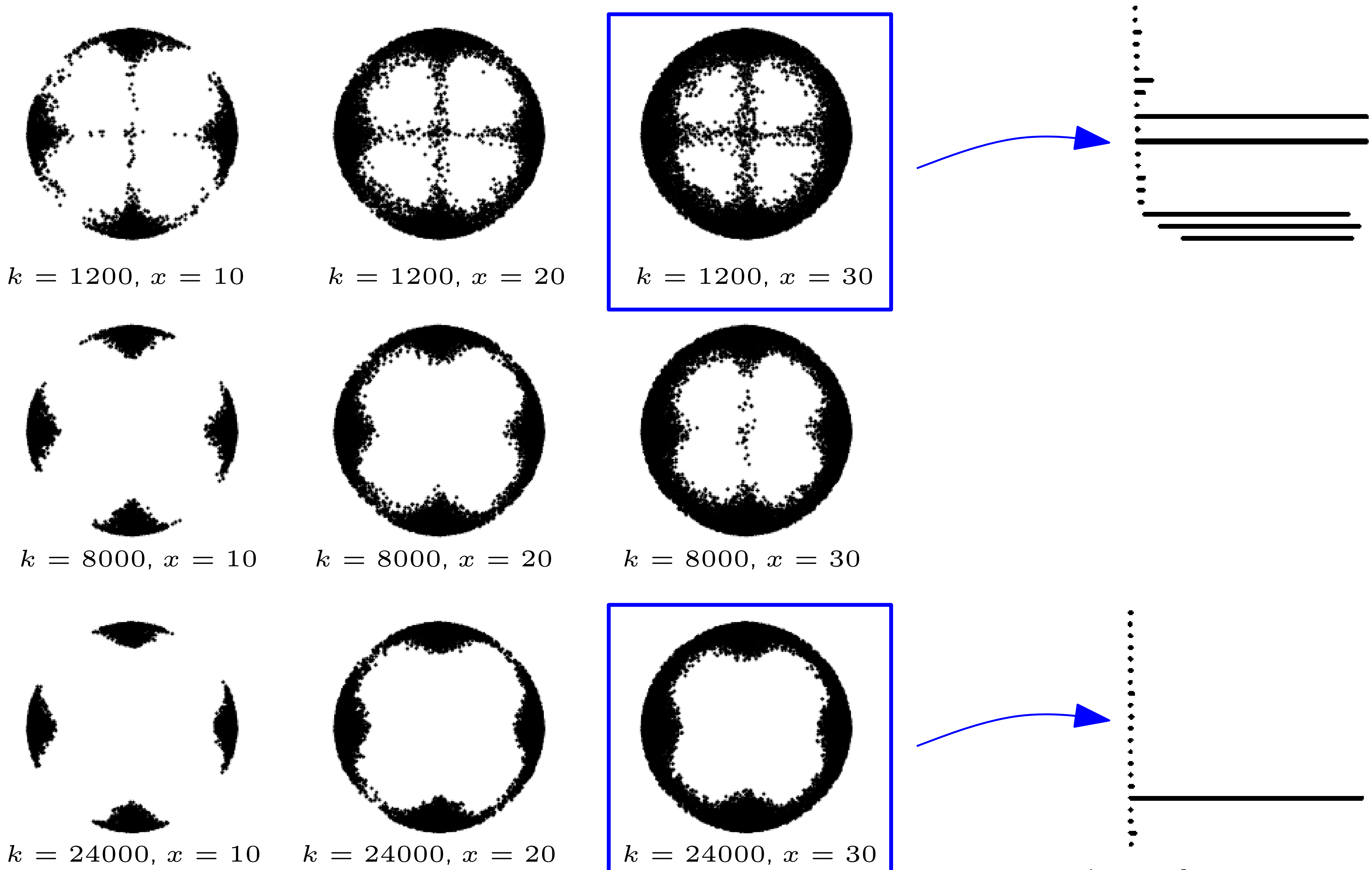
MDS



Isomap

Natural Images Data

Preprocessing: - select bottom $x\%$ of data points according to k-NN distance
- sample 5000 points uniformly at random from filtered point set



(source: [de Silva, Carlsson 04])

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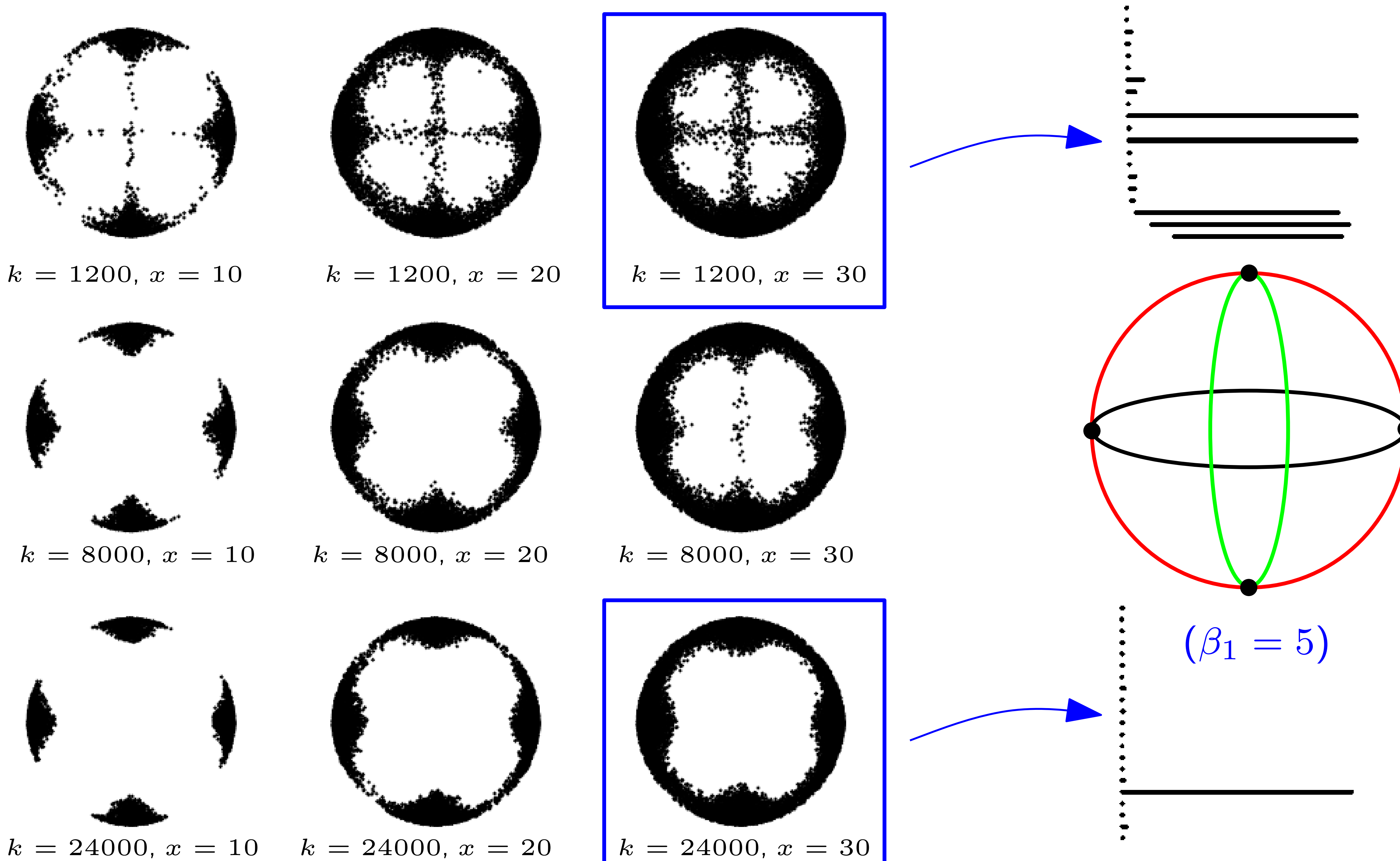

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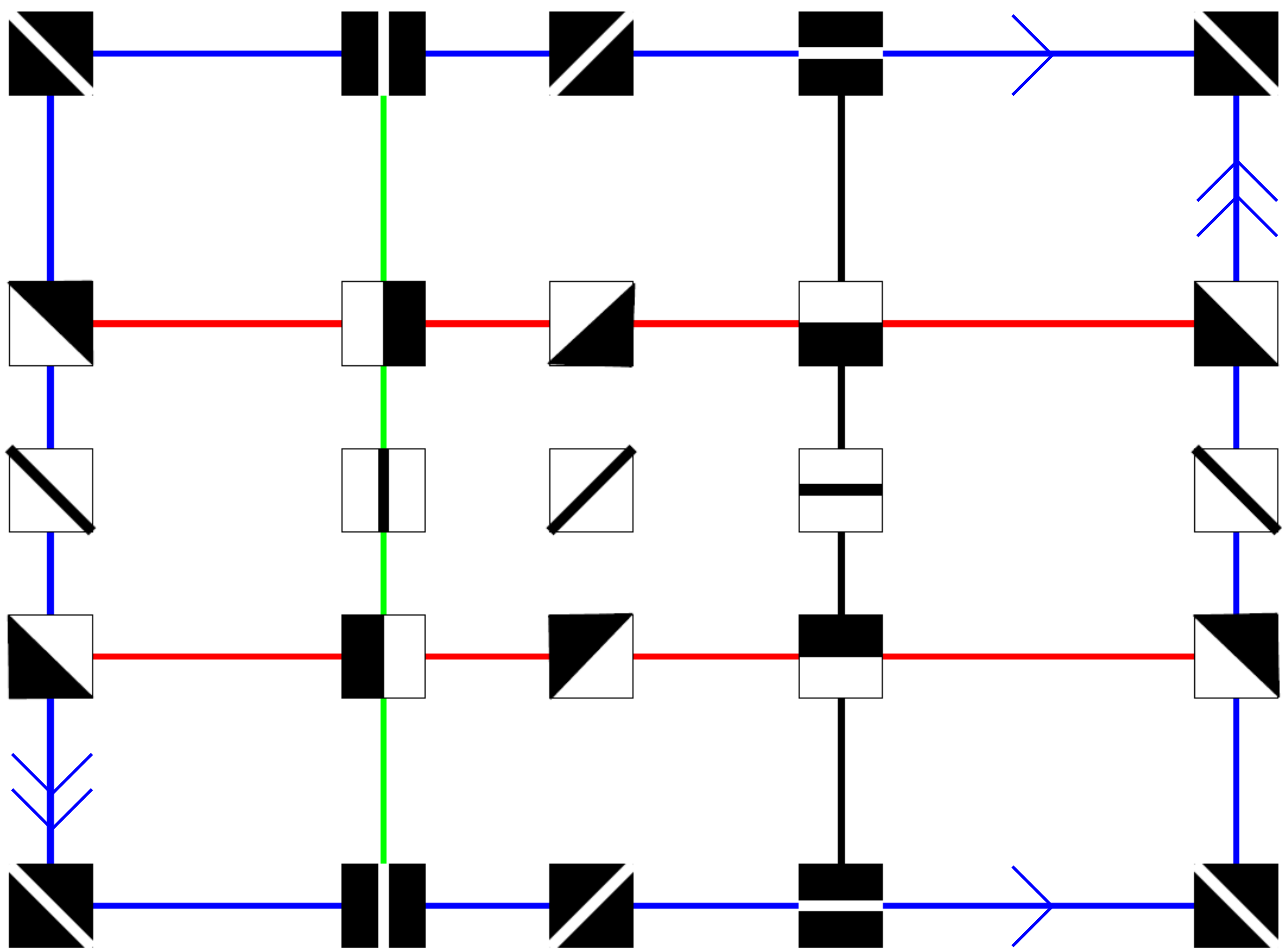

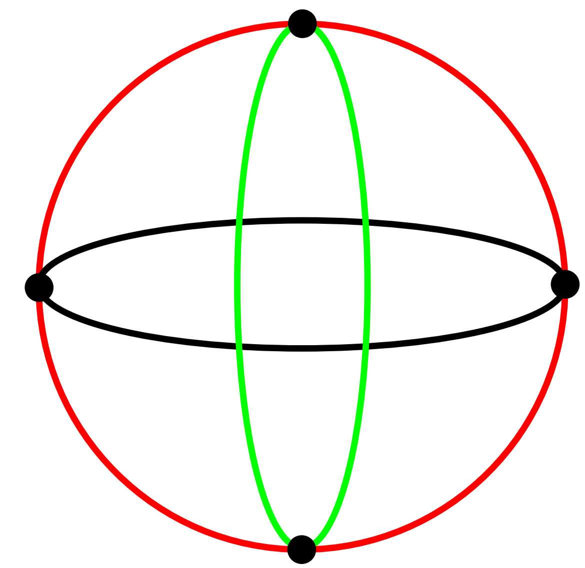

$(\beta_1 = 5)$

(source: [Carlsson, Ishkhanov, de Silva, Zomorodian 2008])

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Preprocessing: - select bottom $x\%$ of data points according to k-NN distance
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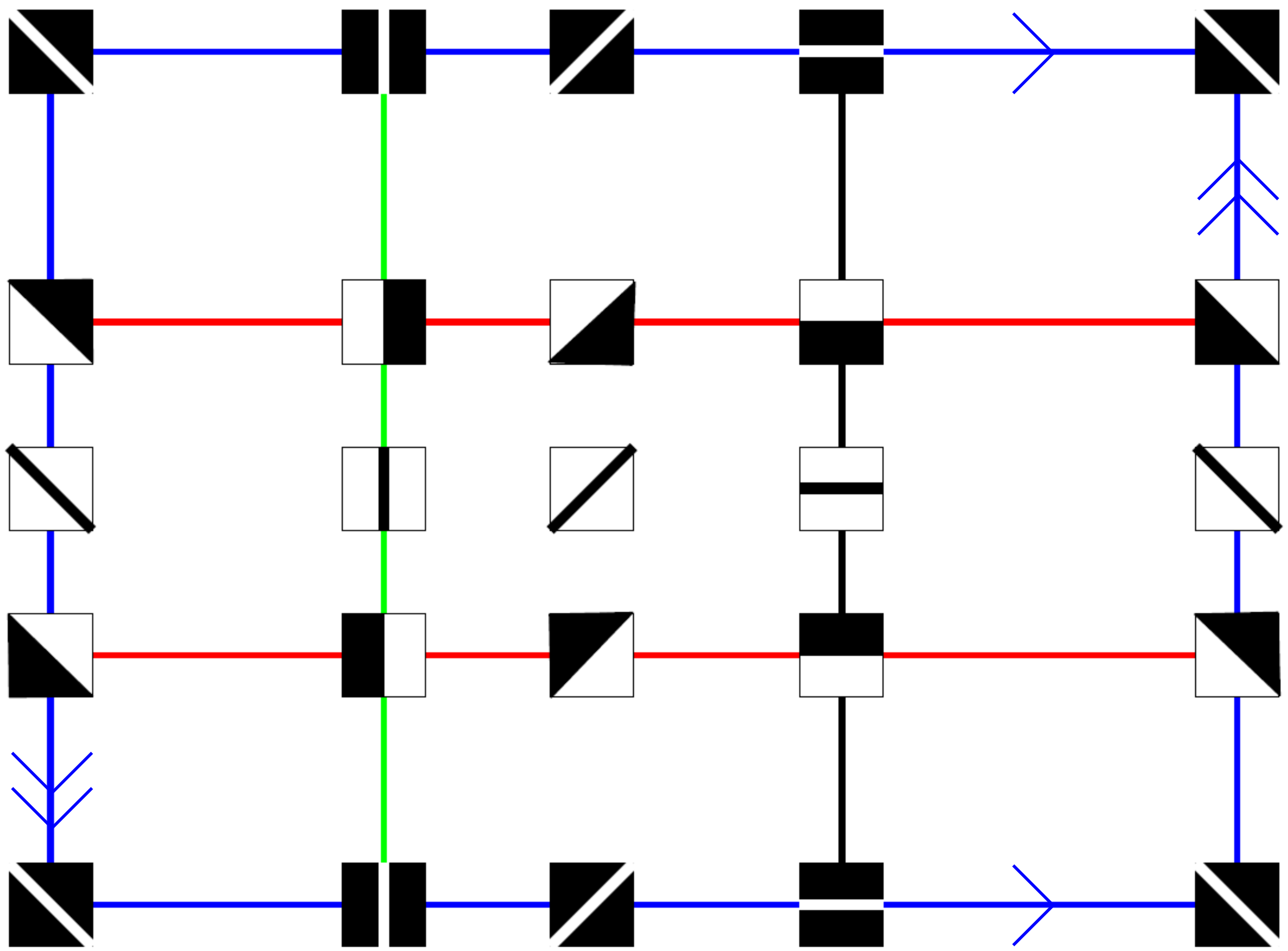
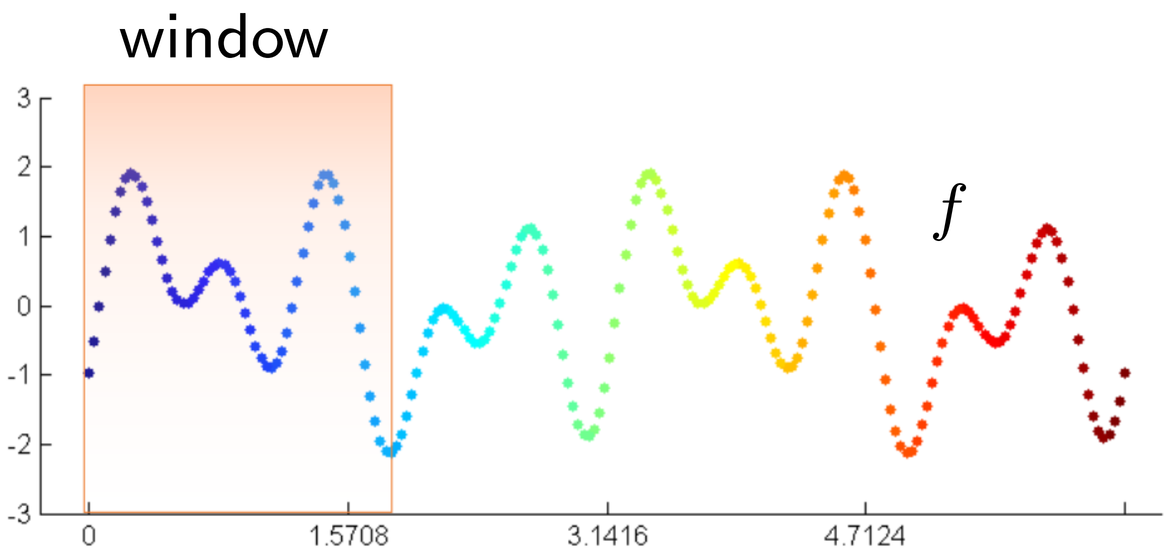


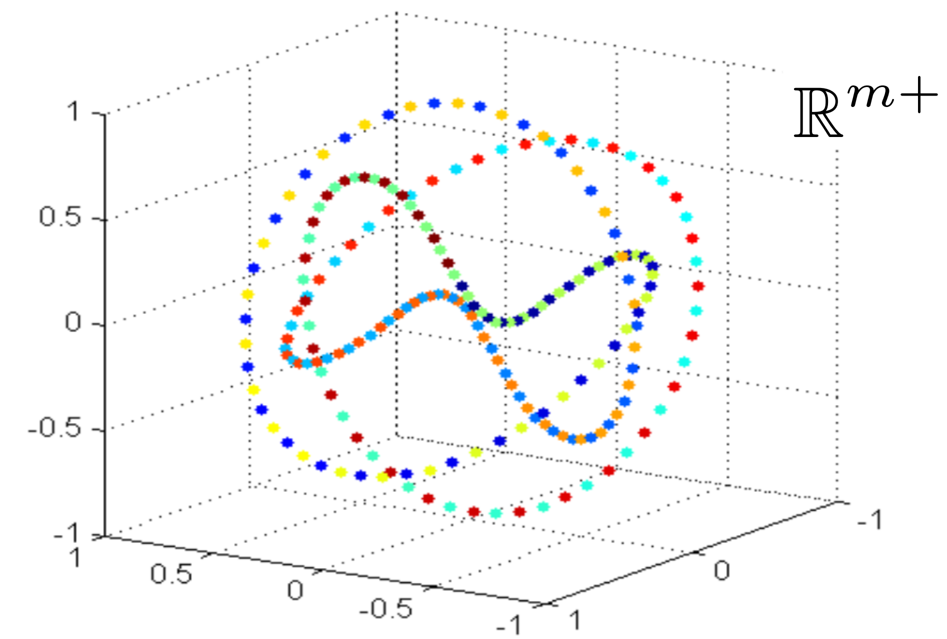
(source: [Carlsson, Ishkhanov, de Silva, Zomorodian 2008])

TDA for time series modeling & analysis



$$\mathrm{TD}_{m,\tau} \Longrightarrow$$

(time-delay embedding)



$f : \mathbb{N} \to \mathbb{R}$

$$\mathrm{TD}_{m,\tau}(f) := \begin{bmatrix} f(t) \\ f(t+\tau) \\ \vdots \\ f(t+m\tau) \end{bmatrix}$$

τ: step / delay

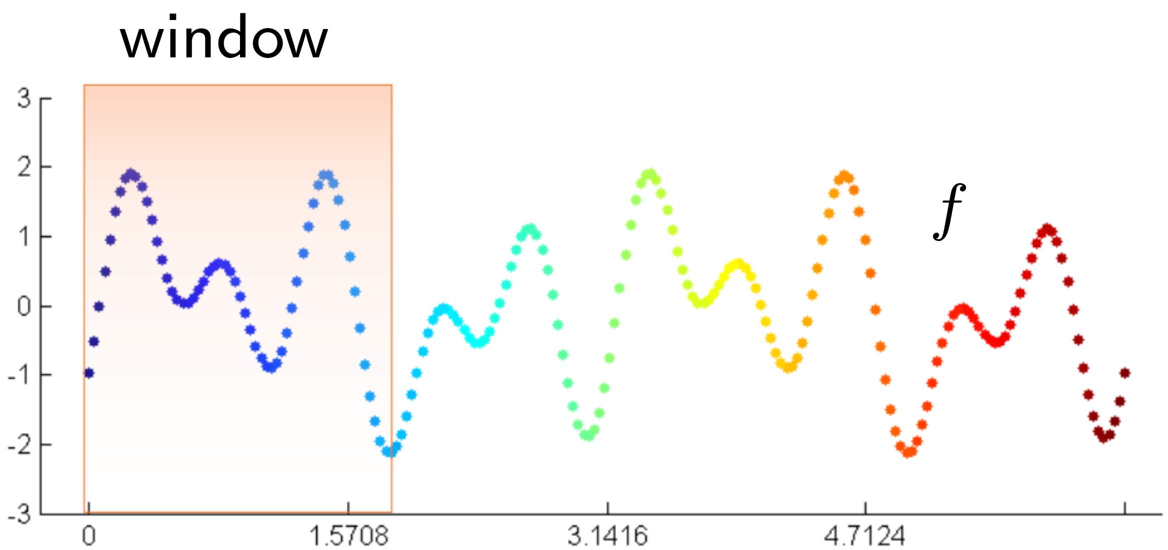
$m\tau$: window size

$m+1$: embedding dimension

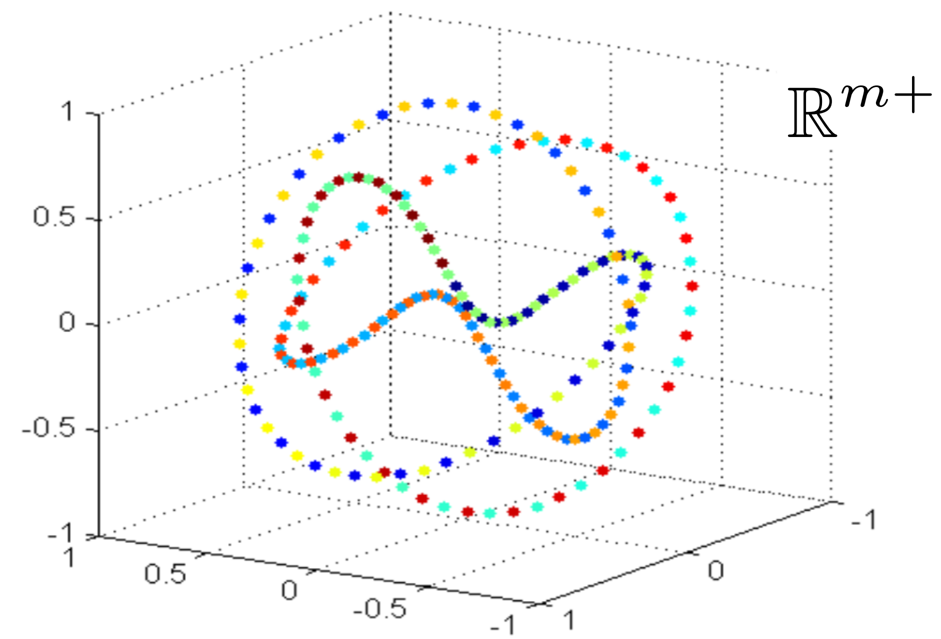
signal	**embedded data**
periodicity	circularity
# prominent harmonics (N)	min. ambient dimension ($m \geq 2N$)
# non-commensurate freq.	intrinsic dimension ($\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$)

[J. Perea et al.:"SW1PerS: Sliding windows and 1-persistence scoring", 2015]

TDA for time series modeling & analysis



$\mathrm{TD}_{m,\tau}$
$\Longrightarrow$
(time-delay embedding)



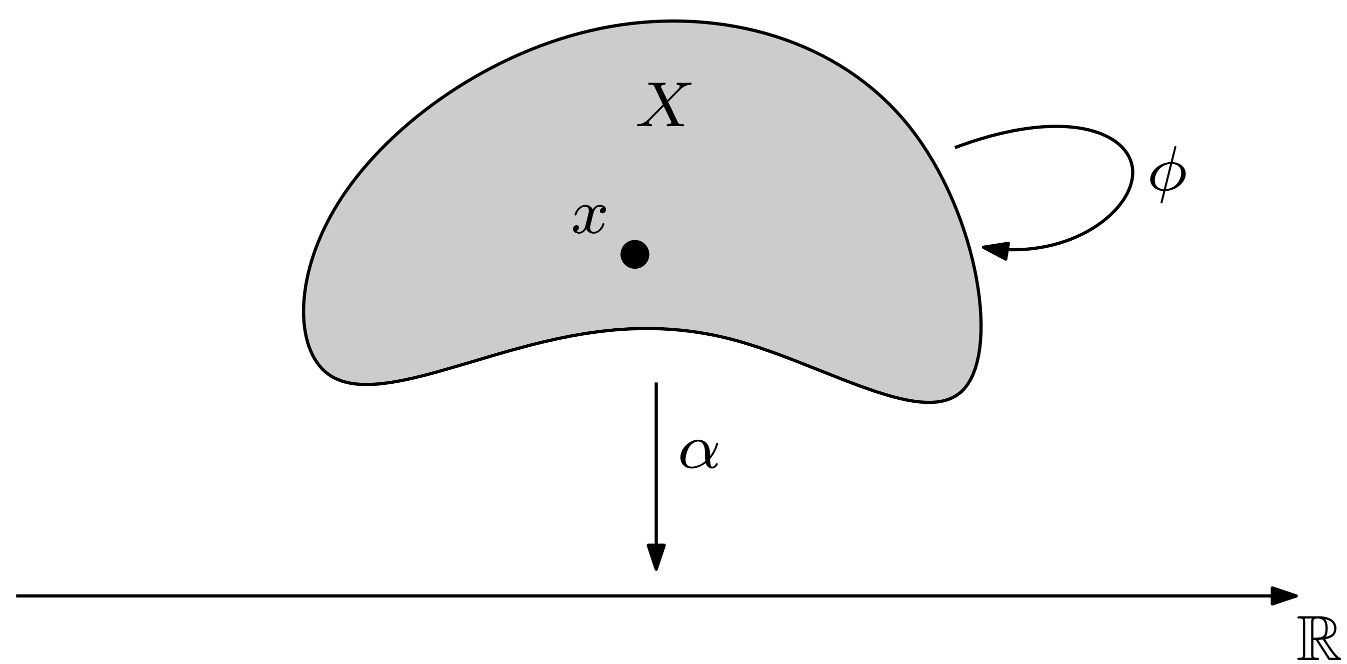
Contributions of TDA:

inference of:

- periodicity
- harmonics
- non-commensurate freq.
- underlying state space

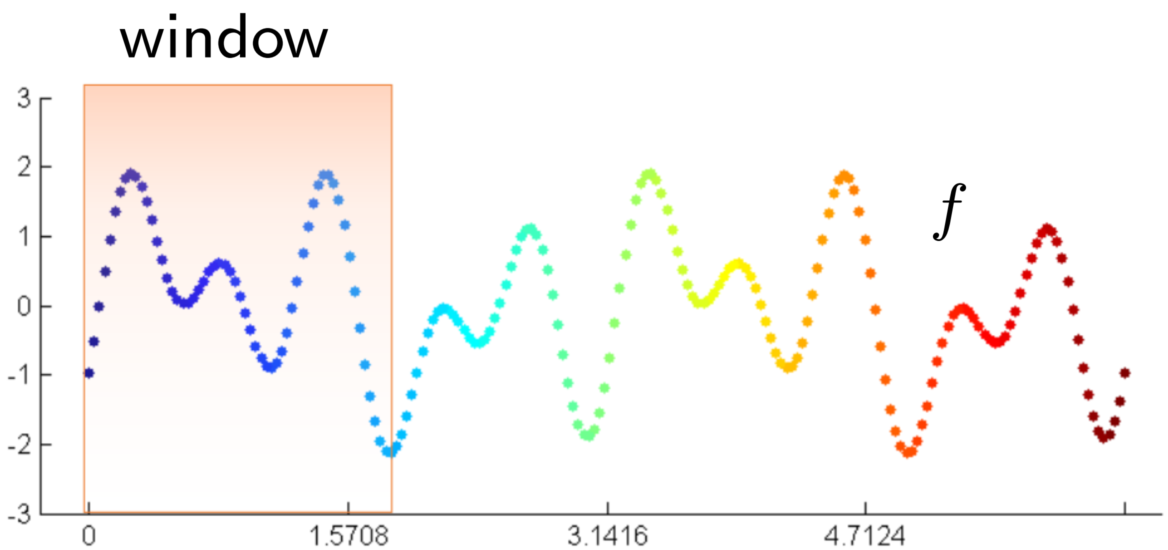
no Fourier transform needed

Dynamical system:



$$f_x(n) := \alpha(\phi^n(x))$$

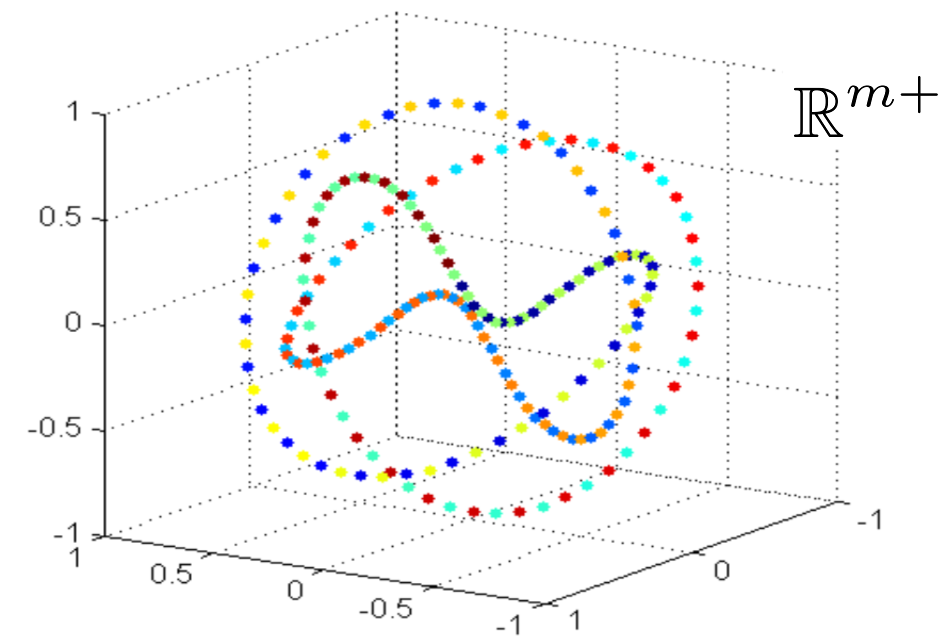
TDA for time series modeling & analysis



$\mathrm{TD}_{m,\tau}$

$\Longrightarrow$

(time-delay embedding)



Contributions of TDA:

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no Fourier transform needed

Dynamical system:

Thm: [Nash, Takens]
Given a Riemannian manifold X of dimension $\frac{m}{2}$, it is a **generic property** of $\phi \in \mathrm{Diff}_2(X)$ and $\alpha \in C^2(X, \mathbb{R})$ that

$$X \to \mathbb{R}^{m+1}$$
$$x \mapsto (\alpha(x), \alpha \circ \phi(x), \cdots, \alpha \circ \phi^m(x))$$

is an embedding.