Topological Persistence (a.k.a. persistent homology)

① Filtrations and persistence modules:

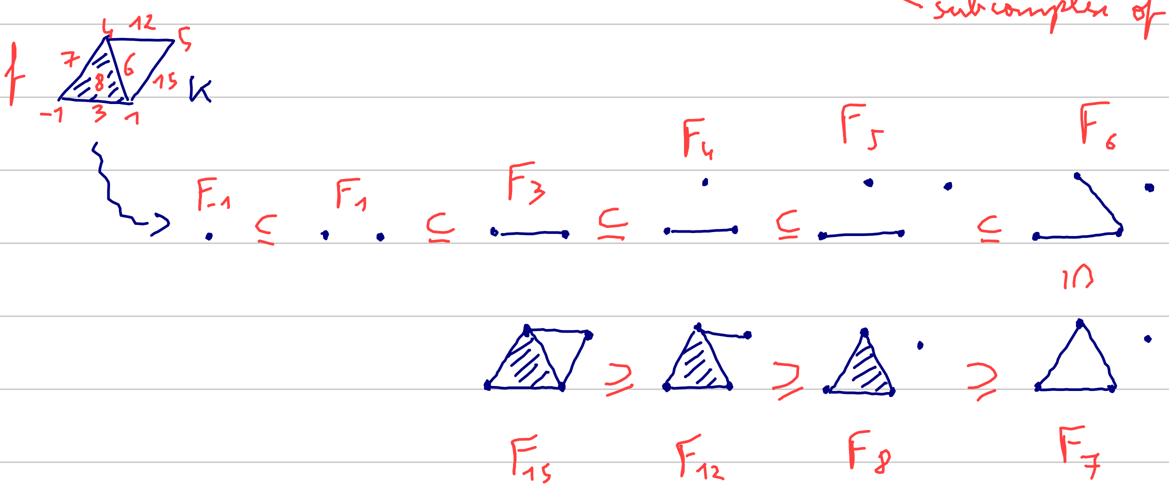
Let $T \subseteq \mathbb{R}$ be a fixed index set.

Def.: A filtration over T is a family $F = (F_t)_{t\in T}$ of nested topological spaces:
$$\forall s \leq t \in T, \quad F_s \subseteq F_t .$$

Examples:

* Sublevel sets of a function $f : X \to \mathbb{R}$:
$$\forall t \in T = \mathbb{R}, \quad F_t := f^{-1}((-\infty, t])$$
* offsets of a compact set $P \subset \mathbb{R}^d$ (i.e. sublevel sets of its distance function):
$$\forall t \in T = \mathbb{R}, \quad F_t = P^t := d_P^{-1}((-\infty, t]) = \{x \in \mathbb{R}^d \mid \min_{p\in P} \|x-p\| \leq t\}$$
* simplicial filtration: given a finite simplicial complex K, and $f : K \to \mathbb{R}$ s.t. $f(\tau) \leq f(\sigma)$ $\forall \tau \subseteq \sigma \in K$, let $T := \operatorname{Im} f$ and $F_t := f^{-1}((-\infty, t]) \subseteq K$.

subcomplex of K



↳ Goal: encode / summarize the evolution of the topology throughout F, for t ranging over T → use homology.

(filtration $\xrightarrow{H_*(\cdot\,;k)}$ persistence module)

Example: Let F be a filtration over T. Apply (singular) homology over a fixed field k to F:

$$\begin{cases} \forall t \in T, \text{ get a } k\text{-vector space } H_*(F_t). \\ \forall s \leq t \in T, \text{ get a } k\text{-linear map } H_*(F_s) \to H_*(F_t) \\ \qquad \text{induced by the inclusion } F_s \hookrightarrow F_t \end{cases}$$

↳ by functoriality of H_*, we have:

$\forall t \in T$, $H_*(F_t) \to H_*(F_t)$ is $id_{H_*(F_t)}$

$\forall s \leq t \leq u \in T$, the following triangle commutes:

$$\begin{tikzcd} H_*(F_s) \arrow[r] \arrow[dr] & H_*(F_t) \arrow[d] \\ & H_*(F_u) \end{tikzcd}$$

i.e. the map $H_*(F_s) \to H_*(F_u)$ is obtained by composing $H_*(F_s) \to H_*(F_t) \to H_*(F_u)$.

Let k be a fixed field.

Def: A <u>persistence module</u> over T is a family $M = (M_t)_{t\in T}$ of k-vector spaces connected by k-linear maps $m_s^t : M_s \to M_t$ for all $s \leq t \in T$, such that:

$$\begin{cases} m_s^s = id_{M_s} & \forall s \in T \\ m_s^u = m_t^u \circ m_s^t & \forall s \leq t \leq u \in T \end{cases}$$

Def: A <u>morphism</u> between persistence modules $\varphi: M \to N$ is defined pointwise: $\forall t \in T$, $\varphi_t : M_t \to N_t$ such that $\forall s \leq t \in T$, the following square commutes:

$$\begin{tikzcd} M_s \arrow[r, "m_s^t"] \arrow[d, "\varphi_s"'] & M_t \arrow[d, "\varphi_t"] \\ N_s \arrow[r, "n_s^t"] & N_t \end{tikzcd}$$

$\varphi: M \to N$ <u>isomorphism</u> if $\varphi_t : M_t \xrightarrow{\simeq} N_t$ $\forall t \in T$.

↳ <u>Goal</u>: encode / summarize the algebraic structure of a persistence module → use <u>decomposition</u>.

② Decompositions:

Idea: decompose M as a direct sum of "simpler" modules.

Def: Given M, N persistence modules over $T \subseteq \mathbb{R}$, their direct sum is defined pointwise:

$$\forall t \in T, \quad (M \oplus N)_t := M_t \oplus N_t$$

$$\forall s \leq t \in T, \quad (m \oplus n)_s^t := m_s^t \oplus n_s^t \quad \left(\rightsquigarrow \text{matrix form: } \begin{bmatrix} m_s^t & 0 \\ 0 & n_s^t \end{bmatrix} \right)$$

Def: M is decomposable if $M \simeq M_1 \oplus M_2$ for some $M_1, M_2 \neq 0$.
Otherwise, M is indecomposable.

Examples: $k^2 \xrightarrow{id} k^2$ decomposes as $k \xrightarrow{id} k \oplus k \xrightarrow{id} k$

$k \xrightarrow{id} k$ is indecomposable

$k \xrightarrow{0} k$ decomposes as $k \to 0 \oplus 0 \to k$

⚠ In contrast to vector spaces, persistence modules are not always semisimple, ie. their submodules may not be summands (there is no basis completion theorem).

↳ example: $0 \to k$ is a submodule of $k \xrightarrow{id} k$ but it is not a summand

$$(\nexists N \text{ s.t. } k \xrightarrow{id} k \simeq N \oplus 0 \to k).$$

$\Rightarrow$ decomposing persistence modules is a (much) harder problem than decomposing vector spaces.

Def: An *interval* of T is a subset $I \subseteq T$ such that $\forall s \le t \le u \in T$, $s, u \in I \Rightarrow t \in I$.

Def: Given an interval $I \subseteq T$, the corresponding *interval module* k_I is defined by:

$$\begin{cases} \forall t \in T,\ (k_I)_t = \begin{cases} k \text{ if } t \in I \\ 0 \text{ otherwise} \end{cases} \\ \forall s \le t \in T,\ (k_I)_s^t = \begin{cases} \mathrm{id}_k \text{ if } s, t \in I \\ 0 \text{ otherwise} \end{cases} \end{cases}$$

↳ interval modules are the basic building blocs for decomposing persistence modules... under some conditions

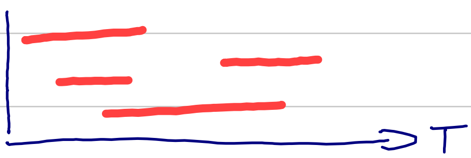
Thm: (Decomposition) [Gabriel / Auslander / Webb / Crawley-Boevey]

A persistence module M over $T \subseteq \mathbb{R}$ decomposes as a direct sum of interval modules: $M \simeq \bigoplus_{j \in J} k_{I_j}$, in either of the following (non-exclusive) cases:

- T is finite, or
- M is pointwise finite-dimensional (pfd), ie. $\dim M_t < \infty \ \forall t \in T$.

Moreover, when it exists, the decomposition is unique up to isomorphism and reordering of the terms in the direct sum.

↳ the intervals $\{ I_j \mid j \in J \}$ involved in the decomposition (when it exists) form the *barcode* of M. It is a *multiset* of intervals as one may have $I_j = I_{j'}$ for some $j \neq j' \in J$ (example: $k^2 \xrightarrow{\mathrm{id}} k^2 \simeq k \xrightarrow{\mathrm{id}} k \oplus k \xrightarrow{\mathrm{id}} k$).



$\boxed{\text{Def.}}$ The <u>persistence diagram</u> of M, denoted by $\mathrm{Dgm}\, M$, is the multiset of points $\{(b_j, d_j) \mid j \in J\}$ where $b_j \le d_j$ denote the endpoints of interval I_j in the decomposition of M.

Proof of decomposition (out of scope)

Assume (for simplicity) that T is finite and M is pfd.
↳ e.g. $\{0, 1, \dots, n\}$

$$M_0 \xrightarrow{m_0^1} M_1 \xrightarrow{m_1^2} \cdots \longrightarrow M_{n-1} \xrightarrow{m_{n-1}^n} M_n$$

Extend M over $\mathbb{N}$ by adding copies of M_n:

$$M_0 \xrightarrow{m_0^1} M_1 \xrightarrow{m_1^2} \cdots \longrightarrow M_{n-1} \xrightarrow{m_{n-1}^n} M_n \xrightarrow{id} M_n \xrightarrow{id} \cdots$$

M then has the structure of a module over $k[t]$:

$$\begin{cases} M \simeq \bigoplus_{t=0}^{\infty} M_t \quad \text{equipped with multiplication by } t: \\ t \cdot (x_0, x_1, \dots) = (0, m_0^1(x_0), m_1^2(x_1), \dots) \end{cases}$$

M is then a finitely generated module over $k[t]$, which is a PID, hence we have the well-known decomposition:

$$M \simeq \underbrace{\bigoplus_{i \in J_{free}} t^{b_i} \cdot k[t]}_{\text{free (infinite cyclic) part}} \oplus \underbrace{\bigoplus_{j \in J_{tor.}} t^{b_j} \cdot \cancel{k[t]}\, / \, t^{(d_j - b_j)} \cdot k[t]}_{\text{torsion (finite cyclic) part}}$$

infinite bars $[b_j, \infty)$ → free (infinite cyclic) part

finite bars $[b_j, d_j)$ → torsion (finite cyclic) part

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③ Computing persistence barcodes / diagrams :

$\boxed{\text{Input}}$: a simplicial filtration F such that:

any finite simplicial filtration can be adapted to satisfy these conditions →

$T = \{0, 1, \dots, n\}$

$F_0 = \emptyset$, $F_n = K$ (finite simplicial complex)

$\forall t$, F_t is a subcomplex of F_{t+1}

$\forall t$, $F_{t+1} \setminus F_t = \{\sigma_t\}$ (only one simplex inserted at each iteration)

↳ simplices of K are in bijection with $\{1, \dots, n\}$ ($\sigma_t \Leftrightarrow t$)

↳ total order on the simplices of K, compatible with incidence relations : $\sigma_s \subseteq \sigma_t \Rightarrow s \leq t$.

$\boxed{\text{Algo}}$: Same as for computing homology, with order on simplices fixed to be the one given by the filtration :

a) Compute matrix of boundary operator :

$$A = \begin{array}{c} \\ \sigma_1 \\ \vdots \\ \sigma_n \end{array} \begin{array}{c} \sigma_1 \; \cdots \; \sigma_n \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right] \end{array}$$

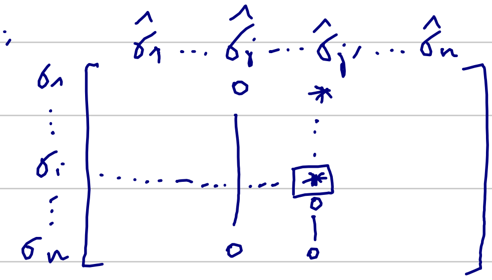
$A_{ij} \neq 0$ iff σ_i is a face of σ_j of codimension 1.

b) $$\text{low}(j) := \begin{cases} 0 \text{ if } A_{ij} = 0 \ \forall i \\ \max\{ i \mid A_{ij} \neq 0 \} \text{ otherwise} \end{cases}$$

c) Reduce A to column-echelon form from left to right:

for $j = 1$ to n do :
- while $\exists \ell < j$ s.t. $\text{low}(\ell) = \text{low}(j) \neq 0$ do:
  - change column c_j into $c_j - \dfrac{A_{\text{low}(j)j}}{A_{\text{low}(j)\ell}} \cdot c_\ell$

d) Interpret the reduced matrix:



* every column j corresponds to a chain

$$\hat\sigma_j = \sigma_j - \sum_{l<j} \lambda_l \sigma_l$$

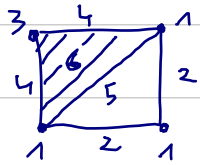
appearing when σ_j is inserted

* if $\hat\sigma_j = 0$ then σ_j is positive (its insertion creates a cycle)
* if $\hat\sigma_j \neq 0$ then σ_j is negative (its insertion trivializes a cycle)

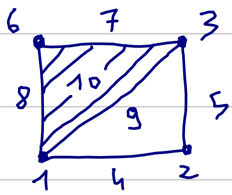
(the cycle created when σ_i was inserted where $i = \mathrm{low}(j)$)

Thus: the summands of $H_*(F)$ are in bijection with the zeroed out columns in the reduced matrix:

- each pair $(\sigma_i, \hat\sigma_j)$ with $i = \mathrm{low}(j)$ generates a finite interval summand $k_{[i,j)}$ in $H_{\dim \sigma_i}(F)$,
- each unpaired $\hat\sigma_j$ generates an infinite interval summand $k_{[j,\infty)}$ in $H_{\dim \sigma_j}(F)$.

Example: $k = \mathbb{Z}/2\mathbb{Z}$, $K =$ 

↳ map simplex set bijectively and monotonously to $\{1, \dots, n\}$:

$F =$ 

↳ boundary matrix:

$$\begin{array}{c|cccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & & & & 1 & & & & 1 & 1 & \\
2 & & & & 1 & 1 & & & & & \\
3 & & & & & 1 & & 1 & & 1 & \\
4 & & & & & & & & & & \\
5 & & & & & & & & & & \\
6 & & & & & & & 1 & 1 & & \\
7 & & & & & & & & & & 1 \\
8 & & & & & & & & & & 1 \\
9 & & & & & & & & & & 1 \\
10 & & & & & & & & & &
\end{array}
\xrightarrow{\text{(reduction)}}
\begin{bmatrix}
 & & & 1 & & & & & & \\
 & & & \underset{(2,4)}{\boxed{1}} & 1 & & & & & \\
 & & & & \underset{(3,5)}{\boxed{1}} & & 1 & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & \underset{(6,7)}{\boxed{1}} & & & \\
 & & & & & & & & & 1 \\
 & & & & & & & & & 1 \\
 & & & & & & & & & \underset{(9,10)}{\boxed{1}} \\
 & & & & & & & & &
\end{bmatrix}$$

Hence the barcode:

$$\begin{cases} H_0: [1,\infty)\ ;\ [2,4)\ ;\ [3,5)\ ;\ [6,7) \\ H_1: [8,\infty)\ ;\ [9,10) \end{cases}$$

↳ after mapping to the original indices:

$$\begin{cases} H_0: [1,\infty)\ ;\ [1,2)\ ;\ [1,2)\ ;\ [3,4) \\ H_1: [4,\infty)\ ;\ [5,6) \end{cases}$$

$\boxed{\text{Note:}}$ For computation by hand, the same approach as with homology can be taken, with a little extra book keeping:

a) fix a total order on simplices that is compatible with filtration and incidence orders.

b) iterate over the simplices in this order, and for each simplex σ_j:

- either it creates a cycle $\Rightarrow$ creates a summand $[j, ?)$ in $H_{\dim \sigma_j}(F)$
- or it kills a cycle (find σ_i that created it) $\Rightarrow$ $[i, ?)$ is completed as $[i, j)$ in $H_{\dim \sigma_i}(F)$

c) Upon termination, replace the remaining "?" in the barcode by "∞".

(4) Stability of persistence barcodes / diagrams:

We have an operator Dgm : $f \longmapsto \mathrm{Dgm}(f)$

real-valued function

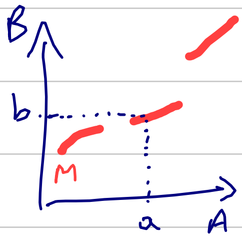
persistence diagram of the filtration of sublevel sets of f (whenever defined)

↳ we want to show that Dgm is Lipschitz continuous:

- metric on functions $f, g : X \to \mathbb{R}$ (for X compact): $\|f-g\|_\infty$
- metric on persistence diagrams: <u>bottleneck distance</u> d_∞.

$\boxed{\text{Def:}}$ A <u>partial matching</u> $A \leftrightarrow B$ is a subset $M \subseteq A \times B$ such that the natural projections $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$, restricted to M, are injective.

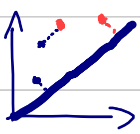
↳ $\begin{cases} \forall a \in B, \exists \leq 1 \ b \in B \text{ s.t. } (a,b) \in M \\ \forall b \in B, \exists \leq 1 \ a \in A \text{ s.t. } (a,b) \in M \end{cases}$



$\boxed{\text{Def:}}$ The <u>p-th cost</u> $c_p(M)$ of a partial matching $M : A \leftrightarrow B$ is given by:

(for $A, B \subseteq \mathbb{R}^2$)

$$c_p(M) := \left(\sum_{(a,b) \in M} \|a-b\|_\infty^p + \sum_{\substack{c \notin \pi_A|_M \\ c \notin \pi_B|_M}} \left(\frac{c_y - c_x}{2} \right)^p \right)^{1/p}.$$



$\boxed{\text{Def:}}$ The <u>p-th distance</u> between $A, B \subseteq \mathbb{R}^2$ is:

$$d_p(A,B) := \inf_{M : A \leftrightarrow B} c_p(M).$$

Note: The bottleneck distance d_∞ is obtained by letting $p \to \infty$ in the above definitions (sums are replaced by max).

Thm: (Stability) [Cohen-Steiner, Edelsbrunner, Harer 2005]
For any pfd functions $f, g : X \to \mathbb{R}$ (i.e. whose sublevel sets have finite-dimensional homology):

$$d_\infty(\mathrm{Dgm}\, f, \mathrm{Dgm}\, g) \leq \|f - g\|_\infty .$$

Def: An <u>ε-interleaving</u> between persistence modules M, N over $\mathbb{R}$ is given by two families of linear maps:

$$\forall t \in \mathbb{R}, \begin{cases} \varphi_t : M_t \to N_{t+\varepsilon} \\ \psi_t : N_t \to M_{t+\varepsilon} \end{cases}$$

such that the following diagrams commute $\forall s \leq t \in \mathbb{R}$:

$$\begin{tikzcd} M_s \arrow[r, "m_s^t"] \arrow[d, "\varphi_s"'] & M_t \arrow[d, "\varphi_t"] \\ N_{s+\varepsilon} \arrow[r, "n_{s+\varepsilon}^{t+\varepsilon}"'] & N_{t+\varepsilon} \end{tikzcd} \qquad \begin{tikzcd} M_{s+\varepsilon} \arrow[r, "m_s^t"] & M_{t+\varepsilon} \\ N_s \arrow[u, "\psi_s"] \arrow[r, "n_s^t"'] & N_t \arrow[u, "\psi_t"'] \end{tikzcd}$$

(φ, ψ are morphisms between shifted persistence modules)

$$\begin{tikzcd} M_s \arrow[rr, "m_s^{s+2\varepsilon}"] \arrow[dr, "\varphi_s"'] & & M_{s+2\varepsilon} \\ & N_{s+\varepsilon} \arrow[ur, "\psi_{s+\varepsilon}"'] & \end{tikzcd} \qquad \begin{tikzcd} & M_{s+\varepsilon} \arrow[dr, "\varphi_{s+\varepsilon}"] & \\ N_s \arrow[ur, "\psi_s"] \arrow[rr, "n_s^{s+2\varepsilon}"'] & & N_{s+2\varepsilon} \end{tikzcd}$$

(φ, ψ compose to shifted identities)

Note: A 0-interleaving is an isomorphism of persistence modules.

Def: The <u>interleaving distance</u> d_i on persistence modules is defined as:

$$d_i(M,N) := \inf\{\varepsilon \geq 0 \mid M, N \text{ are } \varepsilon\text{-interleaved}\}$$

Thm: (Isometry) [Chazal et al. 2009, Lesnick 2011]
For any pfd persistence modules M, N,

$$d_i(M,N) = d_\infty(\mathrm{Dgm}\, M, \mathrm{Dgm}\, N).$$

Thus, persistence barcodes/diagrams are complete algebraic and metric descriptors of persistence modules.

$\boxed{Q}$ How relevant is the interleaving distance d_i on the space of persistence modules?

↳ By the Stability and Isometry theorems, we have $d_i(H_*(F), H_*(G)) \leq \|f-g\|_\infty$ for all pfd functions $f,g : X \to \mathbb{R}$ and their associated sublevel sets filtrations F, G. In fact:

Prop: For any functions $f,g : X \to \mathbb{R}$ (possibly not pfd) and their associated sublevel sets filtrations F, G,

$$d_i(H_*(F), H_*(G)) \leq \|f-g\|_\infty.$$

→ <u>proof</u>: let $\varepsilon = \|f-g\|_\infty$. Then, $\forall t \in \mathbb{R}$ we have

$$F_t \subseteq G_{t+\varepsilon} \quad \text{and} \quad G_t \subseteq F_{t+\varepsilon}.$$

Hence the (commutative) diagrams of inclusions $\forall s \leq t \in \mathbb{R}$:

$$\begin{tikzcd}
F_s \arrow[r] \arrow[d] & F_t \arrow[d] \\
G_{s+\varepsilon} \arrow[r] & G_{t+\varepsilon}
\end{tikzcd}
\qquad
\begin{tikzcd}
F_s \arrow[rr] \arrow[dr] & & F_{s+2\varepsilon} \\
& G_t \arrow[ur] &
\end{tikzcd}$$

and reciprocally. The commutative diagrams of ε-interleaving between $H_*(F)$ and $H_*(G)$ follow then from the functoriality of H_*. ■

$\boxed{\text{Thm:}}$ [Lesnick 2011]

d_i satisfies the following universality property on the space of persistence modules:

For any metric d such that $d(H_*(F), H_*(G)) \leq \|f-g\|_\infty$ for all functions $f, g : X \to \mathbb{R}$, one has $d \leq d_i$.

Thus, d_i is the most discriminative among all stable metrics on the space of persistence modules.

(in the sense that the operator $f \mapsto H_*(F)$ is 1-Lipschitz)