Notes on homology theory

Philosophy: homology was introduced as a tool to compare and classify topological spaces ⇝ introduce relevant related notions than homology itself.

① Comparing topological spaces:

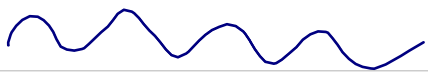
Let X, Y be two topological spaces.

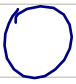
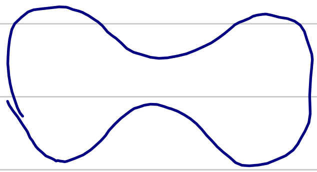
Def: Homeomorphism: $h : X \to Y$ bijective s.t. both h and h^{-1} are continuous.
↳ then X, Y are homeomorphic.

Def: $f, g : X \to Y$ are homotopic (noted $f \sim g$) if $\exists H : [0,1] \times X \to Y$ continuous s.t. $H(0, \cdot) = f$ and $H(1, \cdot) = g$.
↳ H is called a homotopy.

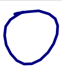
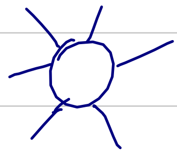
Def: $f : X \to Y$ is a homotopy equivalence if $\exists g : Y \to X$ s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$.
↳ X, Y are said homotopy equivalent.

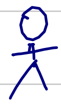
Examples:

— homeo. to 

 homeo. to 

• homotopy equiv. to 

 homotopy equiv. to 

and to 

Note: These notions are introduced for classifying topological spaces (up to homeo., homotopy equiv., etc.)

↳ homotopy theory is a tool for classification.

Prop: X homeo. to $Y \Rightarrow X$ homotopy equiv. to Y.

→ proof: given $h: X \to Y$ homeo., take $f := h$ and $g := h^{-1}$.

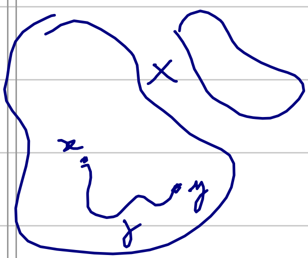
$$\Rightarrow g \circ f = id_X \simeq id_X \text{ and } f \circ g = id_Y \simeq id_Y.$$

∎

② Intuition of homology:

Def 1 A *path* is a continuous map $[0,1] \longrightarrow X$.

Def: $x \sim y$ in X iff $\exists$ path $\gamma : [0,1] \longrightarrow X$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$. (rmk: equivalence relation)

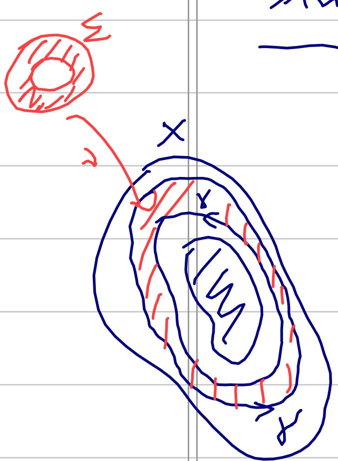


Def: (Path-) connected components of $X \equiv X/\sim$ (equivalence classes of the relation $\sim$).

Alternative formulation: $x \sim y$ in X if $\exists$ path γ s.t. $\gamma(\underbrace{\partial[0,1]}_{=\{0,1\}}) = \{x,y\}$

Similarly: - A *loop* is a map $\gamma : S^1 \longrightarrow X$ that is continuous.

- Two loops $\gamma, \gamma' : S^1 \longrightarrow X$ are equivalent if $\exists$ surface Σ and a map $\partial : \Sigma \longrightarrow X$ s.t. $\partial(\partial\Sigma) = \gamma(S^1) \cup \gamma'(S^1)$.



↳ higher-dimensional generalizations:

- *n-cycle* $\equiv$ map $\gamma : \Sigma \longrightarrow X$ where $\dim \Sigma = n$ and $\partial \Sigma = \varnothing$
- $\gamma \sim \gamma'$ if $\exists\ \partial : \underset{\dim = n+1}{\Omega} \longrightarrow X$ s.t. $\partial(\partial\Omega) = \gamma(\Sigma) \cup \gamma'(\Sigma')$.

↳ leads to $\boxed{\text{cobordism theory}}$ → homology simplifies it via linear algebra

Idea: build cycles by gluing basic building blocks (simplices).

↳ use a discretization of the space.

③ Simplicial complexes:

Def: A graph is given by:
- a vertex set V
- an edge set $E \subseteq \mathcal{P}_2(V)$ — set of pairs $\{v_i, v_j\}$ $(i \neq j)$.

Note: $E \subseteq \mathcal{P}_2(V) \Rightarrow$ every edge has its 2 vertices in the vertex set

↳ direct generalization:

Def: A simplicial complex is given by:
- a vertex set V
- a simplex set $K \subseteq \mathcal{P}(V)$ (all subsets of V) such that:
  - $\emptyset \in K$
  - $\forall \sigma \in K, \forall \tau \subseteq \sigma, \tau \in K$. (subsets of vertices.)

↳ Glossary:
- The complex is often denoted K (vertex set is implicit, assimilated to sets of size 1 in K)
- every $\sigma \in K$ is a simplex
  - ↳ vertex if $\#\sigma = 1$
  - ↳ edge if $\#\sigma = 2$

$\#\cdot$: cardinality

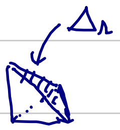
↳ triangle if $\#\sigma = 3$
↳ tetrahedron if $\#\sigma = 4$

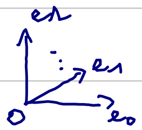
...

- $\tau \leq \sigma \rightsquigarrow \tau$ is a <u>face</u> of σ.
- $\dim \sigma := \#\sigma - 1$ || $\dim K = \sup_{\sigma \in K} \dim \sigma$.
- $K_n = \{\sigma \in K \mid \dim \sigma = n\}$ (k-skeleton)

$\boxed{Q}$ How to turn it into a topological space?

$\boxed{A}$ By drawing it ;-)





$\boxed{\text{Def.}}$ The <u>standard n-simplex</u> Δ_n is defined by:

$$\Delta_n := \operatorname{Conv}\{e_0, e_1, \dots, e_n\} \text{ in } \mathbb{R}^{n+1}$$
$$= \left\{ \sum_{i=0}^{n} \lambda_i e_i \;\middle|\; \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

$\boxed{\text{Def:}}$ A <u>geometric realization</u> of a simplicial complex K is a topological space X together with a family of continuous maps $(f_\sigma : \Delta_{\dim \sigma} \to X)_{\sigma \in K}$ such that:

- $X = \bigcup_{\sigma \in K} \operatorname{Im} f_\sigma$;
- each f_σ is injective;
- $\forall \sigma, \tau \in K$, $\operatorname{Im} f_\sigma \cap \operatorname{Im} f_\tau = \operatorname{Im} f_{\sigma \cap \tau}$ (with the convention that $\operatorname{Im} f_\emptyset = \emptyset$)
- $U \subseteq X$ is open in X iff $f_\sigma^{-1}(U)$ is open in $\Delta_{\dim \sigma}$ for all $\sigma \in K$.

$\boxed{\text{Prop:}}$ All geometric realizations of K are homeomorphic.

↳ $\boxed{\text{Def:}}$ The <u>underlying space</u> $|K|$ is any geometric realization of K.

Def: A space X is <u>triangulable</u> if $\exists K$ such that X and $|K|$ are homeomorphic.
↳ K is then a <u>triangulation</u> of X.

4) Simplicial homology:

Fix a field k.
(e.g. $k = \mathbb{R}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}$, etc.)

4.1 Chains:

Def: n-chain = formal k-weighted sum of n-simplices
$\equiv$ map $K_n \to k$ with <u>finite support</u>.

$$c : K_n \to k \qquad\qquad c := \sum_{\sigma \in K_n} \underset{\in k}{c(\sigma)} \underset{\in K_n}{\sigma}$$
$$\sigma \mapsto c(\sigma)$$

s.t. $c(\sigma) = 0$ except for finitely many σ.

Note: maps $K_n \to k$ with finite support form a k-vector space:

still finite support:
$$(c+c')(\sigma) = c(\sigma) + c'(\sigma) \quad \leftarrow \text{sum in } k$$
$$(\lambda c)(\sigma) = \lambda\, c(\sigma) \quad \leftarrow \text{product in } k$$

→ call $C_n(K)$ this vector space ("space of chains")

4.2 Boundary Operator:

<u>Hyp:</u> $k = \mathbb{Z}/2\mathbb{Z}$
(characteristic = 2)

Def: $\partial_n : C_n(K) \to C_{n-1}(K)$

(def. on basis elements)
$$\sigma = \{v_0, \dots, v_n\} \longmapsto \sum_{i=0}^{n} \underbrace{\{v_0, \dots, \hat{v}_i, \dots, v_n\}}_{\sigma \setminus \{v_i\}}$$
(removed from set)

(extend by linearity)
$$c = \sum_{\sigma \in K_n} c(\sigma)\,\sigma \longmapsto \sum_{\sigma \in K_n} c(\sigma)\, \partial\sigma .$$

Recall: want
"cycles" $\equiv$ zero boundary
"equiv. cycles" $\equiv$ difference is a boundary

$\boxed{\text{Prop:}}$ $\partial_{n-1} \circ \partial_n = 0 \quad \forall n \geq 1.$

proof: show it on basis elements, then will extend to the whole space $C_n(K)$ by linearity.

Let $\sigma = \{v_0, \dots, v_n\} \in C_n(K)$.

$$\partial_n(\sigma) = \sum_{i=0}^{n} \{v_0, \dots, \hat{v_i}, \dots, v_n\} = \sum_{i=0}^{n} \sigma \setminus \{v_i\}.$$

$$\partial_{n-1} \circ \partial_n (\sigma) = \sum_{i=0}^{n} \sum_{\substack{j=0 \\ j \neq i}}^{n} \sigma \setminus \{v_i, v_j\}$$

$$= \sum_{i=0}^{n} \left[\sum_{j<i} \{v_0, \dots, \hat{v_j}, \dots, \hat{v_i}, \dots, v_n\} + \sum_{j>i} \{v_0, \dots, \hat{v_i}, \dots, \hat{v_j}, \dots, v_n\} \right]$$

$$= \sum_{j<i} \{v_0, \dots, \hat{v_j}, \dots, \hat{v_i}, \dots, v_n\} + \sum_{j>i} \{v_0, \dots, \hat{v_i}, \dots, \hat{v_j}, \dots, v_n\}$$

$$= 2 \sum_{j>i} \{v_0, \dots, \hat{v_i}, \dots, \hat{v_j}, \dots, v_n\} = 0.$$

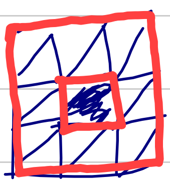
(characteristic $= 2$) ∎

Hence: $\underbrace{\text{Im}\, \partial_n}_{\substack{(n-1)\text{-boundaries} \\ B_{n-1}(K)}} \subseteq \underbrace{\text{Ker}\, \partial_{n-1}}_{\substack{(n-1)\text{-cycles} \\ Z_{n-1}(K)}}$.

$\boxed{\text{Def:}}$ $\boxed{H_n(K) := Z_n(K) / B_n(K)}$ n-th homology group.

Note: $c, c' \in Z_n(K)$ are equivalent ("homologous") iff $(c - c') \in B_n(K)$, i.e. $\exists\, c'' \in C_{n+1}(K)$ s.t. $(c - c') = \partial c''$.

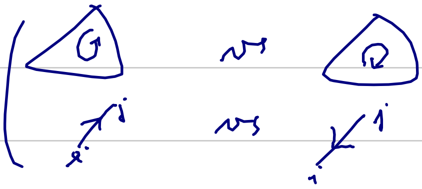
Example:



The two red 1-cycles are homologous, and none is null-homologous.

4.3 Fields k with other characteristics: $\rightarrow$ orient simplices

Orientation: 2 orientations per simplex:



Def: An orientation of $\sigma = \{v_0, \dots, v_n\}$ is an ordering of the vertices, e.g. $[v_0, \dots, v_n]$ or $[v_3, v_1, v_9, \dots]$.

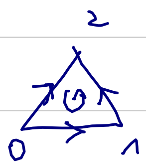
ordering $\swarrow$ permutation $\pi \in \Sigma_{n+1}$

- Two orientations $[v_{\pi(0)}, \dots, v_{\pi(n)}]$ and $[v_{\pi'(0)}, \dots, v_{\pi'(n)}]$ are equivalent if $\mathrm{sgn}(\pi' \circ \pi^{-1}) = +1$.

$\Rightarrow$ $\boxed{\text{2 classes of orientations of } \sigma.}$

Def: $$\partial [i_0, \dots, i_n] = \sum_{j=0}^{n} (-1)^j [i_0, \dots, \hat{i_j}, \dots, i_n].$$

Example:



$$\partial [0,1,2] = [0,1] + [1,2] + [2,0]$$
$$= [0,1] + [1,2] - [0,2]$$
$$\partial [0,2,1] = [2,1] - [0,1] + [0,2]$$
$$= -[0,1] - [1,2] + [0,2]$$
$$= -\partial [0,1,2].$$

Prop: $\partial_{n-1} \circ \partial_n = 0$.

proof: same as before except duplicated terms cancel out due to sign exchange. $\square$

$\rightsquigarrow$ $H_n(K) := \mathrm{Ker}\, \partial_n / \mathrm{Im}\, \partial_{n+1}$ as before.

(5) Algorithms

Input: A finite simplicial complex K, a field k.

Output: $\underbrace{H_r(K; k)}$ $\forall r \geq 0$.
$\rightsquigarrow \simeq k^n$ for some $n \in \mathbb{N} \to$ find n.

$H_r := Z_r / B_r \Rightarrow \dim H_r = \underbrace{\dim Z_r}_{= \dim \ker \partial_r} - \underbrace{\dim B_r}_{= \operatorname{rank} \partial_{r+1}}$

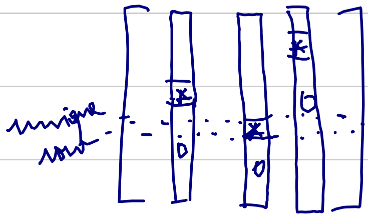
↳ $\forall r$ compute the matrix M_r of ∂_r in the simplex basis.

$$\begin{array}{c|c} & \sigma_1 \;\cdots\; \sigma_{\# K_r} \\ \hline \begin{array}{c} \tau_1 \\ \vdots \\ \tau_{\# K_{r-1}} \end{array} & \begin{bmatrix} \\ & M_r & \\ \\ \end{bmatrix} \end{array} \rightsquigarrow \boxed{\dim H_r = \# K_r - \operatorname{rank} M_r - \operatorname{rank} M_{r+1}}$$

↳ $\forall r$ compute rank M_r.
↳ possible approaches:
- (i) Gaussian elimination $\rightsquigarrow O(m^3)$ where $m = \max\{\# K_r, \# K_{r-1}\}$.
- (ii) Fast matrix multiplication $\rightsquigarrow O(m^\omega)$ where $\omega \in [2; 2.373...]$
- (iii) Projections / fixed point $\rightsquigarrow$ effective algos. in practice.

Note: Gaussian elimination is also efficient in practice because the boundary matrix M_r usually remains sparse throughout the reduction.
↳ near-linear time in practice.

↳ simple algo. for Gaussian elimination: reduce M to column-echelon form up to permutation:



Def: $\mathrm{low}(j) = \begin{cases} \max\{i \mid M_{ij} \neq 0\} \\ 0 \text{ if } M_{ij} = 0 \ \forall i \end{cases}$

(row of lowest non-zero entry in column)

↳ reduce M until $\mathrm{low}\big|_{\{j \mid \mathrm{low}(j) \neq 0\}}$ is <u>injective</u>.

<u>Reduction</u>:

$M = \begin{bmatrix} c_1 & \cdots & c_{\#K_n} \end{bmatrix}$

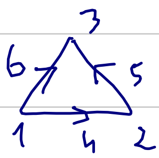
For $j = 1$ to $\#K_n$ do:
  | While $\exists\, i < j$ s.t. $\mathrm{low}(i) = \mathrm{low}(j) \neq 0$ do:
  | | change column content c_j into $c_j - \frac{M_{\mathrm{low}(j)}}{M_{\mathrm{low}(i)}} c_i$.

↳ Upon termination: $\mathrm{rank}\, M = \#K_n - \#\{j \mid \mathrm{low}(j) = 0\}$.

Prop: (i) the algo. terminates because every inner loop reduces $\mathrm{low}(j)$ strictly.

(ii) upon termination, $\mathrm{low}\big|_{\{j \mid \mathrm{low}(j) \neq 0\}}$ is injective by the "while" condition.

<u>Example</u>: Homology of the circle:



$$M_0 \sqcup M_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} & & & -1 & & -1 \\ & \partial_0 = 0 & & \boxed{1} & -1 & \\ & & & & \boxed{1} & \boxed{1} \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right] \end{array}$$

$\square$: low

low is injective until column 5 included

$\Rightarrow$ only column 6 is being reduced:

$c_6 \leftarrow c_6 - c_5$:
$$\begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 1 & -1 & & -1 \\ 2 & \boxed{1} & -1 & \boxed{1} \\ 3 & & \boxed{1} & \end{array}$$

$c_6 \leftarrow c_6 - c_4$:
$$\begin{bmatrix} -1 & & 0 \\ \boxed{1} & -1 & 0 \\ & \boxed{1} & 0 \end{bmatrix}$$

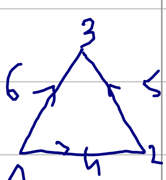
$\Rightarrow$ rank $M_1 = 2$, dim Ker $M_1 = 1$.

Notes:
- operations $c_j \leftarrow c_j - \frac{M_{low(j)}}{M_{low(i)}} c_i$ involve divisions
  $\Downarrow$
  difficult to assert nullity in $\mathbb{R}$
  $\Downarrow$
  use finite (prime) fields instead.
- the pivots in the reduced matrix play an important role

↳ interpretation of the matrix reduction:
- scan 1-simplices in an arbitrary order.
- for each one (say σ):
  - either $\partial\sigma = 0$ in the subcomplex spanned by the simplices of the columns to the left $\rightsquigarrow$ σ creates an 1-cycle $\rightsquigarrow$ β_1 ++

    "homologous"
  - or $\partial\sigma \neq 0$ in that subcomplex $\rightsquigarrow$ the chain $\partial\sigma$ becomes a boundary $\rightsquigarrow$ σ kills an $(k-1)$-cycle $\rightsquigarrow$ β_{k-1} --

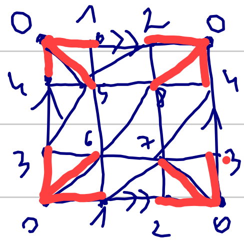
Example: Homology of the circle.



algo:

	1	2	3	4	5	6	
β_0	1	1	1	×	×		$\rightsquigarrow$ 1
β_1						1	$\rightsquigarrow$ 1
β_2	0 throughout (no 2-simplex)						

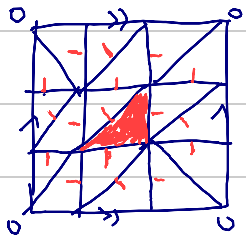
Example: Homology of the torus $S^1 \times S^1$:



pair each red edge with its vertex other than 0 $\longrightarrow$ $\begin{cases}\beta_0 = 1 \\ \beta_1 = 0\end{cases}$

(spanning tree)

→ each remaining edge will create a 1-cycle.



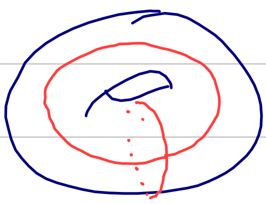
insert [1,2] and [3,4] (β_1++ for each)

then pair each remaining edge with an incident triangle as shown on the left.

$\longrightarrow \beta_1 = 2$

→ the remaining triangle (in red) creates a 2-cycle $\rightarrow \beta_2$++.

In total: $\beta_0 = \beta_2 = 1$, $\beta_1 = 2$.

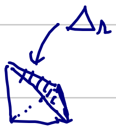


1 connected component
1 handle + 1 tunnel
1 enclosed void

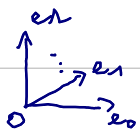
6 Singular homology:

6.1 Singular chains:

Idea: Build cycles by gluing basic building blocs (a.k.a. "singular simplices") together into "chains".



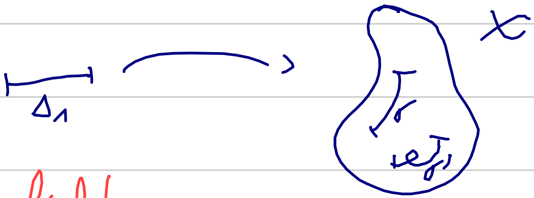
Def: The n-dim. standard simplex Δ_n is defined by:



$$\Delta_n := \mathrm{Conv}\{e_0, e_1, \dots, e_n\} \text{ in } \mathbb{R}^{n+1}$$
$$= \left\{ \sum_{i=0}^{n} \lambda_i e_i \;\middle|\; \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

Def: A n-dim. singular simplex (or singular n-simplex) in X is a continuous map $\sigma : \Delta_n \to X$. $(\sigma \in \mathcal{C}(\Delta_n, X))$

Examples:



Let k be a fixed field.

Def: A n-chain is a formal finite k-weighted sum of singular n-simplices:

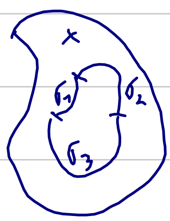
$$c := \sum_{i=1}^{n} \underset{\in k}{\lambda_i} \underset{: \Delta_n \to X}{\sigma_i}$$

↳ equivalent def.: c is a map from the singular n-simplices to k that is zero except on finitely many simplices:

$$c : \mathcal{C}(\Delta_n, X) \to k$$

s.t. $c = 0$ everywhere except on a finite set.

Example:



$c = \sigma_1 + \sigma_2 + \sigma_3$ is a 1-chain.

(note: often we identify each simplex with it's image).

Note: The set of n-chains is the set of k-valued functions over $\mathcal{C}(\Delta_n, X)$ with finite support: $k^{(\mathcal{C}(\Delta_n,X))}$ finite support

$\Rightarrow$ it has a k-vector space structure:

$$\begin{cases} c + c' : & \sigma \mapsto c(\sigma) + c'(\sigma) \\ \lambda c : & \sigma \mapsto \lambda c(\sigma). \end{cases}$$

(finite sums still have finite support).

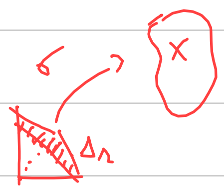
↳ call $C_n(X) := k^{(\mathcal{C}(\Delta_n,X))}$ this (huge) vector space.

6.2 Boundary operator:

Q what is the boundary of a chain?

↳ define the boundary of a simplex

↳ extend to chains by linearity.



defined via the boundary of the standard simplex: if $\sigma : \Delta_n \to X$ then write "$\partial\sigma := \sigma|_{\partial\Delta_n}$".

↳ need to respect orientation

Def: An orientation of Δ_n is an order $[e_{\pi(0)}, \dots, e_{\pi(n)}]$ on the vertices of Δ_n $(\pi \in \Sigma_{n+1})$

↳ Two orientations π, π' are equivalent if $\pi' \circ \pi^{-1} \in \Sigma_{n+1}$ has positive signature.

$\Rightarrow$ 2 equivalence classes of orientations:

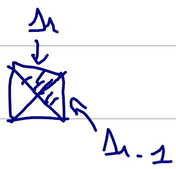
- positive (class of id)
- negative

$\boxed{\text{Def:}}$ $$\partial \Delta_n = \sum_{i=0}^{n} \underbrace{(-1)^i}_{\text{in the field } k} \underbrace{[e_0, \dots \hat{e_i}, \dots, e_n]}_{\text{sequence where } e_i \text{ has been removed}}.$$

where: $(-1)\cdot[w_0,\dots,w_s]$ is the simplex $\mathrm{Conv}\{w_0,\dots,w_s\}$ with the orientation given by vertex order

Rk: $[e_0,\dots,\hat{e_i},\dots,e_n] \neq [e_0,\dots,e_{n-1}]$.

↳ we need to map the simplex $\mathrm{Conv}\{e_0,\dots,e_{n-1}\}$ to the simplex $\mathrm{Conv}\{e_0,\dots,\hat{e_i},\dots,e_n\}$ with proper orientation



↳ let $\varepsilon_i : \Delta_n \to \Delta_{n+1}$

$$e_j \mapsto \begin{cases} e_j & \text{if } j < i \\ e_{j+1} & \text{if } j \geq i \end{cases}$$

Each ε_i maps $[e_0,\dots,e_{n-1}]$ to $[e_0,\dots,\hat{e_i},\dots,e_n]$, which preserves orientation given by increasing vertex labels.

Then:

$\boxed{\text{Def:}}$ Given $\sigma : \Delta_n \to X$, $\partial\sigma := \sum_{i=0}^{n} (-1)^i (\sigma \circ \varepsilon_i)$ $\quad \Delta_{n-1} \to X$

Given $c = \sum_{i=1}^{n} \alpha_i \sigma_i$, $\partial c := \sum_{i=} \alpha_i \partial\sigma_i \in C_{n-1}(X)$.

$\boxed{\text{Prop:}}$ $\forall n \geq 0$, $\partial_n \circ \partial_{n+1} = 0$.

→ proof: We show it on each basis element (simplex).

let $\sigma : \Delta_{n+1} \to X$ be a (oriented) singular $(n+1)$-simplex.

$$\partial_n \circ \partial_{n+1}(\sigma) = \partial_n\left(\sum_{i=0}^{n+1} (-1)^i \sigma \circ \varepsilon_i^{n+1}\right)$$

$$\underset{\text{(linearity)}}{=} \sum_{i=0}^{n+1} (-1)^i \partial_n(\sigma \circ \varepsilon_i^{n+1}) = \sum_{i=0}^{n+1} (-1)^i \sum_{j=0}^{n} (-1)^j (\sigma \circ \varepsilon_i^{n+1} \circ \varepsilon_j^n).$$

$$= \sum_{i=0}^{n+1} \sum_{j=0}^{n} (-1)^{i+j} (\sigma \circ \varepsilon_i^{n+1} \circ \varepsilon_j^n)$$

$$= \sum_{i=0}^{n+1} \sum_{j=0}^{i} (-1)^{i+j} (\sigma \circ \varepsilon_i^{n+1} \circ \varepsilon_j^n) + \sum_{i=0}^{n+1} \sum_{j=i+1}^{n} (-1)^{i+j} (\sigma \circ \varepsilon_i^{n+1} \circ \varepsilon_j^n).$$

For all $i<j$: $[e_0,\dots,e_{n-1}] \xrightarrow{\varepsilon_i^n} [e_0,\dots,\hat{e}_i,\dots,e_n] \xrightarrow{\varepsilon_j^{n+1}} [e_0,\dots,\hat{e}_i,\dots,\hat{e}_{j-1},\dots,e_{n+1}]$

$[e_0,\dots,e_{n-1}] \xrightarrow{\varepsilon_j^n} [e_0,\dots,\hat{e}_j,\dots,e_n] \xrightarrow{\varepsilon_i^{n+1}} [e_0,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+1}]$

↖ the vertex at position e_j after ε_i

$$\Rightarrow \quad \varepsilon_j^{n+1}\circ\varepsilon_i^n = \varepsilon_i^{n+1}\circ\varepsilon_{j-1}^n .$$

ie: $\varepsilon_i^{n+1}\circ\varepsilon_j^n = \varepsilon_{j+1}^{n+1}\circ\varepsilon_i^n \quad \forall i\leq j$.

Hence:

$$\partial_n\circ\partial_{n+1}\sigma = \sum_{i=0}^{n+1}\sum_{j=0}^{i-1}(-1)^{i+j}\left(\sigma\circ\varepsilon_i^{n+1}\circ\varepsilon_j^n\right) + \sum_{i=0}^{n+1}\sum_{j=i}^{n}(-1)^{i+j}\left(\sigma\circ\varepsilon_{j+1}^{n+1}\circ\varepsilon_i^n\right)$$

$j\mapsto j+1$

$$= \sum_{i=0}^{n+1}\sum_{j=0}^{i-1}(-1)^{i+j}\left(\sigma\circ\varepsilon_i^{n+1}\circ\varepsilon_j^n\right) + \sum_{i=0}^{n+1}\sum_{j=i+1}^{n+1}(-1)^{i+j-1}\left(\sigma\circ\varepsilon_j^{n+1}\circ\varepsilon_i^n\right)$$

$$= \sum_{i=0}^{n+1}\sum_{j=0}^{i-1}(-1)^{i+j}\left(\sigma\circ\varepsilon_i^{n+1}\circ\varepsilon_j^n\right) + \sum_{j=0}^{n+1}\sum_{i=0}^{j-1}(-1)^{i+j-1}\left(\sigma\circ\varepsilon_j^{n+1}\circ\varepsilon_i^n\right)$$

$i\leftrightarrow j$

$$= \sum_{i=0}^{n+1}\sum_{j=0}^{i-1}\cancel{(-1)^{i+j}}\left(\sigma\circ\varepsilon_i^{n+1}\circ\varepsilon_j^n\right) + \sum_{i=0}^{n+1}\sum_{j=0}^{i-1}\cancel{(-1)^{i+j-1}}\left(\sigma\circ\varepsilon_i^{n+1}\circ\varepsilon_j^n\right).$$

$$= 0 . \quad \blacksquare$$

$\boxed{\text{Def:}}$ $\forall n\in\mathbb{N}$, $\quad H_n := \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$.

$\boxed{\text{Thm:}}$ (equivalence of homologies)

For any simplicial complex X, $\quad H_n^{\text{simpl}}(X) \underset{(\text{iso.})}{\simeq} H_n^{\text{sing}}(|X|)$.

Note: in order to compare the singular homology of a triangulable space X to that of its triangulations, we need to describe what it becomes through homeomorphisms ↪ define homology of maps.

7 Morphisms:

Let $f : X \to Y$ continuous.

↳ Induced chain map $f^\#$:

$$f^\#_n : \left| \begin{array}{l} C_n(X;k) \longrightarrow C_n(Y;k) \\ \sigma : \Delta_n \to X \longmapsto f \circ \sigma : \Delta_n \to Y \\ \sum_{i=1}^{s} \alpha_i \sigma_i \longmapsto \sum_{i=1}^{s} \alpha_i f^\#_n(\sigma_i) = \sum_{i=1}^{s} \alpha_i \, f \circ \sigma_i . \end{array} \right.$$

Then:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
C_{n+1}(X;k) \arrow[r, "f^\#_{n+1}"] \arrow[d, "\partial^X_{n+1}"] & C_{n+1}(Y;k) \arrow[d, "\partial^Y_{n+1}"] \\
C_n(X;k) \arrow[r, "f^\#_n"] \arrow[d, "\partial^X_n"] & C_n(Y;k) \arrow[d, "\partial^Y_n"] \\
C_{n-1}(X;k) \arrow[r, "f^\#_{n-1}"] \arrow[d] & C_{n-1}(Y;k) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
C_0(X;k) \arrow[r, "f^\#_0"] \arrow[d, "\partial^X_0"] & C_0(Y;k) \arrow[d, "\partial^Y_0"] \\
0 \arrow[r, "0"] & 0
\end{tikzcd}$$

Prop: The above diagram commutes, i.e. $f^\#_n \circ \partial^X_{n+1} = \partial^Y_{n+1} \circ f^\#_{n+1}$ for every $n \in \mathbb{N}$.

→ proof: True on oriented simplices (basis elements):

$$(f^\#_n \circ \partial^X_{n+1})(\sigma) = f^\#_n \left(\sum_{i=0}^{n+1} (-1)^i \, \sigma \circ \varepsilon_i \right)$$

independent of X, Y

$$\underset{\text{(linearity)}}{=} \sum_{i=0}^{n+1} (-1)^i f_n^{\#}(\sigma \circ \varepsilon_i)$$

$$= \sum_{i=0}^{n+1} (-1)^i (f \circ \sigma \circ \varepsilon_i) = \sum_{i=0}^{n+1} (-1)^i (f \circ \sigma) \circ \varepsilon_i$$

$$= \partial_{n+1}^Y (f \circ \sigma) = \partial_{n+1}^Y (f_{n+1}^{\#}(\sigma)). \quad \blacksquare$$

Note: a family of maps $(f_n^{\#})_{n \in \mathbb{N}}$ such as above is called a chain map between chain complexes.

The property implies that $\begin{cases} f_n^{\#}(\operatorname{Ker} \partial_n^X) \subseteq \operatorname{Ker} \partial_n^Y \\ f_n^{\#}(\operatorname{Im} \partial_n^X) \subseteq \operatorname{Im} \partial_n^Y \end{cases}$

$\Rightarrow (f_n^{\#})_{n \in \mathbb{N}}$ induces linear maps $f_n^* : H_n(X; k) \to H_n(Y; k)$ for every $n \in \mathbb{N}$.

$\boxed{\text{Prop:}}$ Singular homology is functorial i.e.:

* Given a topo. space X, $(id_X)_n^* = id_{H_n(X;k)}$ $\forall n \in \mathbb{N}$.
* Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ continuous maps, $(g \circ f)_n^* = g_n^* \circ f_n^*$ for all $n \in \mathbb{N}$.

→ proof: Obvious because already true at the chain complex level. $\blacksquare$

$\boxed{\text{Corollary:}}$ $X \underset{\text{(homeo.)}}{\simeq} Y \Rightarrow H_n(X; k) \underset{\text{(iso)}}{\simeq} H_n(Y; k)$ $\forall n \in \mathbb{N}$.

→ proof: Let $h : X \to Y$ be a homeo, and h^{-1} its inverse.

Then : $\begin{cases} (h^{-1})^* \circ h^* = (h^{-1} \circ h)^* = id_X^* = id_{H_n(X;k)}. \\ h^* \circ (h^{-1})^* = (h \circ h^{-1})^* = id_Y^* = id_{H_n(Y;k)}. \end{cases}$

Therefore, h^* and $(h^{-1})^*$ are inverses of each other, which implies that $h^* : H_n(X; k) \to H_n(Y; k)$ is an iso. $\blacksquare$

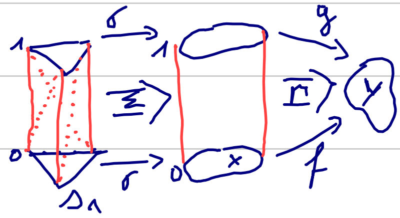
Note: A consequence of the above corollary is that, for any given triangulable space X, all its possible triangulations have homology groups isomorphic to those of X.

Prop: If $f \sim g : X \to Y$ then $f_n^* = g_n^* : H_n(X;k) \to H_n(Y;k)$ for every $n \in \mathbb{N}$.

→ **proof**: For ease of exposition, we assume that $k = \mathbb{Z}/2\mathbb{Z}$

Let $\sigma : \Delta_n \to X$ be a singular n-simplex.

- Consider the prism $[0,1] \times \Delta_n$ and triangulate it in a way that is compatible with $\{0\} \times \Delta_n$ and $\{1\} \times \Delta_n$ (call T the triangulation)



- Let $\Sigma : [0,1] \times \Delta_n \to [0,1] \times X$ be defined by $\Sigma(t,\cdot) = (t, \sigma(\cdot))\ \forall t \in [0,1]$
- Take a homotopy $\Gamma : [0,1] \times X \to Y$ from f to g (i.e. $\Gamma(0,\cdot) = f$ and $\Gamma(1,\cdot) = g$)

Then define the following family of maps $(h_n^\# : C_n(X;k) \to C_{n+1}(Y;k))$:

$$\begin{cases} h_n^\#(\sigma) := \displaystyle\sum_{\substack{\Delta \in T_{n+1} \\ \text{((n+1)-simplices of } T)}} \underbrace{\Gamma \circ \Sigma\big|_\Delta}_{} \\[2em] h_n^\#\left(\displaystyle\sum_{i=1}^{s} \alpha_i \sigma_i\right) := \displaystyle\sum_{i=1}^{s} \alpha_i h_n^\#(\sigma_i) \end{cases}$$

viewed as a map $\Delta_{n+1} \to Y$ after identifying the vertices of Δ with those of Δ_{n+1} (arbitrarily when $k = \mathbb{Z}/2\mathbb{Z}$)

We thus obtain the diagram:

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
C_{n+1}(X;k) \arrow[rr, shift left, "g^{\#}_{n+1}"] \arrow[rr, shift right, "f^{\#}_{n+1}"'] \arrow[d, "\partial^X_{n+1}"] & & C_{n+1}(Y;k) \arrow[d, "\partial^Y_{n+1}"] \\
C_n(X;k) \arrow[rr, shift left, "g^{\#}_n"] \arrow[rr, shift right, "f^{\#}_n"'] \arrow[d, "\partial^X_n"] \arrow[urr, "h^{\#}_n"] & & C_n(Y;k) \arrow[d, "\partial^Y_n"] \\
C_{n-1}(X;k) \arrow[rr, shift left, "g^{\#}_{n-1}"] \arrow[rr, shift right, "f^{\#}_{n-1}"'] \arrow[d] \arrow[urr, "h^{\#}_{n-1}"] & & C_{n-1}(Y;k) \arrow[d] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
C_0(X;k) \arrow[rr, shift left, "g^{\#}_0"] \arrow[rr, shift right, "f^{\#}_0"'] \arrow[d] & & C_0(Y;k) \arrow[d] \\
0 \arrow[rr, shift left, "0"] \arrow[rr, shift right, "0"'] \arrow[urr, "0"] & & 0
\end{tikzcd}$$

$\boxed{\text{claim:}}$ $\forall n \in \mathbb{N}$, $\partial^Y_{n+1} \circ h^{\#}_n + h^{\#}_{n-1} \circ \partial^X_n = f^{\#}_n - g^{\#}_n$.

↳ proof: on each simplex $\sigma : \Delta_n \to X$:

$$\partial^Y_{n+1} \circ h^{\#}_n(\sigma) = \partial^Y_{n+1}\Big(\sum_{\Delta \in T_{n+1}} \Gamma \circ \Sigma|_\Delta\Big) = \sum_{\substack{\Delta \text{ facet of} \\ \text{boundary} \\ \text{of prism}}} \Gamma \circ \Sigma|_\Delta$$

$$= \Gamma \circ \Sigma|_{\substack{\text{lower} \\ \text{boundary} \\ \text{of prism}}} + \Gamma \circ \Sigma|_{\substack{\text{upper} \\ \text{boundary} \\ \text{of prism}}} + \sum_{\substack{\Delta \text{ facet} \\ \text{of the lateral} \\ \text{boundary} \\ \text{of prism}}} \Gamma \circ \Sigma|_\Delta$$

$$= f^{\#}_n + g^{\#}_n + \sum_{\substack{\Delta \text{ facet of} \\ \text{the lateral} \\ \text{boundary} \\ \text{of prism}}} \Gamma \circ \Sigma|_\Delta.$$

$$= f^{\#}_n + g^{\#}_n + \sum_{\Delta' \text{ facet of } \Delta_n} \ \sum_{\substack{\Delta \text{ facet of} \\ \text{prism } [0,1] \times \Delta'}} \Gamma \circ \Sigma|_\Delta$$

$$= f^{\#}_n + g^{\#}_n + \sum_{\substack{\Delta' \text{ facet} \\ \text{of } \Delta_n}} h^{\#}_{n-1}(\sigma|_{\Delta'}) = f^{\#}_n + g^{\#}_n + h^{\#}_{n-1}\Big(\sum_{\substack{\Delta' \text{ facet} \\ \text{of } \Delta_n}} \sigma|_{\Delta'}\Big)$$

$$= f^{\#}_n + g^{\#}_n + h^{\#}_{n-1} \circ \partial^X_n(\sigma). \quad \blacksquare$$

Thus, $(f_n^{\#} - g_n^{\#})(\operatorname{Ker} \partial_n^X) = \partial_{n+1}^Y \circ h_n^{\#}(\operatorname{Ker} \partial_n) + h_{n-1}^{\#} \circ \partial_n^X(\operatorname{Ker} \partial_n)$

$$\subseteq \operatorname{Im} \partial_{n+1}^Y + 0$$

$$= \operatorname{Im} \partial_{n+1}^Y .$$

Hence, $f_n^* - g_n^* = 0$, i.e. $f_n^* = g_n^*$. ∎

$\boxed{\text{Corollary:}}$ $X \underset{\text{(homotopy)}}{\simeq} Y \Rightarrow H_n(X;k) \underset{\text{(iso.)}}{\simeq} H_n(Y;k) \quad \forall n \in \mathbb{N}$.

→ proof: Let $f : X \to Y$ and $g : Y \to X$ form a homotopy equivalence.

Then: $g \circ f \sim id_X \Rightarrow (g \circ f)_n^* = (id_X)_n^* = id_{H_n(X;k)}$.

$$(g \circ f)_n^* = g_n^* \circ f_n^*$$

$f \circ g \sim id_Y \Rightarrow (f \circ g)_n^* = (id_Y)_n^* = id_{H_n(Y;k)}$

$$(f \circ g)_n^* = f_n^* \circ g_n^*$$

Hence, f_n^* and g_n^* are inverse of each other, which implies that $H_n(X;k) \underset{\text{(iso.)}}{\simeq} H_n(Y;k)$. ∎