# Mode Seeking

**Input:** $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$.

**Hyp:** – the $p_i$ are sampled iid according to some unknown probability distribution $\mu$ with (unknown) density $f$ w.r.t. the Lebesgue measure,

– $f$ is regular enough, typically a Morse function: $C^2$– continuous, finitely many critical points, non-degenerate (Hessian matrix is non-singular), all distinct critical values.

Note: Morse functions are generic (dense, open) among $C^2$ functions.

Note: the gradient vector field $x \mapsto \nabla f(x)$ is locally Lipschitz $\Rightarrow$ it can be integrated into a gradient flow $\Phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ whose trajectories (starting at $x \in \mathbb{R}^d$) are solutions of the ODE

$$\begin{cases} \gamma_x'(t) = \nabla f \circ \gamma_x(t) \\ \gamma_x(0) = x \end{cases}$$

(Cauchy–Lipschitz Thm.) $\gamma_x : \mathbb{R}_+ \to \mathbb{R}^d$

**Thm** (from Morse theory) If $f$ is Morse, then almost every point of $\mathbb{R}^d$ ends up at a maximum of $f$ when following the gradient flow (integration of gradient vector field) of $f$.

$\to$ **principle:** cluster (almost) $\mathbb{R}^d$ by the ascending regions of the peaks of $f$.

$\mathrm{Asc}(p) = \{x \in \mathbb{R}^d \mid p \in \mathrm{Im}\, \gamma_x\}$.

$\hookrightarrow$ in practice: simulation by hill – climbing.