

CSC_51056_EP Topological Data Analysis

Final Exam — 3 hours

December 10, 2025

Important:

- The exercises and problem are independent of one another.
- The text of the exam is written in English. Your answers can be written indifferently in French or in English.
- Please keep in mind that the quality of your answers (completeness of the arguments and clarity of their exposition) will be key for the grading.
- All printed documents are allowed. By contrast, computers, cellphones, tablets, pocket calculators, etc., are forbidden.

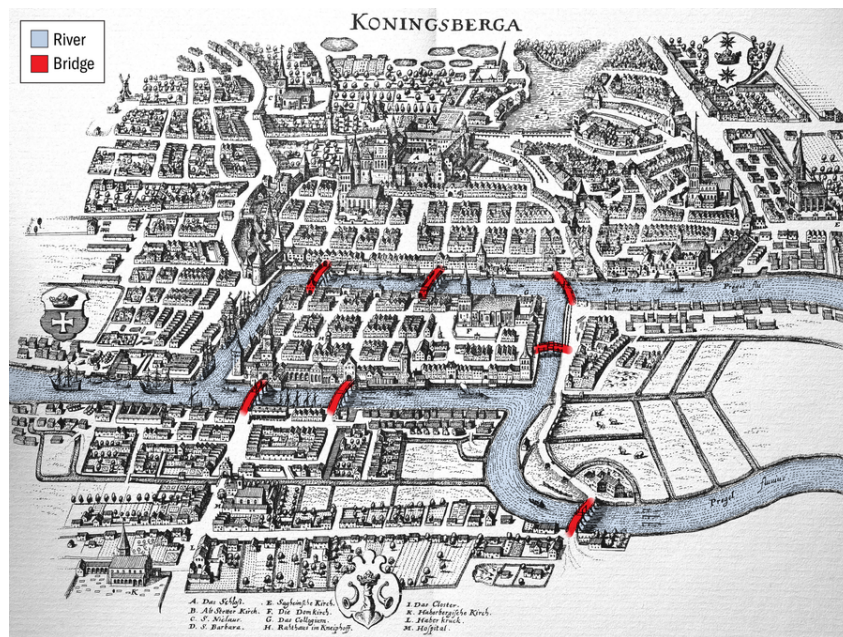


Figure 1: Map of Königsberg in Euler's time, with the seven bridges highlighted in red.
— Credits : Alamy Stock Photo (map), Amanda Montañez (highlighting).

1 The seven bridges of Königsberg

In Euler's time, the city of Königsberg (now Kaliningrad) had four districts connected to one another via seven bridges across the Pregola river—see Figure 1. Schematically, the city can be represented as in Figure 2.

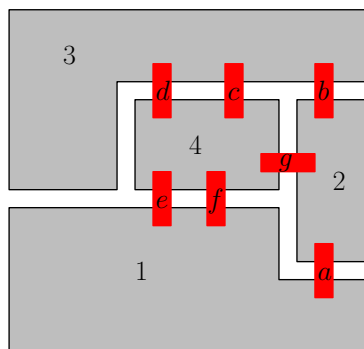


Figure 2: A schematic representation of the four districts (in grey) and seven bridges (in red) of Königsberg.

Question 1. From Figure 2, give a representation of the city as a 1-dimensional cell complex, with one vertex per district and one edge per bridge. You only need to draw the complex, not to describe its combinatorics.

Question 2. Is your cell complex a simplicial complex? If so then explain why. If not then also explain why and turn it into a simplicial complex.

Question 3. Compute formally the homology of the city of Königsberg over the field $\mathbb{Z}/2\mathbb{Z}$. Explain how you proceed and detail your calculations.

Euler's interest was in determining whether one can cross each bridge exactly once while walking through the city. In combinatorial terms, this means asking whether there is a path (i.e., a connected sequence of vertices and edges) in the complex of question 1 that traverses each edge exactly once. Such a path is called a *Eulerian walk*. Note that it is **not** required that the walk be a tour, i.e., that the starting and ending points of the walk be the same.

Euler solved the question in full generality, for any path-connected 1-dimensional cell complex. Here we are going to rephrase his approach in homological terms. Henceforth we fix our ground field to be $\mathbb{Z}/2\mathbb{Z}$ and we consider a path-connected 1-dimensional cell complex G . To each path

$$w = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \dots \xrightarrow{e_{r-1}} v_r$$

in G , possibly with repeated vertices or edges, we associate the simplicial 1-chain

$$\zeta(w) = \sum_{i=0}^{r-1} e_i \in C^1(G; \mathbb{Z}/2\mathbb{Z}).$$

Question 4. Suppose a 1-chain $\zeta \in C^1(G; \mathbb{Z}/2\mathbb{Z})$ comes from a path in G . Then, show that its boundary $\partial_1 \zeta \in C^0(G; \mathbb{Z}/2\mathbb{Z})$ is either zero or the sum of two distinct vertices.

Question 5. Does every 1-chain in $C^1(G; \mathbb{Z}/2\mathbb{Z})$ whose boundary is zero or the sum of two vertices come from a path in G ? If so then prove it, if not then provide a counterexample.

Question 6. Show that any Eulerian walk in G induces the 1-chain $(\sum_{e: \text{edge of } G} e) \in C^1(G; \mathbb{Z}/2\mathbb{Z})$.

Question 7. Is it true that any path in G that induces the 1-chain $(\sum_{e: \text{edge of } G} e) \in C^1(G; \mathbb{Z}/2\mathbb{Z})$ is a Eulerian walk? If so then prove it, if not then provide a counterexample.

Question 8. Does a Eulerian walk exist in the city of Königsberg? If so then provide it, if not then prove it.

Euler proved that, if a connected 1-dimensional cell complex G is such that $\partial_1(\sum_{e:\text{edge of } G} e)$ is either zero or the sum of two vertices in $C^0(G; \mathbb{Z}/2\mathbb{Z})$, then G admits a Eulerian walk. A constructive proof is given by Hierholzer's algorithm, which we will describe hereafter. First, a useful observation:

Question 9. What does our condition (namely, that $\partial_1(\sum_{e:\text{edge of } G} e)$ is either zero or the sum of two vertices in $C^0(G; \mathbb{Z}/2\mathbb{Z})$) mean in terms of the degrees of the vertices in G ? By *degree* of a vertex we mean the number of edges attached to it, with multiplicity (i.e., an edge that forms a loop at a vertex is counted twice for that vertex).

From now on we assume for simplicity that $\partial_1(\sum_{e:\text{edge of } G} e) = 0$, and we want to show that there exists a Eulerian tour, i.e., a Eulerian walk that ends at its starting point. Here is how Hierholzer's algorithm proceeds to compute such a tour (as per Wikipedia):

1. Choose any starting vertex v , and follow a path of edges from that vertex until eventually returning to v .
2. As long as there exists a vertex u that belongs to the current tour but that has adjacent edges not part of the tour, start another trail from u , following unused edges until returning to u , and then join the tour formed in this way to the previous tour.

Question 10. Show that it is impossible to get stuck at any vertex before returning to v during the tour construction at step 1. Explain why the same holds during every new tour construction at step 2.

Question 11. Show that, upon termination, the algorithm has exhausted all the edges of the complex G . Then conclude.

2 Some calculations...

Let the field of coefficients be $\mathbb{Z}/2\mathbb{Z}$. Consider the simplicial filtration depicted in Figure 3, in which every filtration value is assigned to a unique simplex.

Question 12. Compute the barcodes of this filtration in all homology degrees.

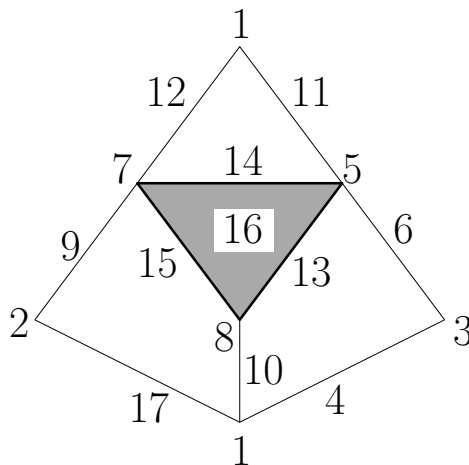


Figure 3: A simplicial filtration.

Question 13. Deduce the homology (over $\mathbb{Z}/2\mathbb{Z}$) of the topological space underlying the simplicial complex of Figure 3.

3 Decomposition of persistence modules

Let T be the set $\{0, 1, \dots, n\}$ equipped with the usual total order on integers, and let M be a pointwise finite-dimensional persistence module over T .

Question 14 (from the course). Show that M decomposes as a direct sum:

$$M \simeq \bigoplus_{j \in J} M^j$$

where each summand M^j is indecomposable.

We will now focus on a single indecomposable summand M^j and show that it is actually an interval module. From now on, we call M itself this indecomposable summand. We write M_i for the constituent vector space of M at index i , and $M_{i,j}: M_i \rightarrow M_j$ for the constituent linear map between indices $i \leq j$.

Without loss of generality, we assume that $M_0 \neq 0$. Let $x \in M_0 \setminus \{0\}$, and consider the submodule X of M generated by x , more precisely:

$$\begin{aligned} \forall 0 \leq i \leq n, \quad X_i &:= M_{0,i}(\langle x \rangle) \\ \forall 0 \leq i \leq j \leq n, \quad X_{i,j} &:= M_{i,j}|_{X_i}, \end{aligned}$$

where $\langle x \rangle$ denotes the 1-dimensional vector subspace of M_0 generated by x .

Question 15. Show that X is a well-defined persistence module.

Question 16. Show that X is an interval module.

Thus, we have identified an interval submodule X of M . We will now show that X is in fact a direct summand of M . If we knew a bit of homological algebra, then we could invoke the fact that X is an injective module, so any short exact sequence of the form $0 \rightarrow X \rightarrow M \rightarrow N \rightarrow 0$ splits, and we would be done. But here we must resort to a more down-to-earth argument and so we will identify an explicit direct-sum complement to X inside M .

Here is a candidate. Let

$$i_0 := \max \{0 \leq i \leq n \mid M_{0,i}(x) \neq 0\}.$$

Consider the submodule Y of M defined as follows:

$$Y_i := \begin{cases} M_i & \text{if } i > i_0 \\ Y_{i_0} \text{ such that } M_{i_0} = X_{i_0} \oplus Y_{i_0} & \text{if } i = i_0 \\ M_{i,i_0}^{-1}(Y_{i_0}) & \text{if } i < i_0, \end{cases}$$

with structural maps $Y_{i,j} := M_{i,j}|_{Y_i}$ for all $0 \leq i \leq j \leq n$.

Question 17. Show that Y is a well-defined persistence module.

Question 18. Show that the submodules X and Y are in direct sum.

Question 19. Show that $M = X + Y$.

This shows that X is a summand of M . Since we assumed M to be indecomposable in first place, we must have $M = X$ and so M is an interval module, as expected.

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