INF556 Topological Data Analysis
Final Exam — 3 hours

December 22, 2023

Important:
- The exercises are independent of one another.
- The text of the exam is written in English. Your answers can be written indifferently in French or in English.
- Please keep in mind that the quality of your answers (completeness of the arguments and clarity of their exposition) will be key for the grading.
- All printed documents are allowed, but all electronic devices are prohibited.

1 Ghost encounter

Question 1. Draw the Reeb graph of the height function on the ghost shown in Figure 1.

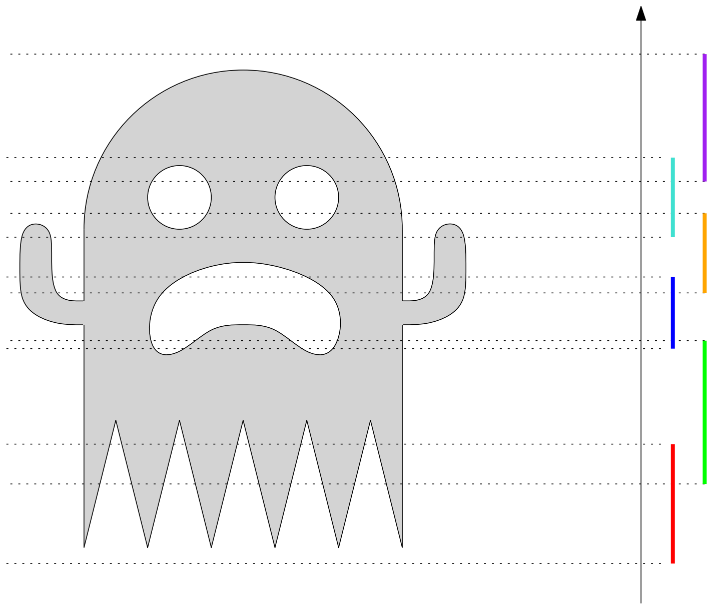

Figure 1: An evil ghost appears! Can you compute its Reeb graph and Mapper to ward it off?

Question 2. Draw the corresponding extended persistence diagram.

Question 3. Draw the Mapper with the cover displayed on the right of Figure 1.

Question 4. Draw the staircase corresponding to the cover and the extended persistence diagram of the Mapper.

2 Pac-Man

Question 5. What is the homology (in all degrees) of the labyrinth in which Pac-Man is moving (Figure 2 left), including the center cell with the ghosts? Give a simple high-level argument to justify your answer. Note that Pac-Man itself, ghosts and white dots (whether big or small) must not be considered as obstacles, while the closed white door of the center cell must be. Note also that the ends of the left and right corridors are connected to each other, so that when Pac-Man disappears at one end he reappears at the other.

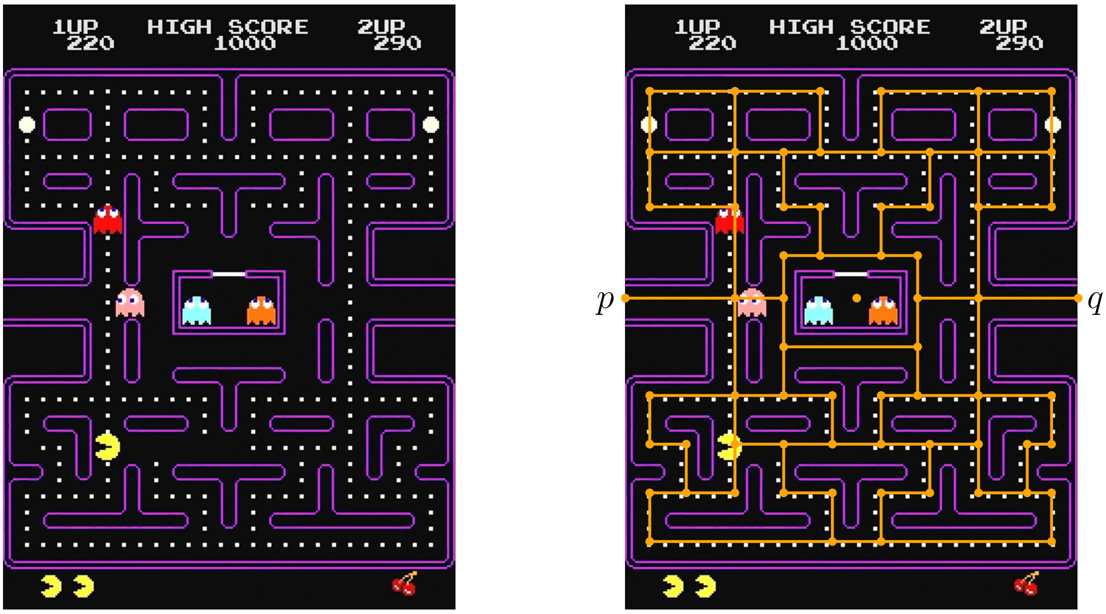

Figure 2: Pac-Man is in trouble! can you help him find his way through the labyrinth?

Question 6. Explain why the homology of the labyrinth is the same as that of the overlaid orange graph in Figure 2 right, in which vertices p, q are identified. For this you can use the following fact: *every bounded planar open set has the same homotopy type as its medial axis.*

Question 7. Use the graph to formally compute the homology (in all degrees) of the labyrinth. Choose your order of insertion well, so that your calculation scales up properly!

3 Fixed points on spheres and projective spaces

Recall Brouwer's fixed point theorem on the closed disk $\mathbb{B}^2$: *every continuous map $f : \mathbb{B}^2 \to \mathbb{B}^2$ has at least one fixed point, i.e., a point $x \in \mathbb{B}^2$ such that $f(x) = x$.* This result does not extend verbatim to the sphere:

Question 8. Give an example of map $f : \mathbb{S}^2 \to \mathbb{S}^2$ that is continuous but has no fixed point.

Nevertheless, a weaker version of Brouwer's theorem holds on the sphere:

Question 9. Show that every continuous map $f : \mathbb{S}^2 \to \mathbb{S}^2$ admits at least one point $x \in \mathbb{S}^2$ such that $f(x) = x$ or $f(x) = -x$.
Hint: you can use the Hairy Ball theorem (*every continuous tangent vector field on $\mathbb{S}^2$ has at least one vanishing point*).

Question 10. Deduce from the previous question that Brouwer's theorem holds on the projective plane $\mathbb{RP}^2$, i.e., every continuous map $\mathbb{RP}^2 \to \mathbb{RP}^2$ has at least one fixed point.
Hint: you can use the following lifting property from homotopy theory[1]: *every continuous map $\phi : \mathbb{S}^2 \to \mathbb{RP}^2$ lifts to a continuous map $\tilde{\phi} : \mathbb{S}^2 \to \mathbb{S}^2$, i.e., $\phi = q_\sim \circ \tilde{\phi}$ where $q_\sim : \mathbb{S}^2 \to \mathbb{RP}^2$ is the quotient map associated to the antipodal equivalence relation: $x \sim y$ iff $x = -y$.*

Note that the above facts hold as well when 2 is replaced by any even dimension d.

4 Persistent homology

Consider the simplicial filtration depicted in Figure 3.

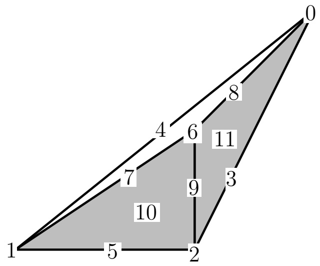

Figure 3: A simplicial filtration.

Question 11. Write the persistence module induced in homology (over $\mathbb{Z}/2\mathbb{Z}$) by this filtration. You can represent its constituent morphisms in matrix form, after choosing bases for its constituent spaces.

Question 12. Compute the boundary matrix of the filtration, apply the Gaussian elimination algorithm to reduce it, then deduce the barcode of the filtration.

5 Bottleneck distance

Question 13. Show that the bottleneck distance satisfies the triangle inequality.

Question 14. Show that the bottleneck distance is only a pseudodistance, that is: exhibit a pair of distinct persistence diagrams whose distance is zero.

6 The natural pseudo-distance

In 2003, Fros*n* and coauthors proposed the following generalization of the supremum norm to functions defined over different domains:

Definition 1. The *natural pseudo-distance* between $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ is defined by

$$\mathrm{d_n}(f, g) = \inf_{h:X\to Y} \|f - g \circ h\|_\infty,$$

where h ranges over all possible homeomorphisms $X \to Y$, and where $\|\cdot\|_\infty$ denotes the usual supremum norm for real-valued functions on X. Note that the pseudo-distance is $+\infty$ when X, Y are not homeomorphic.

[1]This lifting property comes from the facts that $(\mathbb{S}^2, q_\sim)$ is a covering space of $\mathbb{RP}^2$ with trivial fundamental group, and that $\mathbb{S}^2$ is both path-connected and locally path-connected—see e.g. Proposition 1.33 in [Hatcher].

Fros*n* and his friends then went on to prove the following stability result, given a fixed field $\mathbf{k}$:

Theorem 1. *For any tame*[2] *functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$, one has*

$$\mathrm{d}_{\mathrm{b}}^{\infty}(\mathrm{Dgm}\ f, \mathrm{Dgm}\ g) \leq \mathrm{d}_{\mathrm{n}}(f, g).$$

We will show that this is in fact an immediate consequence of the stability theorem seen in class. From now on, let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be fixed tame functions.

Question 15. Given a homeomorphism $h : X \to Y$, show that the map $g \circ h : X \to \mathbb{R}$ is tame.

Question 16. Given a homeomorphism $h : X \to Y$, show that $\mathrm{Dgm}\ g \circ h = \mathrm{Dgm}\ g$.

Question 17. Deduce now the result of Theorem 1.

It is interesting to see if this result extends to weaker notions of equivalence between topological spaces, such as homotopy equivalence:

Question 18. Consider a homotopy equivalence, given by two maps $h : X \to Y$ and $h' : Y \to X$ such that $h' \circ h \sim \mathrm{id}_X$ and $h \circ h' \sim \mathrm{id}_Y$. Do we still have $\mathrm{Dgm}\ g \circ h = \mathrm{Dgm}\ g$? If so, then give a proof, otherwise give a counter-example.

[2] Recall that $f : X \to \mathbb{R}$ is tame if $\dim \mathrm{H}_*(f^{-1}((-\infty, t]); \mathbf{k}) < +\infty$ for all $t \in \mathbb{R}$.

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