An Easy Way to Teach Interior Point Methods

Tamás Terlaky

Faculty of Information Technology and Systems,
Department of Statistics, Stochastic and Operations Research
Delft University of Technology
P.O.Box 5031, 2600 GA Delft, The Netherlands
E-mail: t.terlaky@twi.tudelft.nl

A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details. Weyl, Hermann (1885 - 1955)

Abstract

In this paper the duality theory of Linear Optimization (LO) is built up based on ideas emerged from interior point methods. All we need is elementary calculus. We will embed the LO problem and its dual in a self–dual skew–symmetric problem. Most duality results, except the existence of a strictly complementary solution, are trivial for this embedding problem. The existence of the central path and its convergence to the analytic center of the optimal face will be proved. The proof is based on an elementary, careful analysis of a Newton step.

We show also that if an almost optimal solution on the central path is known, then a simple strongly polynomial rounding procedure provides a strictly complementary optimal solution.

The all-one vector is feasible for the embedding problem and it is an interior point on the central path. This way an elegant solution to the initialization of IPMs is obtained as well. This approach allows to apply any interior point method to the embedding problem while complexity results obtained for feasible interior point methods are preserved.

Keywords: Linear optimization, interior-point methods, self-dual embedding, strictly complementary solution, strongly polynomial rounding procedure, polynomial complexity.

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1 Introduction

Interior point methods (IPMs) are among the most efficient methods for solving linear, and wide classes of convex optimization problems. Since the path-breaking work of Karmarkar [15], much research was invested in IPMs. Many algorithmic variants were developed for Linear Optimization (LO). The new approach forced to reconsider all aspects of optimization problems. Not only the research on algorithms and complexity issues, but implementation strategies, duality theory and research on sensitivity analysis got also a new impulse. After more than a decade of turbulent research, the IPM community reached a good understanding of the basics of IPMs. Several books have been published in the last years that summarize and explore different aspects of IPMs. The seminal work of Nesterov and Nemirovskii [21] provides the most general frame for polynomial IPMs for convex programming. Den Hertog [11] gives a thorough survey of primal and dual path-following IPMs for linear and structured convex optimization problems. Jansen [12] discusses primal-dual target following algorithms for linear optimization and complementarity problems. Wright [31] also concentrates on primal-dual IPMs, with special attention on infeasible IPMs, numerical issues and local, asymptotic convergence properties. The volume [28] contains 13 survey papers that cover almost all aspects of IPMs, their extensions and some applications. The book of Ye [34] is a rich source of polynomial IPMs not only for LO, but for convex optimization problems as well. He extends the IPM theory to derive bounds and approximations for classes of nonconvex optimization problems as well. Finally, Roos, Terlaky and Vial [25] present a thorough treatment of the IPM based theory – duality, complexity, sensitivity analysis – and wide classes of IPMs for LO. This book provides the basis for our discussions in this paper.

Before going in a detailed discussion of our approach, some remarks are made on implementations of IPMs and on extensions and generalizations.

IPMs are implemented with great success in recent years. It is now a common sense, that for large scale, sparse, structured LO problems, IPMs are the method of choice. All leading optimization software systems, like CPLEX, XPRESS-MP and OSL contain implementations of IPMs. The reader can find thorough discussions of implementation strategies in the following papers: [2, 16, 18, 32]. The books [25, 31, 34] devote also a chapter to that subject.

Some of the earlier mentioned books [11, 12, 21, 28, 34] discuss extensions of IPMs for classes of nonlinear problems. In recent years the majority of research is devoted to IPMs for Semidefinite Optimization (SDO). SDO has a wide range of interesting applications not only in such traditional areas as combinatorial optimization [1], but also in control, and different areas of engineering, more specifically structural [8] and electrical engineering [30]. For surveys on algorithmic and complexity issues the reader may consult [5, 6, 7, 4, 21, 22, 24, 27].

Teaching Interior Point Methods

After years of intensive research a deep understanding of IPMs is developed. There are easy to understand, simple variants of polynomial IPMs. The self-dual embedding strategy [13, 25, 35] provides an elegant solution for the initialization problem of IPMs. It is also possible to build up not only the complete duality theory of [25] of LO, but to perform sensitivity analysis [12, 14, 20, 25] on the basis of IPMs. We also demonstrate that IPMs not only converge to an optimal solution (if it exists), but after a finite number of iterations also allow a strongly polynomial rounding procedure [19, 25] to generate exact solutions. This all requires only the knowledge of elementary calculus and can be taught not only in a graduate, but at an advanced undergraduate level as well. Our aim is to present such an approach, based on the one presented in [25].

The paper is structured as follows. First, in Section 2 we briefly review the general LO problem in canonical form and discuss how Goldman and Tucker's [3, 29] self-dual and homogeneous model is derived. In Section 3 the Goldman-Tucker theorem, i.e. the existence of a strictly complementary solution for the skew-symmetric self-dual model will be proved. Here such basic IPM objects, as the interior solution, the central path, the Newton step, the analytic center of polytopes will be introduced. We will show in Section 3.6 that the central path converges to a strictly complementary solution, and in Section 3.7 that an exact strictly complementary solution for LO, or a certificate for infeasibility can be obtained after a finite number of iterations. Our theoretical development is summarized in Section 4. Finally, in Section 5 a general scheme of IPM algorithms is presented.

Notation
$\mathbb{R}^n_+$ will denote the set of nonnegative vectors in $\mathbb{R}^n$. Throughout, we shall use $\|\cdot\|_p$ ($p \in \{1, 2, \infty\}$) to denote the p-norm on $\mathbb{R}^n$, with $\|\cdot\|$ denoting the Euclidean norm $\|\cdot\|_2$. E will denote the identity matrix, e will be used to denote the vector which has all its components equal to one. Given an n-dimensional vector x, we denote by X the $n \times n$ diagonal matrix whose diagonal entries are the coordinates x_j of x. If $x,\ s \in \mathbb{R}^n$ then $x^T s$ denotes the dot product of the two vectors. Further, xs, x^α for $\alpha \in \mathbb{R}$ and $\max\{x, y\}$ will denote the vectors resulting from coordinatewise operations. For any matrix $A \in \mathbb{R}^{m \times n}$, A_j denotes the j-th column of A. Furthermore,

$$\pi(A) := \prod_{j=1}^{n} \|A_j\|.$$

For any index set $J \subset \{1, 2, \ldots, n\}$, $|J|$ denotes the cardinality of J and $A_J \in \mathbb{R}^{m \times |J|}$ the submatrix of A whose columns are indexed by elements in J. Moreover, if $K \subset \{1, 2, \ldots, m\}$, $A_{KJ} \in \mathbb{R}^{|K| \times |J|}$ is the submatrix of A_J whose rows are indexed by elements in K.

2 The Linear Optimization Problem

We consider the general LO problem (P) and its dual (D) given in canonical form:

$$\begin{array}{ll} (P) \qquad & \min\left\{ c^T u \ : \ Au \geq b, \ \ u \geq 0 \right\}, \\ (D) \qquad & \max\left\{ b^T v \ : \ A^T v \leq c, \ \ v \geq 0 \right\}, \end{array}$$

where A is an $m \times k$ matrix, $b, v \in \mathbb{R}^m$ and $c, u \in \mathbb{R}^k$. It is well known that by using only elementary transformations, any given LO problem can easily be transformed into a "minimal" canonical form. These transformations can be summarized as follows:

- introduce slacks in order to get equations (if a variable has a lower and an upper bound, then one or these bounds is considered as an inequality constraint);
- shift the variables with lower or upper bound so that the respective bound becomes 0 and, if needed replace the variable by its negative;
- eliminate free variables;
- use Gaussian elimination to transform the problem into a form where all equations have a singleton column (i.e. choose a basis and multiply the equations by the inverse basis) while dependent constraints are eliminated.

The weak duality theorem for the canonical LO problem is easily proved.

Theorem 1 *Let us assume that $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^m$ are feasible solutions for the primal problem (P) and dual problem (D), respectively. Then one has*

$$c^T u \geq b^T v$$

where equality holds if and only if
(*i*) $u_i(c - A^T v)_i = 0$ *for all* $i = 1, \cdots, k$ *and*
(*ii*) $v_j(Au - b)_j = 0$ *for all* $j = 1, \cdots, m$.[1]

Proof: Using primal and dual feasibility of u and v we may write

$$(c - A^T v)^T u \geq 0 \qquad \text{and} \qquad v^T(Au - b) \geq 0$$

with equality if and only if (*i*), respectively (*ii*) holds. Summing up these two inequalities we have the desired inequality

$$0 \leq (c - A^T v)^T u + v^T(Au - b) = c^T u - b^T v.$$

The theorem is proved. □

One easily derives the following sufficient condition for optimality.

Corollary 2 *Let a primal and dual feasible solution* $u \in \mathbb{R}^n$ *and* $v \in \mathbb{R}^m$ *with* $c^T u = b^T v$ *be given. Then* u *is an optimal solution of the primal problem* (P) *and* v *is an optimal solution of the dual problem* (D). □

The Weak Duality Theorem 1 provides a sufficient condition to check optimality of a feasible solution pair. However, it does not guarantee that, in case of feasibility, an optimal pair with zero duality gap always exists. This is the content of the so-called Strong Duality Theorem that we are going to prove in the next sections by using only simple calculus and basic concepts of IPMs.

As we are looking for optimal solutions of the LO problem with zero duality gap, we need to find a solution of the system formed by the primal and the dual feasibility constraints and by requiring that the dual objective is at least as large as the primal one. By the Weak Duality Theorem 1 we know that any solution of this system is both primal and dual feasible with equal objective values. Thus, by the corollary, they are optimal. By introducing appropriate slack variables the following inequality system is derived.

$$\begin{aligned} Au - z &= b, \quad u \geq 0, \quad z \geq 0 \\ A^T v + w &= c, \quad v \geq 0, \quad w \geq 0 \\ b^T v - c^T u - \rho &= 0, \quad \rho \geq 0. \end{aligned}$$

By homogenizing, the *Goldman-Tucker model* [3, 29] is obtained.

$$\begin{array}{rrrrrl} & Au & -\tau b & -z & & = 0, \quad u \geq 0, \quad z \geq 0 \\ -A^T v & & +\tau c & & -w & = 0, \quad v \geq 0, \quad w \geq 0 \\ b^T v & -c^T u & & & -\rho & = 0, \quad \tau \geq 0, \quad \rho \geq 0. \end{array}$$

This homogeneous system admits the trivial zero solution, but that has no value for our discussions. We are looking for some specific nontrivial solutions of this Goldman-Tucker system.

[1] These conditions are in general referred to as the *complementarity conditions*. Using the coordinatewise notation we may write $u(c - A^T v) = 0$ and $v(Au - b) = 0$. By the weak duality theorem complementarity and feasibility imply optimality.

Clearly any solution with $\tau > 0$ gives a primal and dual optimal pair $(\frac{u}{\tau}, \frac{v}{\tau})$ with zero duality gap, hence ρ must be zero if $\tau > 0$. On the other hand, any optimal pair (u, v) with zero duality gap is a solution of the Goldman-Tucker system with $\tau = 1$ and $\rho = 0$.

One easily verifies that if (v, u, τ, z, w, ρ) is a solution of the Goldman-Tucker system then $\tau\rho > 0$ cannot hold. Indeed, if $\tau\rho$ would be positive then the we would have

$$0 < \tau\rho = \tau b^T v - \tau c^T u = u^T A v - z^T v - u^T A^T v - w^T u = -z^T v - w^T u \leq 0$$

yielding a contradiction.

Finally, if the Goldman-Tucker system admits a feasible solution $(\bar{v}, \bar{u}, \bar{\tau}, \bar{z}, \bar{w}, \bar{\rho})$ with $\bar{\tau} = 0$ and $\bar{\rho} > 0$, then we may conclude that either (P), or (D), or both of them are infeasible. Indeed, $\bar{\tau} = 0$ implies that $A\bar{u} \geq 0$ and $A^T\bar{v} \leq 0$. Further, if $\bar{\rho} > 0$ then we have either $b^T\bar{v} > 0$, or $c^T\bar{u} < 0$, or both. If $b^T\bar{v} > 0$, then by assuming that there is a feasible solution $u \geq 0$ for (P) we have

$$0 < b^T\bar{v} \leq u^T A^T \bar{v} \leq 0$$

which is a contradiction, thus if $b^T\bar{v} > 0$, then (P) must be infeasible. Similarly, if $c^T\bar{u} < 0$, then by assuming that there is a dual feasible solution $v \geq 0$ for (D) we have

$$0 > c^T\bar{u} \geq v^T A\bar{u} \geq 0$$

which is a contradiction, thus if $c^T\bar{u} > 0$, then (D) must be infeasible.

Summarizing the results obtained so far, we have the following theorem.

Theorem 3 *Let a primal dual pair (P) and (D) of LO problems be given. The following statements hold.*

1. *Any optimal pair (u, v) of (P) and (D) with zero duality gap is a solution of the corresponding Goldman-Tucker system with $\tau = 1$.*
2. *If (v, u, τ, z, w, ρ) is a solution of the Goldman-Tucker system then either $\tau = 0$ or $\rho = 0$, i.e. $\tau\rho > 0$ cannot happen.*
3. *Any solution (v, u, τ, z, w, ρ) of the Goldman-Tucker system, where $\tau > 0$ and $\rho = 0$, gives a primal and dual optimal pair $(\frac{u}{\tau}, \frac{v}{\tau})$ with zero duality gap.*
4. *If the Goldman-Tucker system admits a feasible solution $(\bar{v}, \bar{u}, \bar{\tau}, \bar{z}, \bar{w}, \bar{\rho})$ with $\bar{\tau} = 0$ and $\bar{\rho} > 0$, then we may conclude that either (P), or (D), or both of them are infeasible.* $\square$

Our interior point approach will lead us to a solution of the Goldman-Tucker system, where either $\tau > 0$ or $\rho > 0$, avoiding the undesired situation when $\tau = \rho = 0$.

Before proceeding, we simplify our notations. Observe, that the Goldman-Tucker system can be written in the following compact form

$$Mx \geq 0, \qquad x \geq 0, \qquad s(x) = Mx, \tag{1}$$

where

$$x = \begin{pmatrix} v \\ u \\ \tau \end{pmatrix}, \qquad s(x) = \begin{pmatrix} z \\ w \\ \rho \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{pmatrix}$$

is a skew-symmetric matrix, i.e. $M^T = -M$. The Goldman-Tucker Theorem [3, 25, 29] says that system (1) admits a strictly complementary solution. This theorem will be proven in the next section.

Theorem 4 (Goldman, Tucker) *There is a strictly complementary feasible solution x of (1), i.e. for which $x + s(x) > 0$.*

Observe, that this theorem ensures that either case 3 or case 4 of Theorem 3 must occur when one solves the Goldman-Tucker system of LO. This is in fact the strong duality theorem of LO.

Theorem 5 *Let a primal and dual LO problem be given. Exactly one of the following statements hold:*

- *Either problem (P), or (D), or both are infeasible.*
- *(P) and (D) are feasible and there are optimal solutions u^* and v^* such that*
$$c^T u^* = b^T v^*.$$

Proof: Theorem 4 implies that the Goldman-Tucker system of the LO problem admits a strictly complementary solution. Thus, in such a solution, either $\tau > 0$, and in that case item 3 of Theorem 3 implies the existence of an optimal pair with zero duality gap. On the other hand, when $\rho > 0$, item 4 of Theorem 3 proves that either (P) or (D) or both are infeasible. $\square$

Our next goal is to give an elementary constructive proof of Theorem 4. When this project is finished, we have the complete duality theory for LO.

3 The skew-symmetric self-dual model

3.1 Basic properties of the skew-symmetric self-dual model

Following the approach in [25] we make our skew-symmetric model (1) a bit more general. Thus our prototype problem is

$$(SP) \qquad \min \left\{ q^T x \ : \ Mx \geq -q, \quad x \geq 0 \right\},$$

where the matrix $M \in \mathbb{R}^{n \times n}$ is *skew symmetric* and $q \in \mathbb{R}^n_+$. The set of feasible solutions of (SP) is denoted by

$$SP = \{ x \ : \ x \geq 0, \ Mx \geq -q \ \}.$$

By using the assumption that the coefficient matrix M is skew-symmetric and the right-hand-side vector $-q$ is the negative of the objective coefficient vector, one easily verifies that the dual of (SP) is equivalent to (SP) itself, i.e. the problem (SP) is *self-dual.* Due to the self-dual property the following result is trivial.

Lemma 6 *The optimal value of (SP) is zero and (SP) admits the zero vector $x = 0$ as a feasible and optimal solution.*

Given $(x, s(x))$, where $s(x) = Mx + q$ we may write

$$q^T x = x^T (s(x) - Mx) = x^T s(x) = e^T (x s(x)),$$

i.e. for any optimal solution $e^T(x(s(x))) = 0$ implying that the vectors x and $s(x)$ are complementary. For further use, the *optimal set* of (SP) is denoted by

$$SP^* := \{x \ : \ x \geq 0, \ s(x) \geq 0, \ xs(x) = 0\}.$$

A useful property of optimal solutions is given by the following lemma.

Lemma 7 *Let x and y be feasible for (SP). Then x and y are optimal if and only if*

$$xs(y) = ys(x) = xs(x) = ys(y) = 0.$$

Proof: Because M is skew-symmetric we have $(x-y)^T M(x-y) = 0$ which implies that $(x-y)^T(s(x) - s(y)) = 0$. Hence $x^T s(y) + y^T s(x) = x^T s(x) + y^T s(y)$ and this vanishes if and only if x and y are optimal. $\square$

Thus, optimal solutions are complementary in the general sense, i.e. they are not only complementary w.r.t. their own slack vector, but complementary w.r.t. the slack vector for any other optimal solution as well.

All of the above results, including to find a trivial optimal solution were straightforward for (SP). The only nontrivial result what we need to prove is the existence of a strictly complementary solution.

First we prove the existence of a strictly complementary solution if the so-called interior point condition holds.

Assumption 8 (Interior Point Condition (IPC)) *There exists an $x^0 \in SP$ such that*

$$(x^0, s(x^0)) > 0.$$

Before proceeding, we show that this condition can be assumed without loss of generality. If the reader is eager to know the proof of the existence of a strictly complementary solution for the self dual model (SP), he/she might temporarily skip the following subsection and return to it when all the results for the problem (SP) are derived under the IPC.

3.2 IPC for the Goldman-Tucker model

Recall that (SP) is just the abstract model of the Goldman-Tucker problem (1) and our goal is to prove Theorem 4. In order to apply the results of the coming sections we need to modify problem (1) so that the resulting equivalent problem satisfies the IPC.

Self-dual embedding of (1) with IPC

Due to the second statement of Theorem 3, problem (1) cannot satisfy the IPC. However, because problem (1) is just a homogeneous feasibility problem, it can be transformed into an equivalent problem (SP) which satisfies the IPC. This happens by enlarging, i.e. embedding the problem and defining an appropriate nonnegative vector q.

Let us take $x = s(x) = e$. These vectors are positive, but they do not satisfy (1). Let us further define the error vector r obtained this way by

$$r := e - Me, \quad \text{and let} \quad \lambda := n + 1.$$

Then we have

$$\begin{pmatrix} M & r \\ -r^T & 0 \end{pmatrix} \begin{pmatrix} e \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda \end{pmatrix} = \begin{pmatrix} Me + r \\ -r^T e + \lambda \end{pmatrix} = \begin{pmatrix} e \\ 1 \end{pmatrix}.$$

Hence, the following problem

$$\overline{(SP)} \qquad \min \left\{ \lambda\vartheta \; : \; -\begin{pmatrix} M & r \\ -r^T & 0 \end{pmatrix} \begin{pmatrix} x \\ \vartheta \end{pmatrix} + \begin{pmatrix} s \\ \nu \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}; \; \begin{pmatrix} x \\ \vartheta \end{pmatrix}, \begin{pmatrix} s \\ \nu \end{pmatrix} \geq 0 \right\}$$

satisfies the IPC because for this problem the all-one vector is feasible. This problem is in the form of (SP), where

$$\overline{M} = \begin{pmatrix} M & r \\ -r^T & 0 \end{pmatrix}, \qquad \bar{x} = \begin{pmatrix} x \\ \vartheta \end{pmatrix} \qquad \text{and} \qquad \bar{q} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}.$$

We claim that finding a strictly complementary solution to (1) is equivalent to finding a strictly complementary optimal solution to problem $\overline{(SP)}$. This claim is valid, because $\overline{(SP)}$ satisfies the IPC thus, as we will see it admits a strictly complementary optimal solution. Because the objective function is just a constant multiple of ϑ, this variable must be zero in any optimal solution, by Lemma 6. This observation implies the claimed result.

Conclusion: Every LO problem can be embedded in a self-dual problem $\overline{(SP)}$ of the form (SP). This can be done in such a way that $\bar{x} = e$ is feasible for $\overline{(SP)}$ and $\bar{s}(e) = e$. Having a strictly complementary solution of (SP) we either find an optimal solution of the embedded LO problem or that the LO problem does not have an optimal solution.

After this intermezzo, we return to the study of or our prototype problem (SP) by assuming the IPC.

3.3 The level sets of (SP)

Let $x \in SP$ and $s = s(x)$ be a *feasible pair*. Due to self duality, the *duality gap* for this pair is twice the value

$$q^T x = x^T s,$$

however, for the sake of simplicity, the quantity $q^T x = x^T s$ itself will be referred to as *the duality gap*. First we show that the IPC implies the boundedness of the level sets.

Lemma 9 *Let the IPC be satisfied. Then, for each positive K, the set of all feasible pairs (x, s) such that $x^T s \leq K$, is bounded.*

Proof: Because the matrix M is skew-symmetric, we may write

$$\begin{aligned} 0 = (x - x^0)^T M (x - x^0) &= (x - x^0)^T (s - s^0) \\ &= x^T s + (x^0)^T s^0 - x^T s^0 - s^T x^0. \end{aligned}$$

From here we get

$$x_j s_j^0 \leq x^T s^0 + s^T x^0 = x^T s + (x^0)^T s^0 \leq K + (x^0)^T s^0.$$

The proof is complete. $\square$

In particular, this lemma implies that the set of optimal solutions SP^* is bounded as well.[2]

3.4 Central path, optimal partition

First we define the central path [9, 10, 17, 26] of (SP).

Definition 11 *The set of solutions*

$$\{(x(\mu), s(x(\mu))) \ : \ Mx + q = s, \ \ xs = \mu e, \ \ x > 0 \quad \textit{for some} \quad \mu > 0\}$$

is called the central path *of* (SP).

If no confusion is possible, instead of $s(x(\mu))$ the notion $s(\mu)$ will be used. Now we are ready to present our main theorem. This in fact establishes the existence of the central path. At this point our discussion deviates from the one presented in [25]. The proof presented here is more elementary because it does not make use of the logarithmic barrier function.

Theorem 12 *The next statements are equivalent.*
(i) (SP) satisfies the interior point condition;
(ii) For each $\mu > 0$ there exists $(x(\mu), s(\mu)) > 0$ such that

$$\begin{aligned} Mx + q &= s \\ xs &= \mu e. \end{aligned}$$

(iii) For $w > 0$ there exists $(x, s) > 0$ such that

$$\begin{aligned} Mx + q &= s \\ xs &= w. \end{aligned}$$

The solution of these systems are unique.

Before proving this highly important result we introduce the notion of optimal partition and present our main result. The partition (B, N) of the index set $\{1, ..., n\}$ given by

$$\begin{aligned} B := \{i \ : \ x_i > 0, \ \text{ for some } \ x \in SP^*\}. \\ N := \{i \ : \ s(x)_i > 0, \ \text{ for some } \ x \in SP^*\}. \end{aligned}$$

is called the *optimal partition*. By Lemma 7 the sets B and N are disjoint. Our main result says that the central path converges to a strictly complementary optimal solution, and this result proves that $B \cup N = \{1, ..., n\}$. When this result is established, the Goldman-Tucker Theorem 4 for the general LO problem is proved because we use the embedding method presented in Subsection 3.2.

[2] The following result shows that the IPC not only implies the boundedness of the level sets, but the converse is also true. We do not need this property in developing our main results, so this is presented without proof.

Corollary 10 *Let (SP) be feasible. Then the following statements are equivalent:*

(i) the interior point condition is satisfied;

(ii) the level sets of $x^T s$ are bounded;

(iii) the optimal set SP^* of (SP) is bounded. $\square$

Theorem 13 *If the IPC holds then there exists an optimal solution x^* and $s^* = s(x^*)$ of problem (SP) such that $x_B^* > 0$, $s_N^* > 0$ and $x^* + s^* > 0$.*

First we prove Theorem 12.

Proof: We start the proof by demonstrating that the systems in (ii) and (iii) may have at most one solution. Because (ii) is a special case of (iii), it is sufficient to prove uniqueness for (iii).

Let us assume to the contrary that for a certain $w > 0$ there are two vectors $(x, s) \neq (\bar{x}, \bar{s}) > 0$ solving (iii). Then using that the matrix M is skew-symmetric, we may write

$$0 = (x - \bar{x})^T M (x - \bar{x}) = (x - \bar{x})^T (s - \bar{s}) = \sum_{x_i \neq \bar{x}_i} (x - \bar{x})_i (s - \bar{s})_i.$$

Due to $xs = w = \bar{x}\bar{s}$ we have

$$x_i < \bar{x}_i \Longleftrightarrow s_i > \bar{s}_i$$
$$x_i > \bar{x}_i \Longleftrightarrow s_i < \bar{s}_i.$$

By considering these sign properties one easily verifies that the relation

$$0 = \sum_{x_i \neq \bar{x}_i} (x - \bar{x})_i (s - \bar{s})_i < 0$$

should hold, but this is an obvious contradiction. As a result, we may conclude that if the systems in (ii) and (iii) admit a feasible solution, then such a solution is unique.

The Newton step

In proving the existence of a solution for the systems in (ii) and (iii) our main tool will be to analyze the Newton step when applied to the nonlinear systems in (iii).[3]

Let a vector $(x, s) > 0$ with $s = Mx + q$ be given. For a particular $w > 0$ one will find the displacements $(\Delta x, \Delta s)$ that solve

$$M(x + \Delta x) + q = s + \Delta s$$
$$(x + \Delta x)(s + \Delta s) = w.$$

This reduces to

$$M \Delta x = \Delta s$$
$$x \Delta s + s \Delta x + \Delta x \Delta s = w - xs.$$

This equation system is still nonlinear. When we neglect the second order term $\Delta x \Delta s$ the *Newton equation*

$$M \Delta x = \Delta s$$
$$x \Delta s + s \Delta x = w - xs$$

[3]Observe that no preliminary knowledge on any variants of Newton's method is assumed. We just define and analyze the Newton step for our particular situation.

is obtained. This is a linear equation system and the reader easily verifies that the *Newton direction* Δx is the solution of the nonsingular system of equations[4]

$$(M + X^{-1}S)\Delta x = x^{-1}w - s.$$

When we perform a step in the Newton direction with step-length α, for the new solutions (x^+, s^+) we have

$$\begin{aligned} x^+s^+ &:= (x + \alpha\Delta x)(s + \alpha\Delta s) = xs + \alpha(x\Delta s + s\Delta x) + \alpha^2\Delta x\Delta s \\ &= xs + \alpha(w - xs) + \alpha^2\Delta x\Delta s. \end{aligned}$$

This relation clarifies that the local change of xs is determined by the vector $w - xs$. Luckily this vector is known in advance when we apply a Newton step, thus for sufficiently small α we know precisely which coordinates of xs will decrease locally (precisely those for which the related coordinate of $w - xs$ is negative) and which coordinate of xs will increase locally (precisely those for which the related coordinate of $w - xs$ is positive).

The equivalence of the three statements in Theorem 12.
Clearly (ii) is a special case of (iii) and the implication $(ii) \to (i)$ is trivial.

It only remains to be proved that (i), i.e. the IPC, ensures that for each $w > 0$ the nonlinear system in (iii) is solvable. To this end, let us assume that an $x^0 \in SP$ with $(x^0, s(x^0)) > 0$ is given. We will use the notation $w^0 := x^0 s(x^0)$. The claim will be proved in two steps.

Step 1. *For each* $0 < \underline{w} < \overline{w} \in \mathbb{R}^n$ *the following two sets are compact:*

$$\begin{aligned} L_{\overline{w}} &:= \{x \in SP \,:\, xs(x) \le \overline{w}\} \text{ and} \\ U(\underline{w}, \overline{w}) &:= \{w \,:\, \underline{w} \le w \le \overline{w},\ w = xs(x) \text{ for some } x \in L_{\overline{w}}\}. \end{aligned}$$

Let us first prove that $L_{\overline{w}}$ is compact. For each $\overline{w} > 0$, the set $L_{\overline{w}}$ is obviously closed. In order to prove the boundedness of $L_{\overline{w}}$ first we observe that if $0 \le x \in L_{\overline{w}}$, $s(x) \ge 0$, then $x^T s(x) \le e^T \overline{w}$. Further, we have

$$0 = (x - x^0)^T(s - s^0) = x^T s + (x^0)^T s^0 - x^T s^0 - s^T x^0,$$

which in particular implies that for each $1 \le j \le n$,

$$x_j s_j^0 \le x^T s^0 + s^T x^0 = x^T s + (x^0)^T s^0 \le e^T \overline{w} + (x^0)^T s^0.$$

The last relations demonstrate that $L_{\overline{w}}$ is bounded, thus compact.

By definition the set $U(\underline{w}, \overline{w})$ is bounded. We only need to prove that it is closed. Let a convergent sequence $w^i \to \hat{w}$, $w^i \in U(\underline{w}, \overline{w})$, $i = 1, 2, \cdots$ be given. Then clearly $\underline{w} \le \hat{w} \le \overline{w}$ holds. Further, for each i there exists $x^i \in L_{\overline{w}}$ such that $w^i = x^i s(x^i)$. Because the set $L_{\overline{w}}$ is compact, there is an $\hat{x} \in L_{\overline{w}}$ and a convergence subsequence $x^i \to \hat{x}$ (for ease of notation the subsequence is denoted again the same way). Then we have $\hat{x}s(\hat{x}) = \hat{w}$, proving that $U(\underline{w}, \overline{w})$ is closed, thus compact.

Observe, that for each $w \in U(\underline{w}, \overline{w})$ by definition we have an $x \in SP$ with $w = xs(x)$. Due to $w > 0$ this relation implies that $x > 0$ and $s(x) > 0$.

Step 2. *For each* $\hat{w} > 0$, *the system* $Mx + q = s$, $xs = \hat{w}$, $x > 0$ *has a solution.*
If we have $\hat{w} = w^0 = x^0 s(x^0)$, then the claim is trivial. If $\hat{w} \ne w^0$ then we define $\overline{w} :=$

[4] Although it is not advised to use for numerical computations, the Newton direction can be expressed in the closed form $\Delta x = (M + X^{-1}S)^{-1}(x^{-1}w - s)$.

$\max\{\hat{w}, w^0\}$, $\overline{\eta} = \|\overline{w}\|_\infty + 1$, $\underline{w} := \min\{\hat{w}, w^0\}$ and $\underline{\eta} = \frac{1}{2}\min_i \underline{w}_i$. Then $\underline{\eta}e < \hat{w} < \overline{\eta}e$ and $\underline{\eta}e < w^0 < \overline{\eta}e$. Due to the last relation the set $\overline{U} := U(\underline{\eta}e, \overline{\eta}e)$ is nonempty and compact. We define the nonnegative function $d(w) : \overline{U} \to \mathbb{R}$ as

$$d(w) := \|w - \hat{w}\|_\infty.$$

The function $d(w)$ is continuous on the compact set $\overline{U}$, thus it attains its minimum

$$\tilde{w} := \arg\min_{w \in \overline{U}} \{d(w)\}.$$

If $d(\tilde{w}) = 0$, then $\tilde{w} = \hat{w}$ and hence by definition there is an $x \in SP$ satisfying $xs(x) = \hat{w}$ and the claim is proved.

If $d(\tilde{w}) > 0$ then we will show that a damped Newton step from $\tilde{w}$ towards $\hat{w}$ gives a point $w(\alpha) \in \overline{U}$ such that $d(w(\alpha)) < d(\tilde{w})$, contradicting the fact that $\tilde{w}$ minimizes $d(w)$. This situation is illustrated on Figure 1.

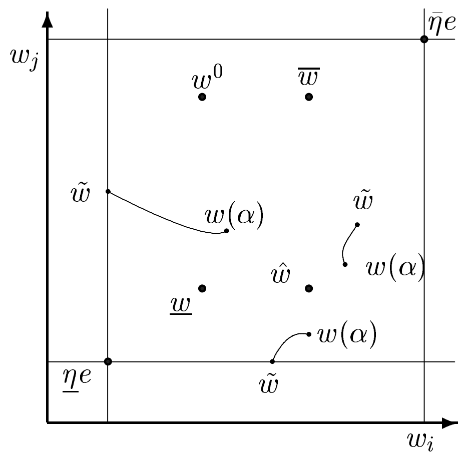

Figure 1: The situation when $\hat{w} \neq \tilde{w}$. A damped Newton step from $\tilde{w}$ to $\hat{w}$ is getting closer to $\hat{w}$. For illustration three possible different $\tilde{w}$ values are chosen.

The Newton step is well defined, because for the vector $\tilde{x} \in SP$ defining $\tilde{w}$ the relations $\tilde{x} > 0$ and $\tilde{s} = s(\tilde{x}) > 0$ hold. A damped Newton step from $\tilde{w}$ to $\hat{w}$ with sufficiently small α results in a point closer (measured by $d(\cdot) = \|\cdot\|_\infty$) to $\hat{w}$, because

$$\begin{aligned} w(\alpha) = x(\alpha)s(\alpha) &:= (\tilde{x} + \alpha\Delta x)(\tilde{s} + \alpha\Delta s) = \tilde{x}\tilde{s} + \alpha(\hat{w} - \tilde{x}\tilde{s}) + \alpha^2 \Delta x \Delta s \\ &= \tilde{w} + \alpha(\hat{w} - \tilde{w}) + \alpha^2 \Delta x \Delta s. \end{aligned}$$

This relation implies that

$$w(\alpha) - \hat{w} = (1 - \alpha)(\tilde{w} - \hat{w}) + \alpha^2 \Delta x \Delta s,$$

i.e. for α small enough[5] all nonzero coordinates of $|w(\alpha) - \hat{w}|$ are smaller than the respective coordinates of $|\tilde{w} - \hat{w}|$. Hence, $w(\alpha)$ is getting closer to $\hat{w}$, closer than $\tilde{w}$. Due to $\underline{\eta}e < \hat{w} < \overline{\eta}e$

[5] The reader easily verifies that any value of

$$\alpha < \min \left\{ \frac{\tilde{w}_i - \hat{w}_i}{\Delta x_i \Delta s_i} \; : \; (\tilde{w}_i - \hat{w}_i)(\Delta x_i \Delta s_i) > 0 \right\}$$

satisfies the requirement.

this result also implies that for the chosen small α value the vector $w(\alpha)$ stays in $\overline{U}$. Thus $\tilde{w} \neq \hat{w}$ cannot be a minimizer of $d(w)$, which is a contradiction.

The proof is complete. $\square$

Now we are ready to prove our main theorem, the existence of a strictly complementary solution, when the IPC holds.

Proof of Theorem 13.
Let $\mu_t \to 0$ $(t = 1, 2, \cdots)$ be a monotone decreasing sequence, hence for all t we have $x(\mu_t) \in L_{\mu_1 e}$. Because $L_{\mu_1 e}$ is compact the sequence $x(\mu_t)$ has an accumulation point x^* and without loss of generality we may assume that $x^* = \lim_{t\to\infty} x(\mu_t)$. Let $s^* := s(x^*)$. Clearly x^* is optimal because

$$x^* s^* = \lim_{t\to\infty} x(\mu_t) s(x(\mu_t)) = \lim_{t\to\infty} \mu_t e = 0.$$

We still have to prove that $(x^*, s(x^*))$ is strictly complementary, i.e. $x^* + s^* > 0$. Let $B = \{i : x_i^* > 0\}$ and $N = \{i : s_i^* > 0\}$. Using that M is skew symmetric, we have

$$0 = (x^* - x(\mu_t))^T (s^* - s(\mu_t)) = x(\mu_t)^T s(\mu_t) - {x^*}^T s(\mu_t) - x(\mu_t)^T s^*,$$

which, by using that $x(\mu_t)_i s(\mu_t)_i = \mu_t$, can be rewritten as

$$\sum_{i\in B} x_i^* s(\mu_t)_i + \sum_{i\in N} s_i^* x(\mu_t)_i = n\mu_t$$
$$\sum_{i\in B} \frac{x_i^*}{x(\mu_t)_i} + \sum_{i\in N} \frac{s_i^*}{s(\mu_t)_i} = n,$$

By taking the limit as μ_t goes to zero we obtain that

$$|B| + |N| = n,$$

i.e. (B, N) is a partition of the index set. Hence $(x^*, s(x^*))$ is a strictly complementary solution. The proof of Theorem 13 is complete. $\square$

As we mentioned earlier, this result is powerful enough to prove the strong duality theorem of LO in the strong form, including strict complementarity, i.e. the Goldman-Tucker Theorem 4 for SP and for (P) and (D).

Our next step is to prove that the accumulation point x^* is unique.

3.5 Convergence to the analytic center

In this subsection we prove that the central path has only one accumulation point, i.e. it converges to a unique point, the so-called analytic center [26] of the optimal set SP^*.

Definition 14 *Let $\bar{x} \in SP^*$, $\bar{s} = s(\bar{x})$ maximize the product*

$$\prod_{i\in B} x_i \prod_{i\in N} s_i$$

over $x \in SP^*$. Then $\bar{x}$ is called the analytic center *of SP^*.*

Theorem 15 *The limit point x^* of the central path is the analytic center of SP^*.*

Proof: The same way as in the proof of Theorem 13 we derive

$$\sum_{i\in B}\frac{\bar{x}_i}{x_i^*}+\sum_{i\in N}\frac{\bar{s}_i}{s_i^*}=n.$$

Now we apply the arithmetic-geometric-mean inequality to derive

$$\left(\prod_{i\in B}\frac{\bar{x}_i}{x_i^*}\prod_{i\in N}\frac{\bar{s}_i}{s_i^*}\right)^{\frac{1}{n}}\leq\frac{1}{n}\left(\sum_{i\in B}\frac{\bar{x}_i}{x_i^*}+\sum_{i\in N}\frac{\bar{s}_i}{s_i^*}\right)=1.$$

Hence,

$$\prod_{i\in B}\bar{x}_i\prod_{i\in N}\bar{s}_i\leq\prod_{i\in B}x_i^*\prod_{i\in N}s_i^*$$

proving that x^* is the analytic center of SP^*. The proof is complete. □

3.6 Identifying the optimal partition

The condition number

In order to give bounds on the size of the variables along the central path we need to find a quantity that in some sense characterizes the set of optimal solutions. For an optimal solution $x\in SP^*$ we have

$$xs(x)=0,\qquad\text{and}\qquad x+s(x)\geq 0.$$

Our next question is about the size of the nonzero coordinates of optimal solutions. Following the definitions in [25, 34] we define a condition number of the problem (SP) which characterizes the magnitude of the nonzero variables on the optimal set SP^*.

Definition 16 *Let us define*

$$\sigma^x:=\min_{i\in B}\max_{x\in SP^*}\{x_i\}\quad\sigma^s:=\min_{i\in N}\max_{x\in SP^*}\{s(x)_i\}.$$

Then the condition number *of (SP) is defined as*

$$\sigma=\min\{\sigma^x,\sigma^s\}=\min_i\max_{x\in SP^*}\{x_i+s(x)_i\}.$$

To determine the condition number σ is in general more difficult then to solve the optimization problem itself. However, we can give an easily computable lower bound for σ. This bound depends only on the problem data.

Lemma 17 (Lower bound for σ:) *If M and q are integral*[6] *and all the columns of M are nonzero, then*

$$\sigma\geq\frac{1}{\pi(M)},$$

where $\pi(M)=\prod_{i=1}^n\|M_i\|$.

[6]If the problem data is rational, then by multiplying by the least joint multiple of the denominators an equivalent LO problem with integer data is obtained.

Proof: The proof is based on Cramer's rule and on the estimation of determinants by using Hadamard's inequality. Let (x, s) be an optimal solution. Without loss of generality we may assume that the columns of the matrix $D = (-M, E)$ corresponding to the nonzero coordinates of (x, s) are linearly independent. If they are not independent, then by using Gaussian elimination we can reduce the solution to get one with linearly independent columns. Let us denote this index set by J. Further, let the index set K be such that D_{KJ} is a nonsingular square submatrix of D. Such K exists, because the columns in D_K are linearly independent. Now we have $D_{KJ}x_J = q_K$, and hence, by Cramer's rule,

$$x_j = \frac{\det\left(D_{KJ}^{(j)}\right)}{\det\left(D_{KJ}\right)}, \quad \forall j \in J,$$

where $D_{KJ}^{(j)}$ denotes the matrix obtained when the j-th column in D_{KJ} is replaced by q_K. Assuming that $x_j > 0$ then, because the data is integral, the numerator in the quotient given above is at least one. Thus we obtain $x_j \geq \frac{1}{\det(D_{KJ})}$. By Hadamard's inequality[7] the last determinant can be estimated by the product of the norm of its columns, what can further be bounded by the product of the norms of all the columns of the matrix M. □

The condition that none of the columns of the matrix M is a zero vector is not restrictive. For the general problem (SP) a zero column M_i would imply that $s_i = q_i$ for each feasible solution, thus the pair (x_i, s_i) could be removed. More important is that for our embedding problem $\overline{(SP)}$ none of the columns of the coefficient matrix

$$\begin{pmatrix} M & r \\ -r^T & 0 \end{pmatrix}$$

is zero. By definition we have $r = e - Me$ nonzero, because $e^T r = e^T e - e^T Me = n$. Moreover, if $M_i = 0$, then by using that the matrix M is skew symmetric we have $r_i = 1$, thus the i-th column of the coefficient matrix is again nonzero.

The size of the variables along the central path

Now, by using the condition number σ we are able to derive lower and upper bounds for the variables along the central path. Let (B, N) be the optimal partition of the problem (SP).

Lemma 18 *For each positive* μ *one has*

$$\begin{aligned} x_i(\mu) &\geq \frac{\sigma}{n} \quad i \in B, \qquad x_i(\mu) \leq \frac{n\mu}{\sigma} \quad i \in N, \\ s_i(\mu) &\leq \frac{n\mu}{\sigma} \quad i \in B, \qquad s_i(\mu) \geq \frac{\sigma}{n} \quad i \in N. \end{aligned}$$

Proof: Let (x^*, s^*) be optimal, then by orthogonality we have

$$\begin{aligned} (x(\mu) - x^*)^T(s(\mu) - s^*) &= 0, \\ x(\mu)^T s^* + s(\mu)^T x^* &= n\mu, \\ x(\mu)_i s_i^* \leq x(\mu)^T s^* &\leq n\mu, \quad 1 \leq i \leq n. \end{aligned}$$

[7]**Hadamard's inequality:** Let G be a nonsingular $n \times n$ matrix. Then the inequality

$$\det(G) \leq \prod_{i=1}^{n} \|G_i\|$$

holds.

Since $s_i^* \geq \sigma$ and $x_i(\mu)s_i(\mu) = \mu$, for $i \in N$, we have

$$x_i(\mu) \leq \frac{n\mu}{s_i^*} \leq \frac{n\mu}{\sigma} \quad \text{and} \quad s_i(\mu) \geq \frac{\sigma}{n}, \;\; i \in N.$$

The proofs of the other bounds are analogous. □

Identifying the optimal partition
The bounds presented in Lemma 18 make it possible to identify the optimal partition (B, N), when μ is sufficiently small. We just have to calculate the μ value that ensures that the coordinates going to zero are certainly smaller than the coordinates that converge to a positive number.

Corollary 19 *If we have a central solution $x(\mu) \in SP$ with*

$$\mu < \frac{\sigma^2}{n^2}$$

then the optimal partition (B, N) can be identified.

The results of Lemma 18 and Corollary 19 can be generalized to the situation when a vector (x, s) is not on, but just in a certain neighborhood of the central path. In order to keep our discussion short, we do not go in those details. The interested reader is referred to [25].

3.7 Rounding to an exact solution

Our next goal is to find a strictly complementary solution. This could be done by moving along the central path as $\mu \to 0$. Here we show that we do not have to do that, we can stop at a sufficiently small $\mu > 0$, and round off the current "almost optimal" solution to a strictly complementary optimal one. We need some new notation. Let the optimal partition be denoted by (B, N), let $\omega := \|M\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |M_{ij}|$ and $\pi := \pi(M) = \prod_{i=1}^n \|M_i\|$.

Lemma 20 *Let M and q be integral and all the columns of M be nonzero. If $(x, s) := (x(\mu), s(x(\mu)))$ is a central solution with*

$$x^T s = n\mu < \frac{\sigma^2}{n^{\frac{3}{2}}(1+\omega)^2\pi}, \qquad \textit{which certainly holds if} \qquad n\mu \leq \frac{1}{n^{\frac{3}{2}}(1+\omega)^2\pi^3},$$

then by a simple rounding procedure a strictly complementary optimal solution can be found in $\mathcal{O}(n^3)$ arithmetic operations.

Proof: Let $x := x(\mu) > 0$ and $s := s(x) > 0$ be given. Let we simply set the small variables x_N and s_B to zero. Then we will correct the so created error and estimate the size of the correction. For (x, s) we have

$$M_{BB}x_B + M_{BN}x_N + q_B = s_B, \tag{2}$$

but by rounding x_N and s_B to zero the error $\hat{q}_B = s_B - M_{BN}x_N$ occurs. Similarly, we have

$$M_{NB}x_B + M_{NN}x_N + q_N = s_N \tag{3}$$

but by rounding x_N and s_B to zero the error $\hat{q}_N = -M_{NN}x_N$ occurs.

Let us first estimate $\hat{q}_B$ and $\hat{q}_N$ by using the results of Lemma 18. For $\hat{q}_B$ we have

$$\|\hat{q}_B\| \leq \sqrt{n}\|\hat{q}_B\|_\infty \leq \sqrt{n}\|s_B - M_{BN}x_N\|_\infty \leq \sqrt{n}\|(E, -M_{BN})\|_\infty \left\| \begin{matrix} s_B \\ x_N \end{matrix} \right\|_\infty \leq \sqrt{n}(1+\omega)\tfrac{n\mu}{\sigma} = \tfrac{n^{\frac{3}{2}}\mu(1+\omega)}{\sigma}. \tag{4}$$

We give a bound for the infinity norm of $\hat{q}_N$ as well:

$$\|\hat{q}_N\|_\infty = \| - M_{NN}x_N\|_\infty \leq \|M_{NN}\|_\infty \|x_N\|_\infty \leq \omega\frac{n\mu}{\sigma}. \tag{5}$$

Now we are going to correct these errors by adjusting x_B and s_N. Let us denote the correction by ξ for x_B and by ζ for s_N, further let $(\hat{x}, \hat{s})$ be given by $\hat{x}_B := x_B + \xi > 0$, $\hat{x}_N = 0$, $\hat{s}_B = 0$ and $\hat{s}_N := s_N + \zeta > 0$.

If we know the correction ξ of x_B, then from equation (3) the necessary correction ζ of s_N can easily be calculated. Equation (2) does not contain s_N, thus by solving the equation

$$M_{BB}\xi = \hat{q}_B$$

the corrected value $\hat{x}_B = x_B - \xi$ can be obtained.

First we observe that the equation $M_{BB}\xi = \hat{q}_B$ is solvable, because any optimal solution x^* satisfies $M_{BB}x_B^* = -q_B$, thus we may write

$$\begin{aligned} M_{BB}\xi &= M_{BB}(x_B - x_B^*) \\ &= -q_B + s_B - M_{BN}x_N + q_B \\ &= s_B - M_{BN}x_N \quad = \quad \hat{q}_B. \end{aligned}$$

This equation system can be solved by Gaussian elimination. The size of ξ obtained this way can be estimated by applying Cramer's rule and Hadamard's inequality, the same way as we have estimated σ in Lemma 17. If M_{BB} is zero, then we have $q_B = 0$ and $M_{BN}x_N = s_B$, thus rounding x_N and s_B to zero does not produce any error here, hence we can choose $\xi = 0$. If M_{BB} is not the zero matrix, then let $\overline{M}_{BB}$ be a maximal nonsingular square submatrix of M_{BB} and let $\bar{q}_B$ be the corresponding part of $\hat{q}_B$. By using the upper bounds on x_N and s_B by Lemma 18 we have

$$|\xi_i| = \frac{|\det(\overline{M}_{BB}^{(i)})|}{|\det(\overline{M}_{BB})|} \leq |\det(\overline{M}_{BB}^{(i)})| \leq \|\bar{q}_B\| \, |\det(\overline{M}_{BB})| \leq \frac{n^{\frac{3}{2}}\mu(1+\omega)}{\sigma}\pi,$$

where (4) was used in the last estimation. This result, due to $\|x_B\|_\infty \geq \frac{\sigma}{n}$, implies that $\hat{x}_B = x_B + \xi > 0$ certainly holds if $n\mu < \frac{\sigma^2}{n^{\frac{3}{2}}(1+\omega)\pi}$.

Finally, we simply correct s_N by using (3), i.e. we define $\zeta := -\hat{q}_N - M_{NB}\xi$. We still must ensure that

$$\hat{s}_N := s_N - \hat{q}_N - M_{NB}\xi > 0.$$

Using again the bounds given in Lemma 17, the bound (5) and the estimate on ξ, one easily verifies that

$$\|\hat{q}_N + M_{NB}\xi\|_\infty \leq \|(E, M_{NB})\|_\infty \left\| \begin{matrix} \hat{q}_N \\ \xi \end{matrix} \right\|_\infty \leq (1+\omega)\max\left\{\omega\frac{n\mu}{\sigma}, \frac{n^{\frac{3}{2}}\mu(1+\omega)\pi}{\sigma}\right\} = \frac{n^{\frac{3}{2}}\mu(1+\omega)^2\pi}{\sigma}.$$

Thus, due to $\|s_N\|_\infty \geq \frac{\sigma}{n}$, the vector $\hat{s}_N$ is certainly positive if

$$\frac{\sigma}{n} > \frac{n^{\frac{3}{2}}\mu(1+\omega)^2\pi}{\sigma}.$$

This is exactly the first inequality given in the lemma. The second inequality follows by observing that $\pi\sigma \geq 1$, by Lemma 17.

The proof is completed by noting that the solution of an equation system by using Gaussian elimination, some matrix-vector multiplications and vector-vector summations, all with a dimension not exceeding n, is needed to perform our rounding procedure. Thus the computational complexity of our rounding procedure is at most $\mathcal{O}(n^3)$. □

Note, that this rounding results can also be generalized to the situation when a vector (x, s) is not on, but just in a certain neighborhood of the central path. For details the reader is referred again to [25].[8]

4 Summary of the theoretical results

Let us return to our general LO problem in canonical form

$$(P) \qquad \min\left\{c^T u \;:\; Au - z = b,\; u \geq 0,\; z \geq 0\right\}$$

$$(D) \qquad \max\left\{b^T v \;:\; A^T v + w = c,\; v \geq 0,\; w \geq 0\right\},$$

where the slack variables are already included in the problem formulation. In what follows we recapitulate the results obtained so far.

- In Section 2 we have seen that to solve the LO problem it is sufficient to find a strictly complementary solution to the Goldman-Tucker model

$$\begin{array}{rrrrrl} & Au & -\tau b & -z & & = 0 \\ -A^T v & & +\tau c & & -w & = 0 \\ b^T v & -c^T u & & & -\rho & = 0 \end{array}$$

$$v \geq 0,\; u \geq 0,\; \tau \geq 0,\; z \geq 0,\; w \geq 0,\; \rho \geq 0.$$

- This *homogeneous* system always admits the zero solution, but we need a solution for which $\tau + \rho > 0$ holds.
- If (u^*, z^*) is optimal for (P) and (v^*, w^*) for (D) then $(v^*, u^*, 1, z^*, w^*, 0)$ is a solution for the Goldman-Tucker model with the requested property $\tau + \rho > 0$. See Theorem 3.
- Any solution of the Goldman-Tucker model (v, u, τ, z, w, ρ) with $\tau > 0$ yields an optimal solution pair (scale the variables (u, z) and (v, w) by $\frac{1}{\tau}$) for LO. See Theorem 3.

[8] This result makes clear that when one solves an LO problem by using an IPM, the iterative process can be stopped at a sufficiently small value μ. At that point a strictly complementary optimal solution can easily be identified.

- Any solution of the Goldman-Tucker model (u, z, v, w, τ, ρ) with $\rho > 0$ provides a certificate of primal or dual infeasibility. See Theorem 3.
- If $\tau = 0$ in every solution (v, u, τ, z, w, ρ) then (P) and (D) have no optimal solutions with zero duality gap.
- The Goldman-Tucker model can be transformed into a skew-symmetric self-dual problem (SP) satisfying the IPC. See Section 3.2.
- If problem (SP) satisfy the IPC then
  - the central path exists (see Theorem 12);
  - the central path converges to a strictly complementary solution (see Theorem 13);
  - the limit point of the central path is the analytic center of the optimal set (see Theorem 15);
  - if the problem data is integral and a solution on the central path with a sufficiently small μ is given, then the optimal partition (see Corollary 19) and an exact strictly complementary optimal solution (see Lemma 20) can be found.
- These results give a constructive proof of Theorem 4.
- This way, as we have seen in Section 2, the Strong Duality Theorem 5 is proved.

The above summary shows that we have completed our project. The duality theory of LO is built up by using only elementary calculus and fundamental concepts of IPMs.

In the rest of the paper a generic IP algorithm is presented.

5 A general scheme of IP algorithms

In this section a glimpse of the main elements of IPMs is given. We keep on working with our model problem (SP). In Sections 2 and 3.2 we have shown that a general LO problem can be transformed into a problem of the form (SP), and that problem satisfies the IPC. Some notes are due to the linear algebra involved. We know that the size of the resulting embedding problem (SP) is more than doubled comparing to the size of the original LO problem. In spite of the size increase the linear algebra can be organized so that the computational cost of an IPM iteration stays essentially the same.

Let us consider the problem (cf. page 7)

$$\overline{(SP)} \qquad \min\left\{\lambda\vartheta \;:\; -\begin{pmatrix} M & r \\ -r^T & 0 \end{pmatrix}\begin{pmatrix} x \\ \vartheta \end{pmatrix} + \begin{pmatrix} s \\ \nu \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix};\; \begin{pmatrix} x \\ \vartheta \end{pmatrix}, \begin{pmatrix} s \\ \nu \end{pmatrix} \geq 0\right\}, \tag{6}$$

where $r = e - Me$, $\lambda = n + 1$ and the matrix M is given by (1). This problem satisfies the IPC, because the all one vector $(x^0, \vartheta^0, s^0, \nu^0) = (e, 1, e, 1)$ is a feasible solution, moreover it is also on the central path by taking $\mu = 1$. In other words, it is a positive solution of the equation system

$$
\begin{aligned}
-\begin{pmatrix} M & r \\ -r^T & 0 \end{pmatrix}\begin{pmatrix} x \\ \vartheta \end{pmatrix} + \begin{pmatrix} s \\ \nu \end{pmatrix} &= \begin{pmatrix} 0 \\ \lambda \end{pmatrix}; \quad \begin{pmatrix} x \\ \vartheta \end{pmatrix}, \begin{pmatrix} s \\ \nu \end{pmatrix} \geq 0 \\
\begin{pmatrix} x \\ \vartheta \end{pmatrix}\begin{pmatrix} s \\ \nu \end{pmatrix} &= \begin{pmatrix} \mu e \\ \mu \end{pmatrix},
\end{aligned} \tag{7}
$$

which defines the central path of problem $\overline{(SP)}$. As we have seen, for each $\mu > 0$, this system has a unique solution. However, in general this solution cannot be calculated exactly. Therefore we are making Newton steps to get approximate solutions.

Newton step:

Let us assume that an interior point $(x, \vartheta, s, \nu) > 0$ is given. We want to find the solution of (7) for a given $\mu \geq 0$, in other words we want to determine the displacements

$$(\Delta x, \Delta\vartheta, \Delta s, \Delta\nu)$$

so that

$$
-\begin{pmatrix} M & r \\ -r^T & 0 \end{pmatrix}\begin{pmatrix} x+\Delta x \\ \vartheta+\Delta\vartheta \end{pmatrix} + \begin{pmatrix} s+\Delta s \\ \nu+\Delta\nu \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}; \quad \begin{pmatrix} x+\Delta x \\ \vartheta+\Delta\vartheta \end{pmatrix}, \begin{pmatrix} s+\Delta s \\ \nu+\Delta\nu \end{pmatrix} \geq 0
$$

$$
\begin{pmatrix} x+\Delta x \\ \vartheta+\Delta\vartheta \end{pmatrix}\begin{pmatrix} s+\Delta s \\ \nu+\Delta\nu \end{pmatrix} = \begin{pmatrix} \mu e \\ \mu \end{pmatrix}.
$$

By neglecting the second order terms $\Delta x \Delta s$ and $\Delta\vartheta\Delta\nu$, and the nonnegativity constraints, the Newton equation system is obtained (cf. page 10)

$$
\begin{array}{lllll}
-M\Delta x & -r\Delta\vartheta & +\Delta s & & = 0 \\
r^T\Delta x & & & +\Delta\nu & = 0 \\
s\Delta x & & +x\Delta s & & = \mu e - xs \\
& \nu\Delta\vartheta & & +\vartheta\Delta\nu & = \mu - \vartheta\nu.
\end{array} \tag{8}
$$

We start by making some observations. For any vector (x, ϑ, s, ν) that satisfies the equality constraints of (6) we have

$$x^T s + \vartheta\nu = \vartheta\lambda.$$

Applying this to the solution obtained after making a Newton step we may write

$$(x+\Delta x)^T(s+\Delta s) + (\vartheta+\Delta\vartheta)^T(\nu+\Delta\nu) = (\vartheta+\Delta\vartheta)\lambda.$$

By rearranging the terms we have

$$(x^T s + \vartheta\nu) + (\Delta x^T\Delta s + \Delta\vartheta\Delta\nu) + (x^T\Delta s + s^T\Delta x + \vartheta\Delta\nu + \nu\Delta\vartheta) = \vartheta\lambda + \Delta\vartheta\lambda.$$

As we mentioned above, the first term in the left hand side sum equals to $\vartheta\lambda$, while from (8) we derive that the second sum is zero. From the last equations of (8) one easily derives that the third expression equals to $\mu(n+1) - x^T s - \vartheta\nu = \mu\lambda - \vartheta\lambda$. This way the equation $\mu\lambda - \vartheta\lambda = \Delta\vartheta\lambda$ is obtained, i.e. an explicit expression for $\Delta\vartheta$,

$$\Delta\vartheta = \mu - \vartheta$$

is derived. This value can be substituted in the last equation of (8) to derive the solution

$$\Delta\nu = \frac{\mu}{\vartheta} - \nu - \frac{\nu(\mu - \vartheta)}{\vartheta},$$

i.e.

$$\Delta\nu = \frac{\mu(1-\nu)}{\vartheta}.$$

On the other hand, Δs can be expressed from the third equation of (8) as

$$\Delta s = \mu X^{-1} e - s - X^{-1} S \Delta x,$$

where X and S are the diagonal matrices containing the coordinates of the vectors x and s in their respective diagonals. Finally, substituting all these values in the first equation of (8) we have

$$M\Delta x + X^{-1} S \Delta x = \mu X^{-1} e - s - (\mu - \vartheta) r,$$

i.e. Δx is the unique solution of the positive definite system[9]

$$(M + X^{-1} S)\Delta x = \mu X^{-1} e - s - (\mu - \vartheta) r.$$

Having determined the displacements, we can make a (possibly damped) Newton step to update our current iterate:

$$\begin{aligned} x &:= x + \Delta x \\ \vartheta &:= \vartheta + \Delta\vartheta = \mu \\ s &:= s + \Delta s \\ \nu &:= \nu + \Delta\nu. \end{aligned}$$

We have seen that the central path is our guide to a strictly complementary solution. However, due to the nonlinearity of the equation system determining the central path, we cannot stay on the central path with our iterates, regardless that our initial interior point is perfectly centered. For this reason we need some centrality, or with other words proximity, measures that enable us to control and keep our iterates in an appropriate neighborhood of the central path.

Proximity measures

Let the vectors $\bar{x}$ and $\bar{s}$ be composed from x and ϑ, and from s and ν respectively. Note that on the central path all the coordinates of the vector $\bar{x}\bar{s}$ are equal. This observation indicates that the proximity measure

$$\delta_c(\bar{x}\bar{s}) := \frac{\max(\bar{x}\bar{s})}{\min(\bar{x}\bar{s})},$$

[9]Observe, that although the dimensions of problem $\overline{(SP)}$ are larger than problem (SP), to determine the Newton step for both systems requires essentially the same computational effort.

Note also, that the special structure of the matrix M (see (1)) can be utilized when one solves this positive definite linear system. For details the reader is referred to [2, 25, 31, 35].

where $\max(\bar{x}\bar{s})$ and $\min(\bar{x}\bar{s})$ denotes the largest and smallest coordinate of the vector $\bar{x}\bar{s}$, is an appropriate measure of centrality. In the literature of IPMs various centrality measures were developed (see the books [11, 12, 25, 31, 35]). Here we present just another one, extensively used in [25]:

$$\delta_0(\bar{x}\bar{s},\mu) := \frac{1}{2}\left\| \left(\frac{\bar{x}\bar{s}}{\mu}\right)^{\frac{1}{2}} - \left(\frac{\mu}{\bar{x}\bar{s}}\right)^{\frac{1}{2}} \right\|.$$

Both of these proximity measures allow us to design polynomial IPMs.

Generic Interior Point Newton Algorithm

Input:
A proximity parameter κ;
an accuracy parameter $\varepsilon > 0$;
a variable damping factor α;
update parameter θ, $0 < \theta < 1$;
$(\bar{x}^0, \bar{s}^0)$, $\mu^0 \le 1$ s.t. $\delta(\bar{x}^0\bar{s}^0;\mu^0) \le \kappa$.

```
begin
  x̄ := x̄⁰; s̄ := s̄⁰; μ := μ⁰;
  while (n+1)μ ≥ ε do
  begin
    μ := (1 − θ)μ;
    while δ(x̄, s̄; μ) ≥ κ do
    begin
      x̄ := x̄ + αΔx̄;
      s̄ := s̄ + αΔs̄;
    end
  end
end
```

The following crucial issues remain: how to choose the centrality parameter κ, how to update μ and how to damp the Newton step when needed.

To conclude our discussions, three sets of parameters are presented that ensure that the resulted IPMs are polynomial. The proofs of complexity can e.g. be found in [25]. Recall that $\overline{(SP)}$ admits the all one vector as a perfectly centered initial solution with $\mu = 1$.

The first algorithm is a *primal-dual logarithmic barrier algorithm with full Newton steps*, studied e.g. in [25]. This IPM enjoys the best complexity known to date. Let us make the following choice:

- $\delta(\bar{x}\bar{s},\mu) := \delta_0(\bar{x}\bar{s},\mu)$, this measure is zero on the central path;
- $\mu^0 := 1$;
- $\theta := \frac{1}{2\sqrt{n+1}}$;
- $\kappa = \frac{1}{\sqrt{2}}$;
- $(\Delta\bar{x}, \Delta\bar{s})$ is the solution of 8;
- $\alpha = 1$.

Theorem 21 (Theorem II.52 in [25]) *With the given parameter set the full step Newton algorithm requires not more than*

$$\left\lceil 2\sqrt{n+1}\log\frac{n+1}{\varepsilon}\right\rceil$$

iterations to produce a feasible solution $(\bar{x},\bar{s})$ for $\overline{(SP)}$ such that $\delta_0(\bar{x}\bar{s},\mu)\leq\kappa$ and $(n+1)\vartheta\leq\varepsilon$.

The second algorithm is a *large update primal-dual logarithmic barrier algorithm*, studied also e.g. in [25]. Among our three algorithms, this is the most practical. Let us make the following choice:

- $\delta(\bar{x}\bar{s},\mu) := \delta_0(\bar{x}\bar{s},\mu)$, this measure is zero on the central path;
- $\mu^0 := 1$;
- $0 < \theta < \frac{n+1}{n+1+\sqrt{n+1}}$;
- $\kappa = \frac{\sqrt{R}}{2\sqrt{1+\sqrt{R}}}$, where $R = \frac{\theta\sqrt{n+1}}{1-t}$;
- $(\Delta\bar{x},\Delta\bar{s})$ is the solution of 8;
- α is the result of a line search, when along the search direction the primal-dual logarithmic barrier function

  $$\bar{x}^T\bar{s}-(n+1)\sum_{i=1}^{n+1}\log\bar{x}_i\bar{s}_i$$

  is minimized.

Theorem 22 (Theorem II.74 in [25]) *With the given parameter set the large update primal-dual logarithmic barrier algorithm requires not more than*

$$\left\lceil \frac{1}{\theta}\left\lceil 2\left(1+\sqrt{\frac{\theta\sqrt{n+1}}{1-t}}\right)^4\right\rceil \log\frac{n+1}{\varepsilon}\right\rceil$$

iterations to produce a feasible solution $(\bar{x},\bar{s})$ for $\overline{(SP)}$ such that $\delta_0(\bar{x}\bar{s},\mu)\leq\kappa$ and $(n+1)\vartheta\leq\varepsilon$.

When we choose $\theta=\frac{1}{2}$, then the total complexity becomes $\mathcal{O}\left(n\log\frac{n+1}{\varepsilon}\right)$, while the choice $\theta=\frac{K}{\sqrt{n+1}}$, with any fixed positive value K gives $\mathcal{O}\left(\sqrt{n}\log\frac{n+1}{\varepsilon}\right)$ complexity.

Other versions of this algorithm were studied in [23], where the analysis of large update methods was based purely on the use of the proximity $\delta_0(\bar{x}\bar{s},\mu)$.

The last algorithm is the *Dikin step algorithm* studied in [25]. This is one of the simplest IPMs, with an extremely elementary complexity analysis. The prize for simplicity is that the polynomial complexity result is not the best possible. Let us make the following choices:

- $\delta(\bar{x}\bar{s},\mu) := \delta_c(\bar{x}\bar{s})$, this measure is always larger than or equal to 1;
- $\mu^0 := 0$, this implies that μ stays equal to zero, thus θ is irrelevant;
- $\kappa = 2$;
- $(\Delta\bar{x},\Delta\bar{s})$ is the solution of (8) when the right hand sides of the last two equations are replaced by $-\frac{x^2s^2}{\|\bar{x}\bar{s}\|}$ and $-\frac{\vartheta\nu}{\|\bar{x}\bar{s}\|}$, respectively;

- $\alpha = \frac{1}{2\sqrt{n+1}}$.

Theorem 23 (Theorem I.27 in [25]) *With the given parameter set the Dikin step algorithm requires not more than*

$$\left\lceil 2(n+1)\log\frac{n+1}{\varepsilon}\right\rceil$$

iterations to produce a feasible solution $(\bar{x}, \bar{s})$ *for* $\overline{(SP)}$ *such that* $\delta_c(\bar{x}\bar{s}) \leq 2$ *and* $(n+1)\vartheta \leq \varepsilon$.

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