Introduction to Integer Linear Programming

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Definitions

- Mathematical programming formulation:

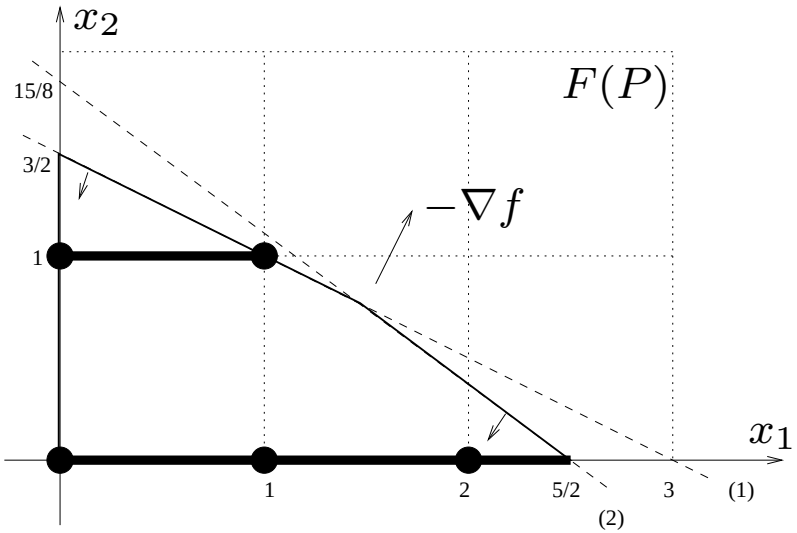
$$\left.\begin{array}{rl} \min_x & c^{\mathsf{T}}x + d^{\mathsf{T}}y \\ \text{s.t.} & Ax + By \leq b \\ & x \geq 0, y \geq 0, \\ & x \in \mathbb{Z}^n \end{array}\right\} [P] \qquad (1)$$

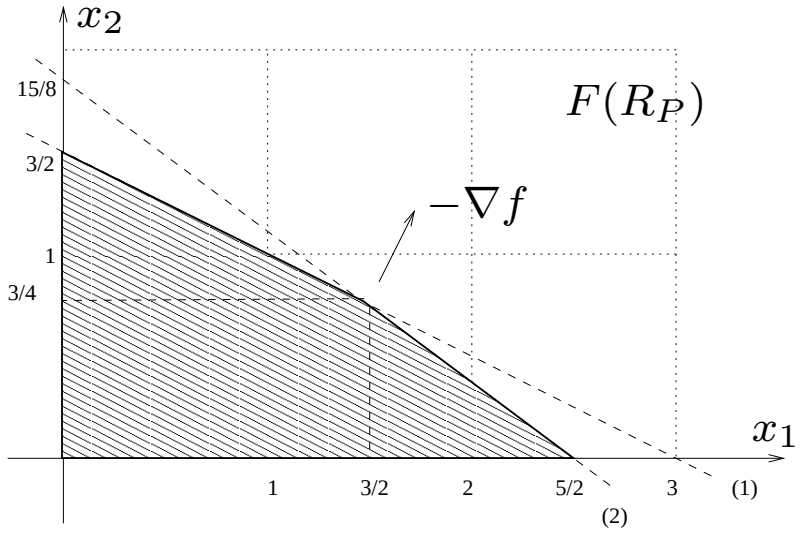
- The *linear* (or *continuous*) *relaxation* R_P of P is obtained by P relaxing (i.e. removing) the integrality constraints
- Let $F(P)$ be the feasible region of P: we have $F(P) \subseteq F(R_P)$
- Let (x^*, y^*) be the solution of P and $(\bar{x}, \bar{y})$ be the solution of R_P; then $c^{\mathsf{T}}\bar{x} + d^{\mathsf{T}}\bar{y} \leq c^{\mathsf{T}}x^* + d^{\mathsf{T}}y^*$: R_P is a *lower bounding problem* w.r.t. P

Simple example

Consider example:

$$\left.\begin{array}{ll} \min & -2x_1 - 3x_2 \\ & x_1 + 2x_2 \leq 3 \\ & 6x_1 + 8x_2 \leq 15 \\ & x_1 \in \mathbb{R}_+,\ x_2 \in \mathbb{Z}_+ \end{array}\right\}$$

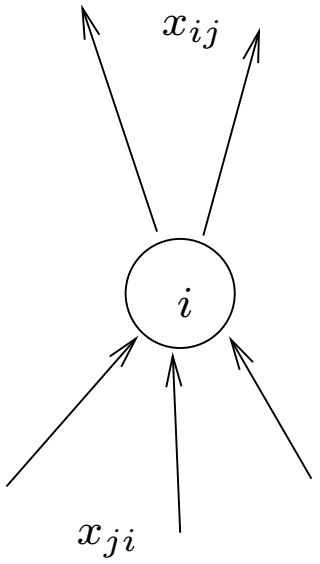




Maximum flow problem

Given a network on a directed graph $G = (V, A)$ with a source node s, a destination node t, and integer capacities u_{ij} on each arc (i, j). We have to determine the maximum amount of integral material flow that can circulate on the network from s to t. The variables $x_{ij} \in \mathbb{Z}$, defined for each arc (i, j) in the graph, denote the number of flow units.

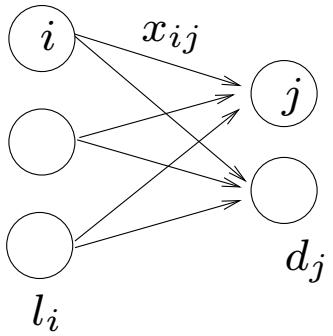
$$\left.\begin{array}{rll} & \max_x & \displaystyle\sum_{(s,i)\in A} x_{si} \\ \forall\, i \le V,\ \begin{array}{l} i \neq s \\ i \neq t \end{array} & & \displaystyle\sum_{(i,j)\in A} x_{ij} = \sum_{(j,i)\in A} x_{ji} \\ \forall (i,j) \in A & & 0 \le x_{ij} \le u_{ij} \\ \forall (i,j) \in A & & x_{ij} \in \mathbb{Z} \end{array}\right\}$$



Transportation problem

Let x_{ij} be the (discrete) number of product units transported from plant $i \leq m$ to customer $j \leq n$ with respective unit transportation cost c_{ij} from plant i to customer j. We model the problem of determining x minimizing the total cost, subject to production limits l_i at plant i and demand d_j at customer j, as follows:

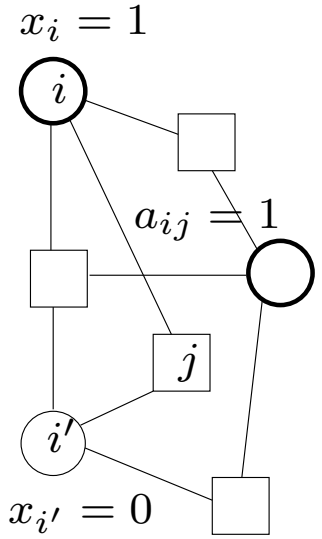
$$
\left.
\begin{array}{rrcl}
 & \min_x \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & & \\
\forall\, i \leq m & \sum_{j=1}^{n} x_{ij} & \leq & l_i \\
\forall\, j \leq n & \sum_{i=1}^{m} x_{ij} & \geq & d_j \\
\forall\, i, j & x_{ij} \in \mathbb{Z}_+ & &
\end{array}
\right\}
$$



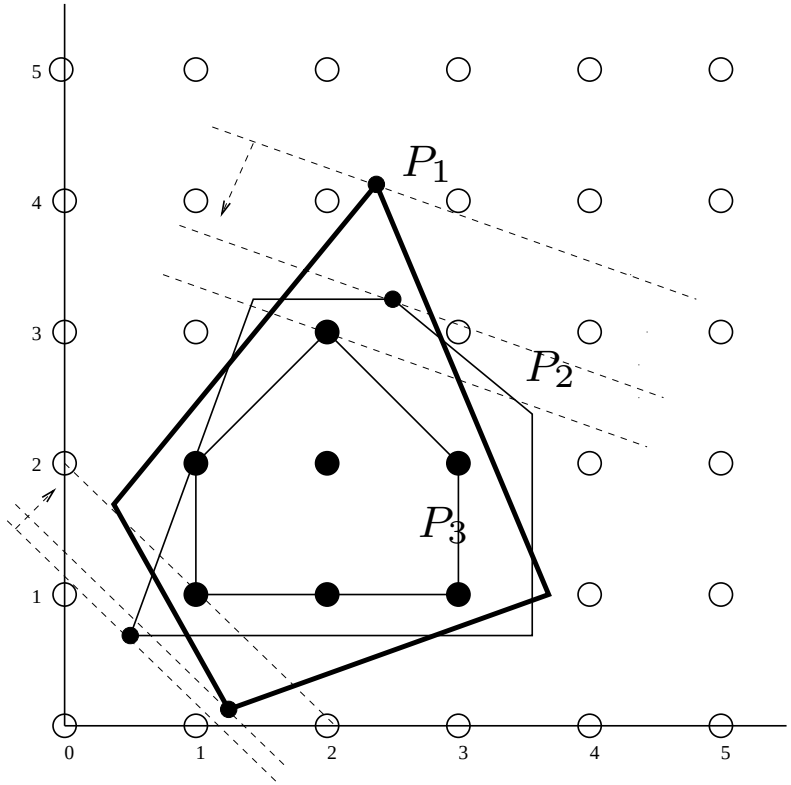
Set Covering problem

Let $x_i = 1$ if a servicing facility will be built on geographical region $i \leq m$ and 0 otherwise. The cost of building a facility on region i is f_i, and $a_{ij} = 1$ if a facility on region i can serve town $j \leq n$, and 0 otherwise. We need to determine $x \in \{0,1\}^m$ so that each town is serviced by at least one facility and the total cost is minimum.

$$\left.\begin{array}{rrcl} \min_x & \sum_{i=1}^{m} f_i x_i & & \\ \forall\, j \leq n & \sum_{i=1}^{m} a_{ij} x_i & \geq & 1 \\ \forall\, i \leq m & x_i & \in & \mathbb{Z}_+ \end{array}\right\}$$



Good and ideal formulations



The smaller is $F(R_P)$, the bigger (better) is the lower bound produced by R_P. As $F(R_{P_3}) \subset F(R_{P_2})$ and $F(R_{P_1}) \subset F(R_{P_2})$, the formulation P_3 is better than P_1 If and P_2.

Here P_3 is the best possible (ideal) formulation.

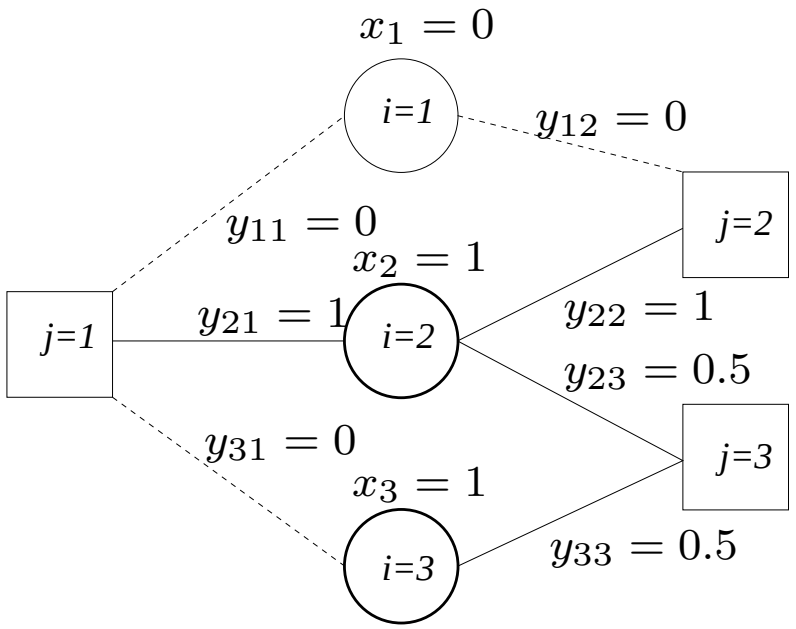
Formally, R_{P_3} defines the *convex hull* of P.

$P = \{x^1, \ldots, x^t\}$, then $conv(P) = \{x :$
$x = \sum_{i=1}^{t} \lambda_i x^i,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_i \geq 0, \forall i = 1, \ldots, t\}$.

Uncapacitated Facility Location problem

Similar to the Set Covering Problem, except for the addition of the variable transportation costs c_{ij}, which arise if the demand of town j is fully served by facility i. Let y_{ij} be the fraction of demand of town j served by facility i.

$$\left.\begin{aligned}
&\min_{x,y} \sum_{i=1}^{m} f_i x_i + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} y_{ij} \\
&\forall j \le n, \quad \sum_{i=1}^{m} y_{ij} = 1 \\
&\forall i \le m, \quad \sum_{j=1}^{m} y_{ij} \le n x_i \\
&\forall i \le m, \forall j \le n, \quad y_{ij} \ge 0, \\
&\forall i \le m, \quad x_i \in \{0,1\}.
\end{aligned}\right\}$$



UFL problem II

We can change constraints

$$\forall i \leq m, \quad \sum_{j=1}^{m} y_{ij} \leq n x_i \quad [R_1]$$

to constraints

$$\forall i \leq m, \forall j \leq n, \quad y_{ij} \leq x_i. \quad [R_2]$$

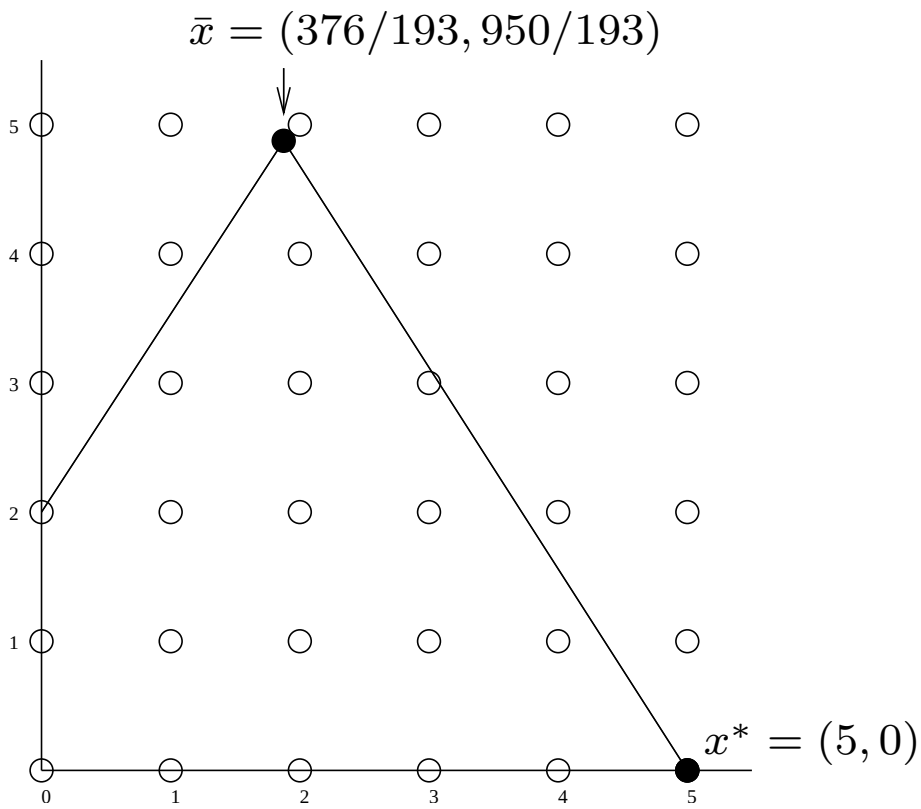
Formulation R_2 is better than R_1 as $F(R_2) \subset F(R_1)$. We can verify it by showing $F(R_2) \subseteq F(R_1)$ and finding a point $(x, y) \in F(R_1) \setminus F(R_2)$.

Rounding heuristic

There is a strong relation between an integer program and its linear relaxation.

But just rounding the solution $\bar{x}$ of the LP relaxation does not always produce good results. Consider the integer program:

$$\max 1.00x_1 + 0.64x_2$$
$$50x_1 + 31x_2 \leq 250$$
$$3x_1 - 2x_2 \geq -4$$
$$x_1, x_2 \in \mathbb{Z}_+$$



Main algorithmic ideas

- If we can say a priori that $\bar{x} \in \mathbb{Z}^n$ then can solve P by simply solving R_P (*total unimodularity property*).
- Add constraints to get P' such that $\bar{x}' \in \mathbb{Z}^n$ (*cutting planes algorithm*).
- Solve by “smart” enumeration of all solutions (*Branch-and-Bound algorithm*).
- Combine adding constraints and enumeration (*Branch-and-Cut algorithm*).
- Modern Integer Programming solvers (like *Cplex*) use the Branch-and-Cut algorithm.

Total unimodularity I

- Consider system $Bx = b$ where $B = (b_{ij})$ is invertible $n \times n$ s.t. $b_{ij} \in \mathbb{Z}$ for all i, j
- Solve for x, get $B^{-1}b$
- From inverse matrix formula, infer $B^{-1} = \frac{1}{|B|}C$ with C integral
- If $|B| \in \{1, -1\}$ then $x = B^{-1}b = \pm Cb \in \mathbb{Z}^n$
- A square invertible matrix B s.t. $|B| = \pm 1$ is *unimodular*
- An $m \times n$ matrix A s.t. every square submatrix has determinant in $\{-1, 0, 1\}$ is *totally unimodular* (TUM)
- Theorem: if A is TUM, then for all $b \in \mathbb{R}^n$, every vertex of the polyhedron $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ is integral. *Intuititively, every vertex can be written as $B^{-1}b$ for B square submatrix of A*

Total unimodularity II

- If A is TUM, A^{T} and $(A|I)$ are TUM
- *TUM Sufficient conditions*. An $m \times n$ matrix A is TUM if:
  1. for all $i \leq m$, $j \leq n$ we have $a_{ij} \in \{0, 1, -1\}$;
  2. each column of A contains at most 2 nonzero coefficients;
  3. there is a partition R_1, R_2 of the set of rows such that for each column j, $\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} = 0$.
- Example: take $R_1 = \{1, 3, 4\}$, $R_2 = \{2\}$

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}$$

Total unimodularity III

- Consider digraph $G = (V, A)$ and a nonnegative flow $x_{ij} \in \mathbb{R}_+$ on each arc; the flow conservation equations

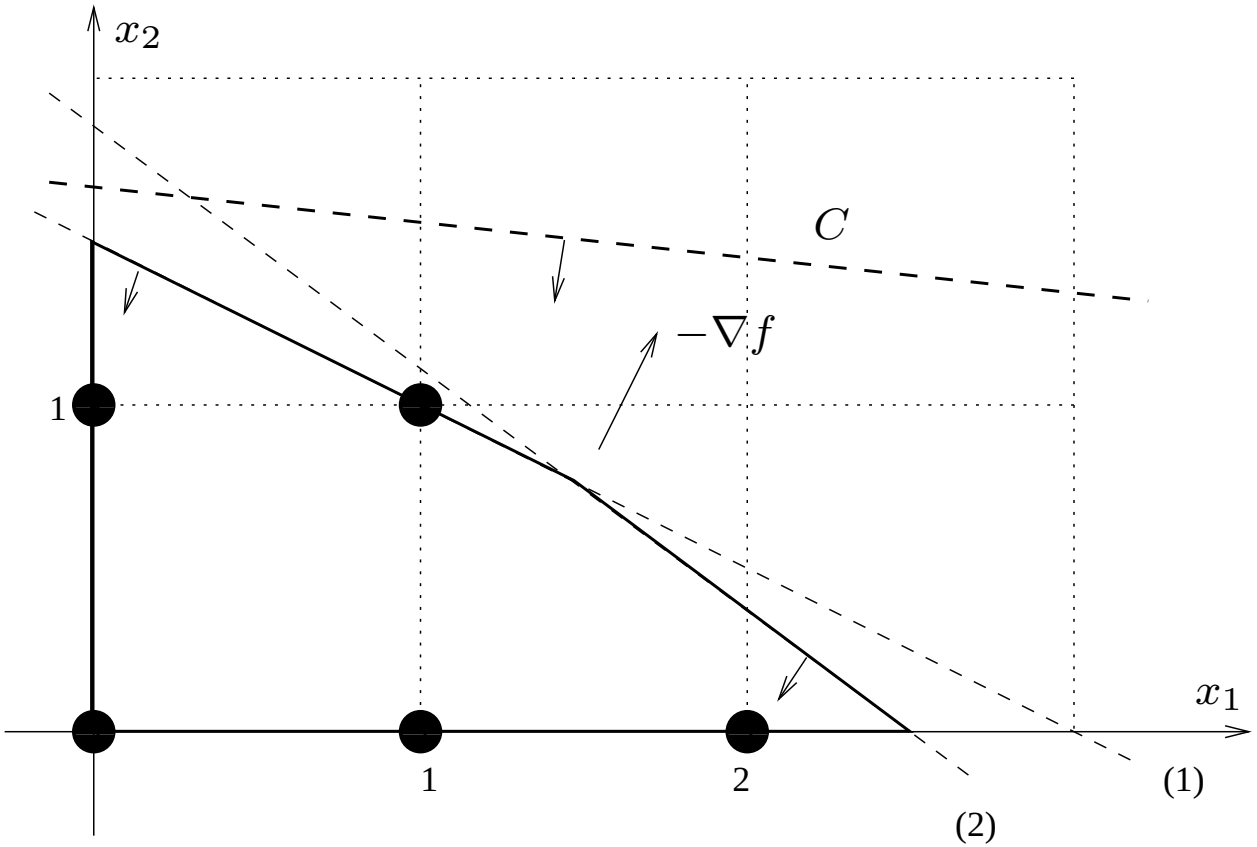
$$\forall i \in V \quad \sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = 0$$

yield a TUM matrix (which row partition?)

- Maximum flow problem can be solved to integrality by simply solving the continuous relaxation with the simplex algorithm
- Constraints of the transportation problem also form a TUM matrix. Partition: $R_1 = \{\sum_{j=1}^{n} x_{ij} \leq l_i\}_{i\leq m}$, $R_2 = \{\sum_{i=1}^{m} x_{ij} \geq d_j\}_{j\leq n}$.
- Constraints of the set covering problem do not form a TUM. To prove this, you just need to find a counterexample.

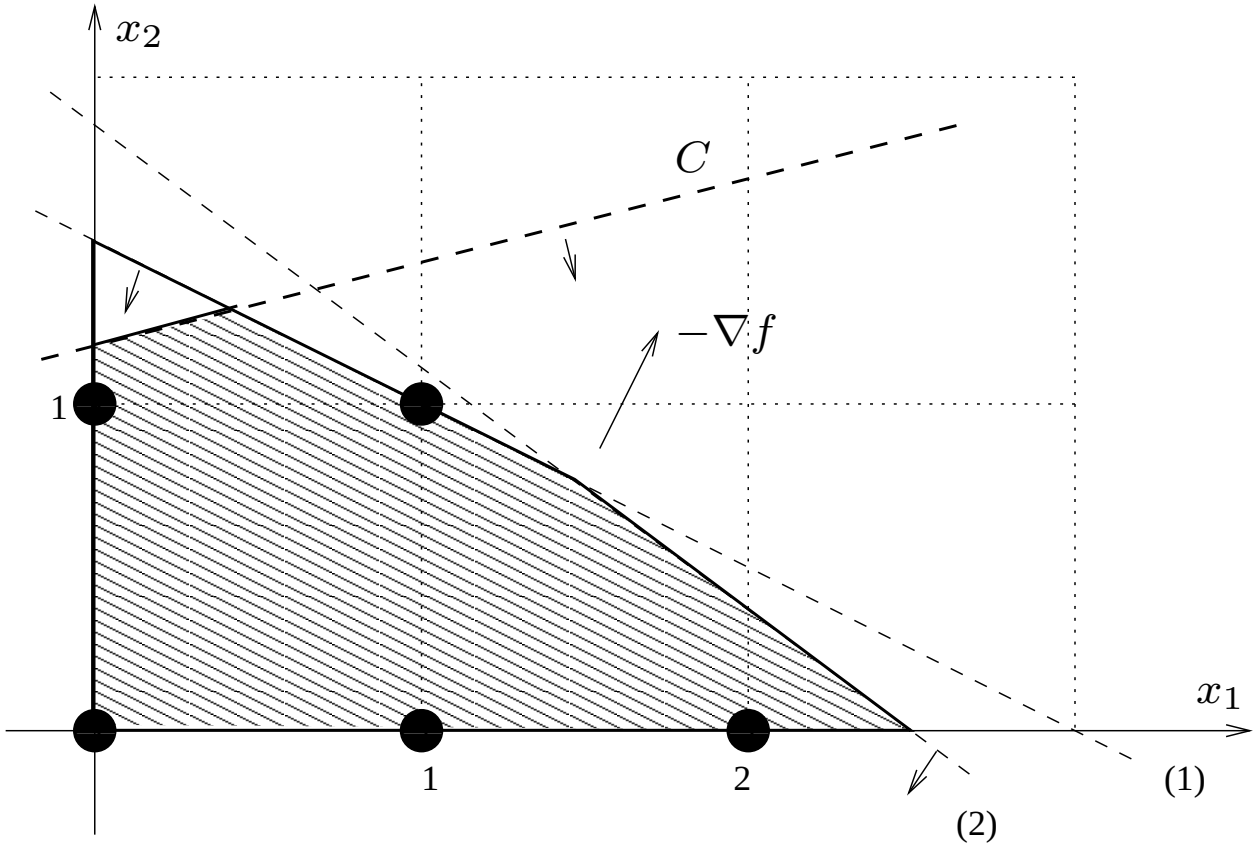
Cutting planes: definitions I

A constraint $C \equiv \pi^{\mathsf{T}} x \leq \pi_0$ is *valid* for P if
$\forall\, x' \in F(P)\ (\pi^{\mathsf{T}} x' \leq \pi_0)$



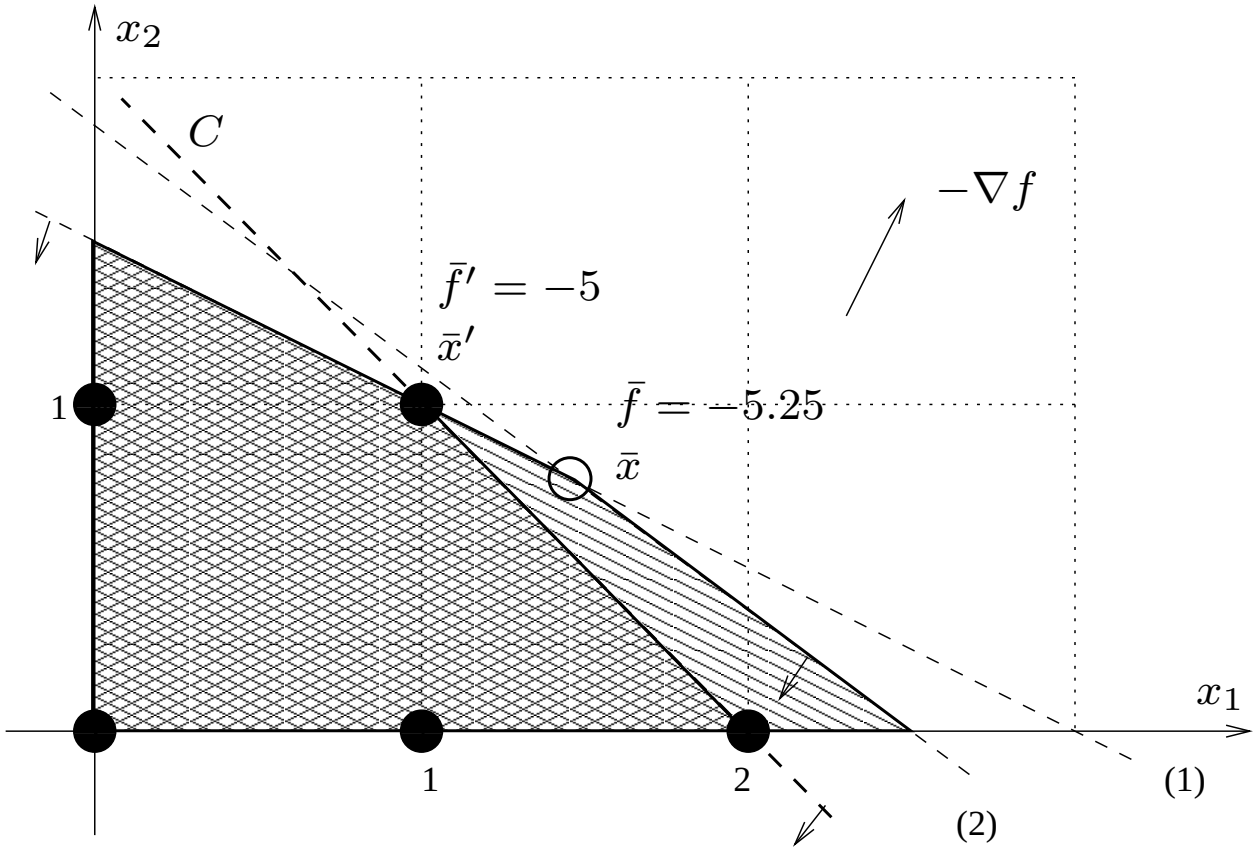
Cutting planes: definitions II

Let P' be problem P with the added valid constraint C. C is a *cutting plane* for P if $F(R_{P'}) \subseteq F(R_P)$



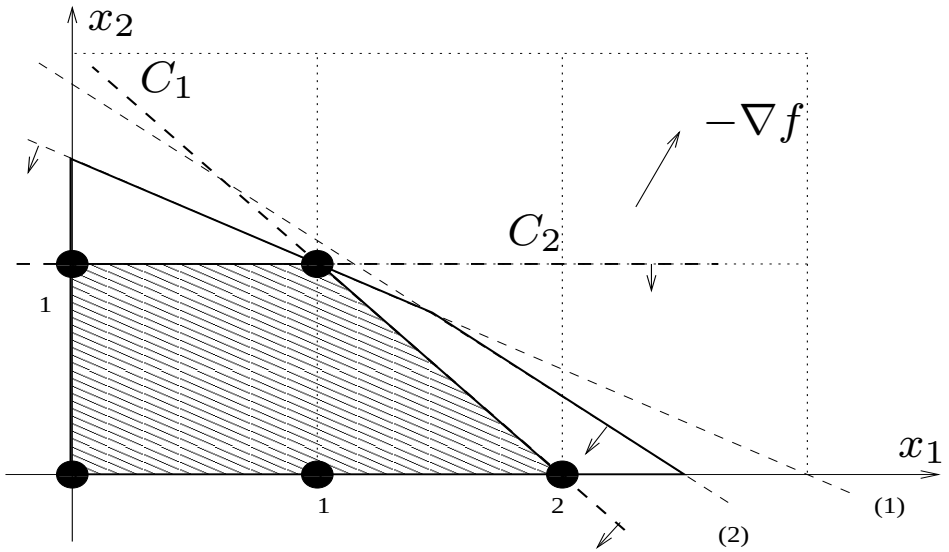
Cutting planes: definitions III

Let $\bar{x}$ be the optimal solution for R_P. A cutting plane $C: \ \pi^{\mathsf{T}} x \leq \pi_0$ is a *valid cut* for $\bar{x} \in R_P$ if $\pi^{\mathsf{T}} \bar{x} > \pi_0$.



Convex hull

- To have a description of the convex hull of $F(P)$, we need a finite number of valid constraints for P.



- Computing the convex hull for $F(P)$ is in general harder than solving P.
- The idea of the cutting plane algorithm is to add valid cuts progressively and resolve the LP relaxation each time until we obtain an integer solution. Thus, we add only those cuts we need.

Cutting Plane Algorithm

- Overall strategy:
  1. Solve R_P, get relaxed solution $\bar{x}$
  2. If $\bar{x} \in \mathbb{Z}^n$ problem is solved, exit
  3. Use solution $\bar{x}$ of R_P to construct a valid cut C for P
  4. Add the constraint C to the formulation of P
  5. Go back to 1
- The most important step of the algorithm: step 3 (*separation problem*).
- Cutting Plane algorithms may depend on the particular problem structure or be completely general.
- Independent of problem structure: *Gomory cutting planes*.
- Problem structure: *Row generation for the TSP*.

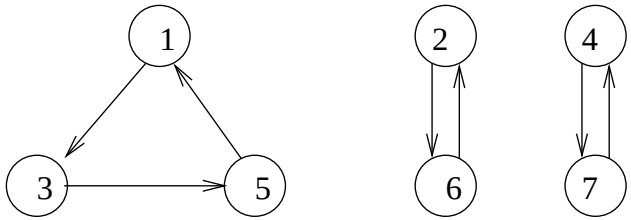
Row generation for TSP I

- TSP formulation has an exponential number of constraints (one for each proper subset of $\{1, \ldots, n\}$)
- Continuous relaxation solution becomes unmanageable as n grows
- Try relaxing (i.e., removing) problematic constraints

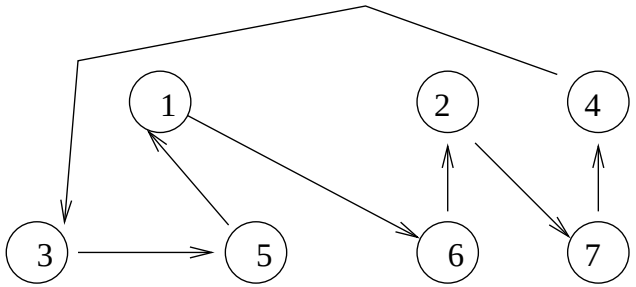
$$\forall S \subsetneq \{1, \ldots, n\} \quad \sum_{i \neq j \in S} x_{ij} \leq |S| - 1$$

- Obtain IP with a TUM matrix whose solutions are sets of disjoint cycles

Relaxed solution:



Optimal solution:

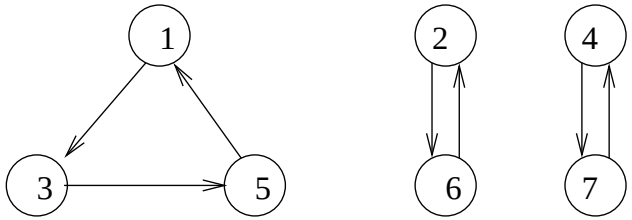


Row generation for TSP II

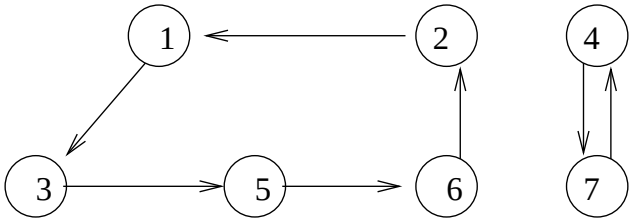
- Consider enforcing problematic constraints one by one in a Cutting Plane algorithm: how do we solve the separation problem?
- In example above, problematic constraint with $S = \{1, 2\}$ yields: $x_{12} + x_{21} \leq 1$
- The relaxed solution above has $x_{12} = x_{21} = 0$
- The constraint is valid but it is not a valid cut (i.e. current solution will not change when constraint is added to the continuous relaxation of the problem)
- Does not solve the separation problem
- Generate valid cuts by identifying disjoint cycles
- Usually requires considerably fewer than 2^n added constraints

Row generation for TSP III

- In relaxed solution, $S = \{1, 3, 5\}$ is a disjoint cycle



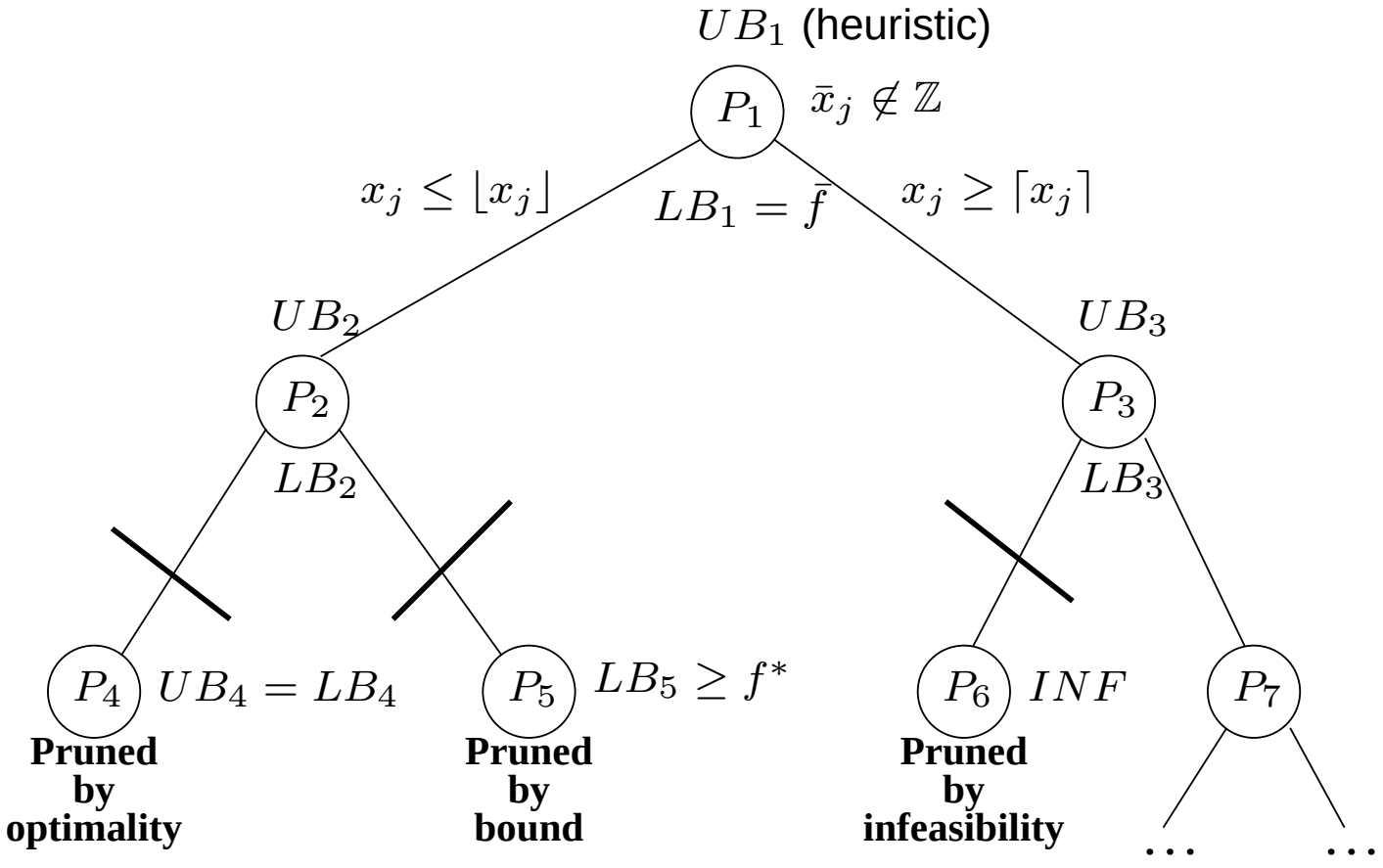
- Enforce constraint for S (at most $|S| - 1 = 2$ arcs in complete digraph on S):
  $x_{13} + x_{31} + x_{15} + x_{51} + x_{35} + x_{53} \leq 2$
- Get:



- In above relaxed solution, $S = \{4, 7\}$ is a disjoint cycle, enforce constraint $x_{47} + x_{74} \leq 1$
- Get optimal solution

Branch-and-Bound I

Here we use the “divide and conquer” approach. If we cannot solve a problem, we break it into easier subproblems. We do it using an enumeration tree.



Branch-and-Bound II

1. Initialize list problem $L = \{P\}$, best objective function value $f^* = \infty$, $x^* =$ "infeasible"
2. If $L = \emptyset$, terminate with solution x^*
3. Select a subproblem Q from L and remove it from L
4. (*Bound*) Solve R_Q to find solution $\bar{x}$ with objective value $\bar{f}$
5. If R_Q is infeasible, back to 2 (prune by infeasibility)
6. If $\bar{f} \geq f^*$, Q cannot contain optimal solution, back to 2 (prune by bound)
7. If $\bar{x}$ is integral and $\bar{f} < f^*$: update $x^* = \bar{x}$, $f^* = \bar{f}$, back to 2 (prune by optimality)
8. (*Branch*) Select a fractional component $\bar{x}_j$, generate two subproblems from Q with added constraints $x_j \leq \lfloor \bar{x}_j \rfloor$ and $x_j \geq \lceil \bar{x}_j \rceil$ respectively, add them to L, then back to 2

Branch-and-Bound III

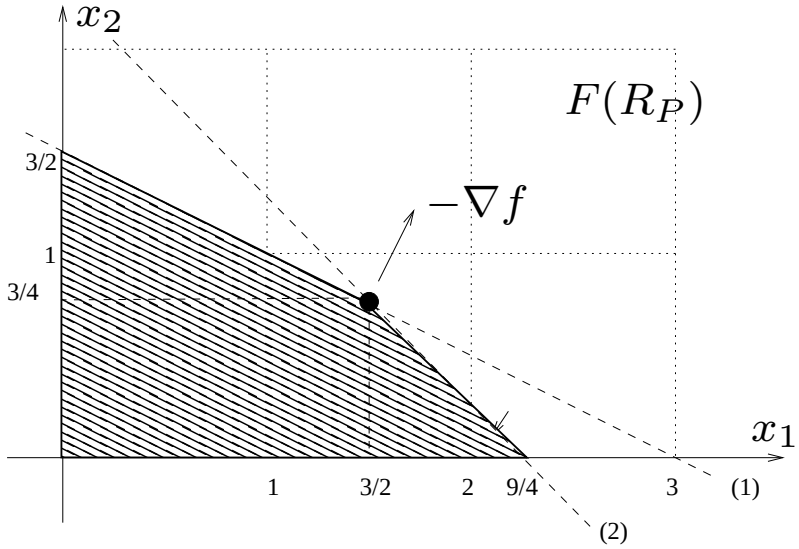
- How do we choose a subproblem Q from L (step 3)?
- How do we select a fractionary component $\bar{x}_j$ from $\bar{x}$ (step 8)?
- No “best answer”, depends on problem structure.
- Choice of subproblem: associate $LB = \bar{f}$ to each generated problem, then choose subproblem with minimum LB.
- Choice of fractionary component: choose the component with fractionary value closest to 0.5.

BB example I

Consider simple example:

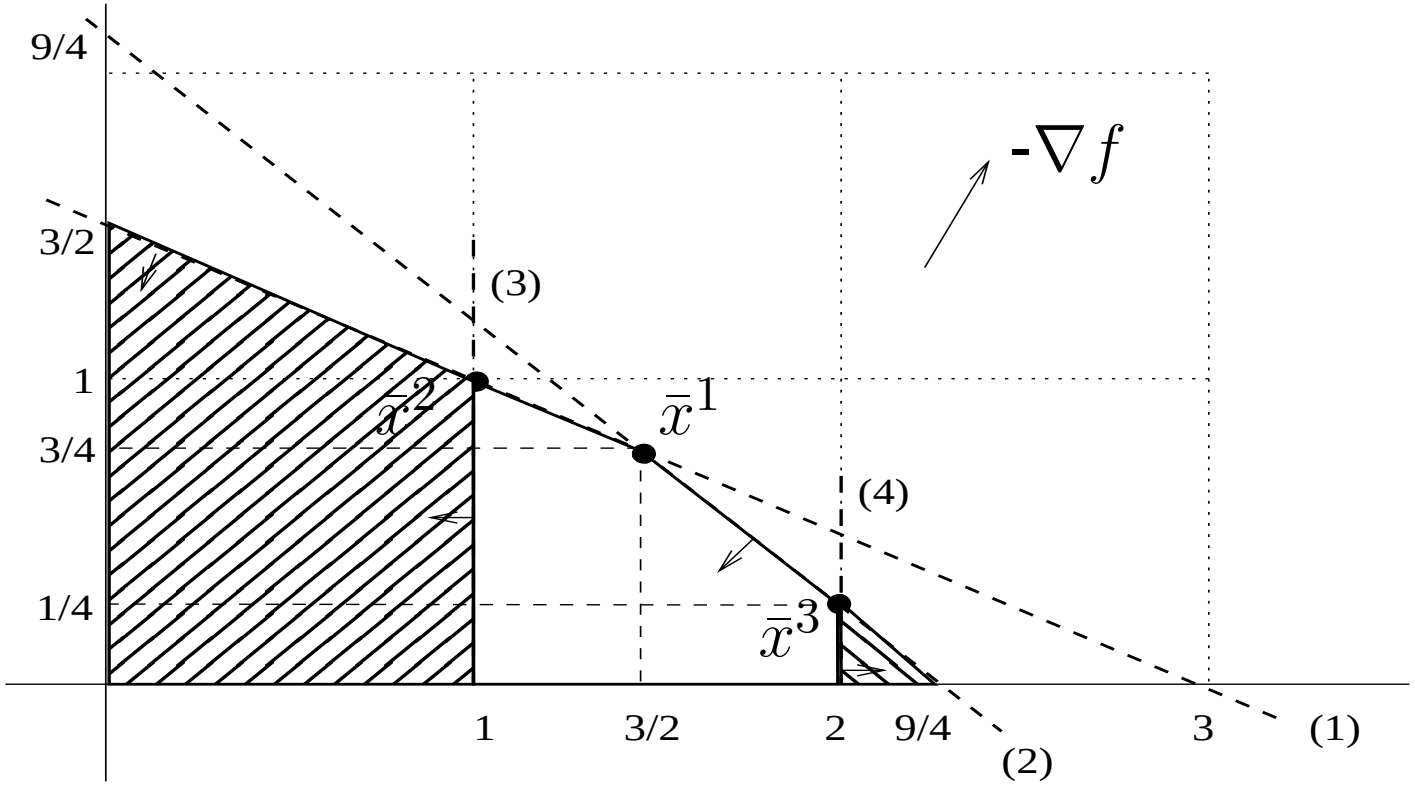
$$\left.\begin{array}{rrcl} \min & -2x_1 - 3x_2 & & \\ & x_1 + 2x_2 & \leq & 3 \\ & 4x_1 + 4x_2 & \leq & 9 \\ & x_1, x_2 & \in & \mathbb{Z}_+ \end{array}\right\}$$

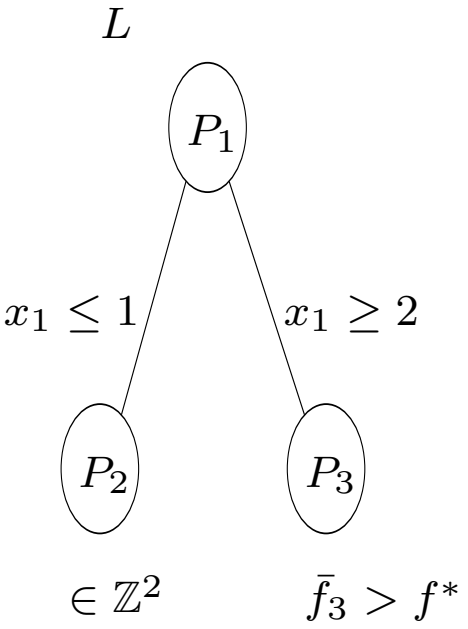
Solution of R_P is at $\bar{x} = (3/2, 3/4)$ with $\bar{f} = -21/4$



BB example II

$\bar{x}^i$ = solution of R_{P_i}, $\bar{f}_i$ = optimal objective value of R_{P_i}, $\forall i$





P_2	P_1	P_3
$\bar{x}^2 = (1, 1)$	$\bar{x}^1 = (1.5, 0.75)$	$\bar{x}^3 = (2, 0.25)$
$\bar{f}_2 = -5$	$\bar{f}_1 = -5.25$	$\bar{f}_3 = -4.75$

Branch-and-Cut

- In the Branch-and-Bound algorithm, before branching, we generate valid cuts for the current fractional solution $\bar{x}$.
- The cuts are generated until there is no much progress on the value $\bar{f}$ of the objective function.
- Cuts can be general or problem specific.
- Solvers, like *Cplex* generate cuts by default.
- Most used classes of general cuts: *Gomory cuts*, *(flow) cover cuts*.

Gomory inequalities

- Let $X = P \cap \mathbb{Z}^n$, where $P = \{x \in \mathbb{R}^n_+ : \ Ax \leq b\}$, A is an $m \times n$ matrix with columns $(a_1, \ldots, a_n)$, and $u \in \mathbb{R}^m_+$.
- $\sum_{j=1}^{n} ua_j x_j \leq ub$ is valid for P as $u \geq 0$;
- $\sum_{j=1}^{n} \lfloor ua_j \rfloor x_j \leq ub$ is valid for P, as $x \geq 0$;
- $\sum_{j=1}^{n} \lfloor ua_j \rfloor x_j \leq \lfloor ub \rfloor$ is valid for X, as x is integer.
- Using this procedure, we can generate **all** valid inequalities for an integer program.

Cover inequalities

- Let $X = \left\{ x \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}$, $a_j \geq 0, \forall j \leq n$, $b \geq 0$, $N = \{1, 2, \ldots, n\}$.
- Set $C \subseteq N$ is a *cover* if $\sum_{j \in C} a_j > b$.
- If $C \subseteq N$ is a cover, then the cover inequality

$$\sum_{j \in C} x_j \leq \mid C \mid -1$$

is valid for X.

Course material

- C. Papadimitriou, K. Steiglitz, *Combinatorial Optimization: Algorithms and Complexity*, Dover, New York, 1998
- L. Wolsey, *Integer Programming*, John Wiley & Sons, Inc, New York, 1998.