Algorithms for finding minimum fundamental cycle bases in graphs

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27 February 2004

Abstract

We describe new heuristics for solving the problem of finding the fundamental cycle bases of minimum cost in a simple, undirected, biconnected graph G. Since each spanning tree of G is associated to a fundamental cycle basis, edge swaps are iteratively performed on the current spanning tree so as to improve the cost of the corresponding fundamental cycle basis. Furthermore, we establish graph-theoretical structural results that allow an efficient implementation of our algorithms.

1 Introduction

Let $G = (V, E)$ be a simple, undirected and biconnected graph with n nodes and m edges, weighted by a non-negative cost function $w : E \to \mathbb{R}^+$, which is extended to sets of edges in the natural way (if $F \subseteq E$, $w(F) = \sum_{e \in F} w(e)$). A set of cycles in the graph is a *cycle basis* if it is a basis of the cycle vector space. The cost of a set of cycles is the sum of the costs of all cycles in the set. Given any spanning tree of G with edge set $T \subseteq E$, the edges in T are called *branches* of the tree, and those in $E \backslash T$ are called the *chords* of G with respect to T. Any chord uniquely identifies a cycle consisting of the chord

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itself and the unique path in T connecting the two nodes incident on the chord. These $m - n + 1$ cycles are called *fundamental cycles* and they form a *Fundamental Cycle Basis* (FCB) of G with respect to T. It was shown that a cycle basis is fundamental if and only if each cycle in the basis contains at least one edge which is not contained in any other cycle in the basis [9]. Finding the Minimum Fundamental Cycle Basis (MIN FCB) of a graph is an $\mathcal{NP}$-hard problem [2]. Furthermore, it does not admit a polynomial-time approximation scheme unless $\mathcal{P} = \mathcal{NP}$; a $(4+\varepsilon)$-approximation algorithm was found for complete graphs, and a $2^{O(\sqrt{\log n \log \log n})}$-approximation algorithm for arbitrary graphs [7].

Interest in minimum FCBs arises in a variety of application fields, such as electrical circuit testing [1], periodic timetable planning [6] and generating minimal perfect hash functions [3].

2 Edge-swapping local search and metaheuristics

Our local search for the MIN FCB problem is based on an iterative improvement of a current spanning tree, obtained by performing edge swaps. We start from an initial spanning tree grown by adding nodes to the tree in such a way that short fundamental cycles are completed early in the process (based on [8]). At each iteration, we identify the edge swap between branch and chord that leads to the largest decrease in FCB cost. This edge-swapping operation is inserted in a local search procedure.

Consider any given spanning tree T of G. For each branch $e \in T$, the *fundamental cut* of G induced by e is the edge set $\delta^e_T = \{\{u,v\} \in E \mid u \in S^e_T, v \in \bar{S}^e_T\}$, where S^e_T, $\bar{S}^e_T$ is the node partition induced by the removal of e from T. For any chord $f \in \delta^e_T$, let $\pi = (e, f)$ be the edge swap which consists in removing e while adding f to T. Denote by πT the resulting spanning tree. Now for each such edge swap π we calculate the cost difference Δ_π between the FCB of T and that of πT. Let Δ_{opt} be the largest such difference, and π_{opt} be the correspoding edge swap. The local search iteratively identifies π_{opt} and updates the current T with $\pi_{\text{opt}} T$ while π_{opt} is not the identity.

Applying an edge swap to a spanning tree may change the fundamental cycles and cut structure considerably. Hence, efficient procedures are needed to determine the cuts $\delta^e_{\pi T}$ for all $e \in \pi T$, and to compute Δ_π from the data at the previous iteration, namely from T, π and the cuts δ^e_T, for $e \in T$.

Some of the following structural properties are straightforward, others can be proved by careful case enumeration.

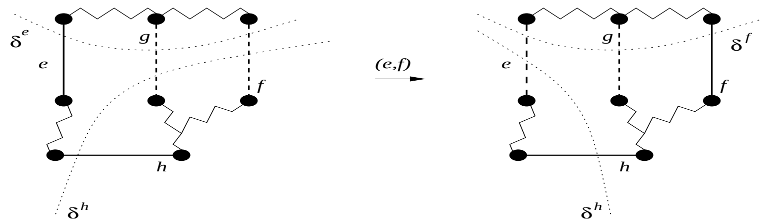

Fig. 1. Let $g \in \delta^h \cap \delta^e$. Then $g \notin \pi(\delta^h)$.

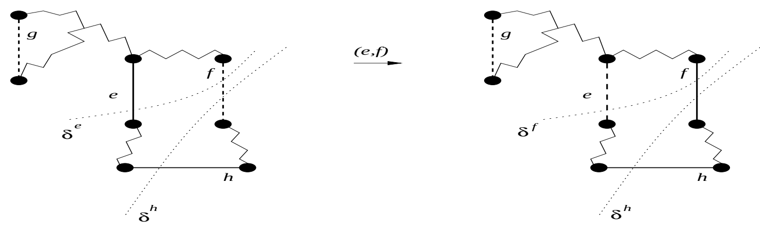

Fig. 2. Let $g \notin \delta^h \cup \delta^e$. Then $g \notin \pi(\delta^h)$.

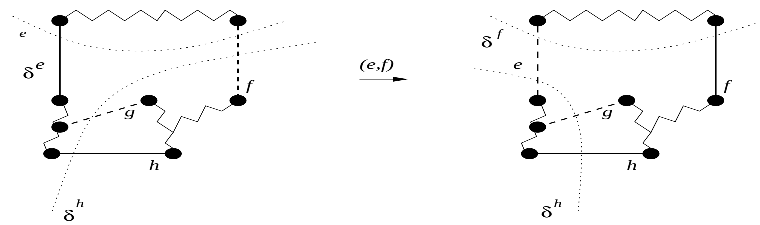

Fig. 3. Let $g \in \delta^h$ and $g \notin \delta^e$. Then $g \in \pi(\delta^h)$.

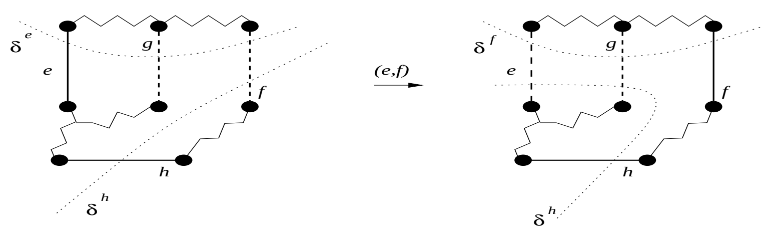

Fig. 4. Let $g \notin \delta^h$ and $g \in \delta^e$. Then $g \in \pi(\delta^h)$.

Efficient cut structure update:

- any edge swap $\pi = (e, f)$ applied to a spanning tree T, where $e \in T$ and $f \in \delta^e_T$, changes a cut δ^h_T if and only if $f \in \delta^h_T$;
- $\delta^h_{\pi T}$ can be determined by taking the symmetric difference $\delta^h_T \triangle \delta^e_T$ (see Figures 1-4 for a graphical sketch of the proof).

Efficient cycle structure update (notation: γ^h_T is the unique fundamental cycle of G w.r.t. the chord h):

- if $h \notin \delta^e_T$, then γ^h_T is unchanged by π;
- if $h \in \delta^e_T$, then $\gamma^h_{\pi T}$ can be determined by taking the symmetric difference $\gamma^h_T \triangle \gamma^f_T$.

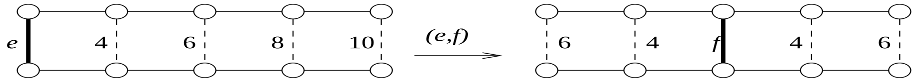

Fig. 5. All edge weights are equal to 1 and the numbers indicated on the chords correspond to the costs of the corresponding fundamental cycles. The cut on the left has a difference between cheapest and most expensive cycle of $10 - 4 = 6$; after the edge swap the difference is $6 - 4 = 2$.

It can be verified that the complexity of identifying the best edge swap π_{opt} and applying it to T to obtain πT is $O(m^2 n^2)$.

The implementation of the local search algorithm described above is computationally intensive. For large-scale problems, we would like to test the edge swap only for a small subset of pairs e, f while minimizing the chances of missing pairs which yield large cost decreases. A good strategy is to focus on branches inducing fundamental cuts whose edges define fundamental cycles with "unbalanced" costs, i.e., with a large difference between the cheapest and the most expensive of those fundamental cycles. See Fig. 5 for a simple example.

To try to escape from local minima, we have included the above edge-swap move within two well-known metaheuristics: variable neighbourhood search (VNS) [4] and tabu search (TS) [5]. We used a basic implementation of VNS. Our implementation of the Tabu search, on the other hand, is a blend of classic TS and VNS. If π_{opt} is the identity, an edge swap that worsens the FCB cost is applied to the current solution and inserted in a tabu list. If all possible edge swaps are tabu or a pre-determined number of successive non-improving moves is exceeded, t random edge swaps are applied to the current spanning tree. The number t increases until a pre-determined limit is reached, and is then re-set to 1. The procedure runs until a given termination condition is met.

3 Some computational results

We ran extensive tests over three classes of graphs.

(1) *Square mesh graphs with unit edge costs.* These are $n \times n$ square meshes with nodes positioned at (p, q) where $p, q \in \mathbb{Z}$ and $0 \leq p, q < n$ (n^2 vertices and $2n(n-1)$ edges). Because of the high degree of symmetry of the graph topology and the uniform edge costs, these are considered hard instances where previous constructive heuristics [2,3] performed badly, with FCB costs being on average three times as large as those of the solutions produced by our algorithms.

(2) *Random simple Euclidean weighted graphs.* The nodes are positioned randomly on a 20×20 square centered at the origin. Each edge between pair of nodes is randomly generated with probability p and cost equal to the Euclidean distance between its adjacent nodes. Our solutions were on average 50% better than those obtained with previous constructive methods [2,3]. For small instances (10-15 nodes) our local search actually found the optimal solutions.

(3) *Application to periodic timetabling.* This application is described in [6]. Finding minimum FCBs of appropriate graphs leads to a more compact MIP formulation of a certain type of Periodic Event Scheduling Problem (PESP). We were able to find solutions between 5% to 20% better than those found by C. Liebchen using a purpose-built modification of Deo's methods.

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