Barrier certificate with Interval Analysis

A review of Adel's PhD thesis

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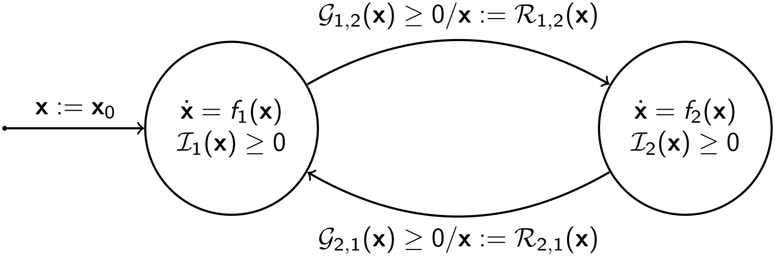
Main goal

Formal verification of safety properties of hybrid systems described by hybrid automata.



Barrier certificate has been considered to achieve this goal and a 2 step process has been considered:

1. for continuous-time dynamical systems
2. then for hybrid automata

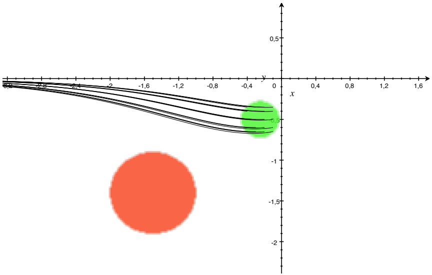
Safety of continuous dynamical systems

Consider a non-linear dynamical system S

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{d})$$

with $\mathbf{d} \in \mathcal{D}$ a constant and bounded disturbance

S is safe iff all trajectories starting from the initial region do not reach the unsafe region.

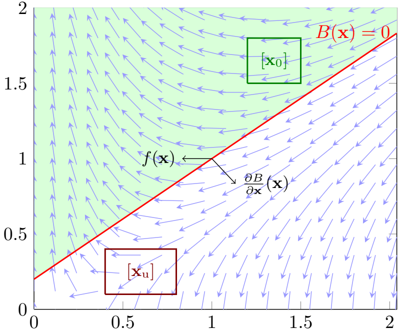


Barrier certificates

Main idea of [Prajna&Jadbabaie HSCC04]

A barrier is a **function** separating

- the unsafe region $\mathcal{X}_u$
- all trajectories starting form the initial region $\mathcal{X}_0$.



does not require computation of reachable set.

Conditions on Barrier Function

To be a valid barrier functions, [Prajna & Jadbabaie, HSCC04] shows that $B(\mathbf{x})$ has to satisfy

$$\begin{cases} B(\mathbf{x}) \le 0 & \forall \mathbf{x} \in \mathcal{X}_0 \\ B(\mathbf{x}) > 0 & \forall \mathbf{x} \in \mathcal{X}_u \\ B(\mathbf{x}) = 0 \Rightarrow \left\langle \frac{\partial B}{\partial x}(\mathbf{x}), f(\mathbf{x}, \mathbf{d}) \right\rangle < 0 & \forall \mathbf{x} \in \mathcal{X} \end{cases}$$

Finding a barrier function is difficult in general

Parametric Barrier Function

In [Prajna & Jadbabaie HSCC04], parametric barrier functions $B(\mathbf{x}, \mathbf{p})$ are considered. They have to satisfy

$$\exists \mathbf{p} \in \mathcal{P} :$$

$$\begin{cases} B(\mathbf{x}, \mathbf{p}) \leq 0 & \forall \mathbf{x} \in \mathcal{X}_0 \\ B(\mathbf{x}, \mathbf{p}) > 0 & \forall \mathbf{x} \in \mathcal{X}_u \\ B(\mathbf{x}, \mathbf{p}) = 0 \Rightarrow \left\langle \frac{\partial B}{\partial x}(\mathbf{x}, \mathbf{p}), f(\mathbf{x}, \mathbf{d}) \right\rangle < 0 & \forall \mathbf{x} \in \mathcal{X} \end{cases}$$

Example

- $B_1(\mathbf{x}, \mathbf{p}) = p_0 x_0 + p_1 x_1 + p_2$
- $B_2(\mathbf{x}, \mathbf{p}) = p_0 \ln(x_0) + p_1 x_1 + p_2$

In [Prajna & Jadbabaie HSCC04] only polynomial dynamical systems and polynomial barrier functions are considered,
with the 3rd constraint relaxed into

$$\left(\left\langle \frac{\partial B}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{p}), f\left(\mathbf{x}, \mathbf{d}\right)\right) \right\rangle < 0 \right) \qquad \forall \mathbf{x} \in \mathcal{X}$$

Designing barriers via interval analysis

We assume that the sets $\mathcal{X}_0$ and $\mathcal{X}_\mathrm{u}$ are defined by

$$\mathcal{X}_0 = \{\mathbf{x} \in \mathcal{X} \mid g_0(\mathbf{x}) \leqslant 0\}$$
$$\mathcal{X}_u = \{\mathbf{x} \in \mathcal{X} \mid g_u(\mathbf{x}) \leqslant 0\}.$$

with $g_0 : \mathcal{X} \to \mathbb{R}$ and $g_u : \mathcal{X} \to \mathbb{R}$ two known functions

Quantified Constraint Satisfaction Problem approach

Theorem

If $\exists \mathbf{p} \in \mathcal{P}$ *such that* $\forall \mathbf{x} \in \mathcal{X}$, $\forall \mathbf{d} \in \mathcal{D}$

$$
\begin{aligned}
\xi\left(\mathbf{x}, \mathbf{p}, \mathbf{d}\right) = & \left(g_0(\mathbf{x}) > 0 \vee B(\mathbf{x}, \mathbf{p}) \leqslant 0\right) \\
& \wedge \left(g_u(\mathbf{x}) > 0 \vee B(\mathbf{x}, \mathbf{p}) > 0\right) \\
& \wedge \left(B(\mathbf{x}, \mathbf{p}) \neq 0 \vee \left\langle \frac{\partial B}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{p}), f(\mathbf{x}, \mathbf{d}) \right\rangle < 0 \right)
\end{aligned}
$$

then the dynamical system is safe

Note: this offers a convenient way to treat each constraint of the conjunction in the same way.

Starting point

Consider some function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ and some box $[\mathbf{z}] \in \mathbb{IR}^k$.

CSC-FPS [Jaulin & Walter 1996] is designed to determine whether

$$\exists \mathbf{p} \in [\mathbf{p}]\,,\ \forall \mathbf{x} \in [\mathbf{x}]\,,\ g\left(\mathbf{x}, \mathbf{p}\right) \in [\mathbf{z}]$$

We consider this approach here.

CSC-FPS consists of:

- FPS (Feasible Point Searcher): explores parameter space $\mathcal{P} = [\mathbf{p}]$ to find some satisfying $\mathbf{p}$.
- CSC (Computable Sufficient Condition): checks whether $\mathbf{p}$ satisfies the constraint for all $\mathbf{x} \in \mathcal{X} = [\mathbf{x}]$

Additional contributions

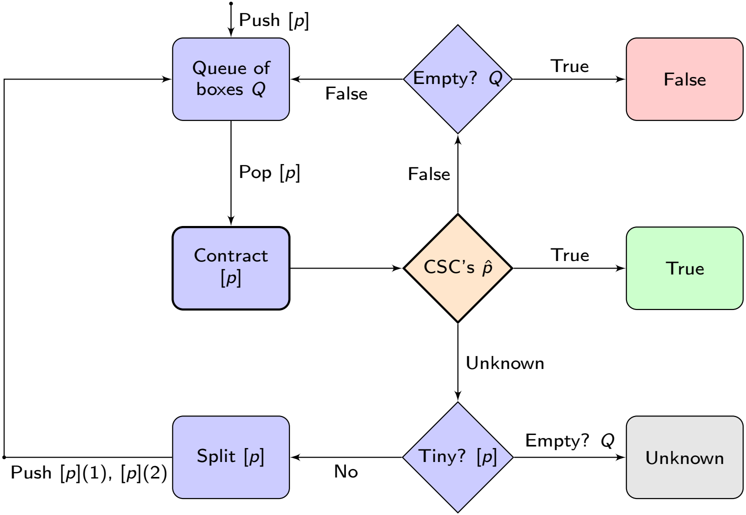
- **Adaptation:** Extending to handle conjunction of constraints, all are of the form

$$\tau\left(\mathbf{x},\mathbf{p},\mathbf{d}\right)=\left(u(\mathbf{x},\mathbf{p})\in\mathcal{A}\right)\vee\left(v(\mathbf{x},\mathbf{p},\mathbf{d})\in\mathcal{B}\right).$$

- **Improvements:** Enhance the algorithm by adding contractors operators

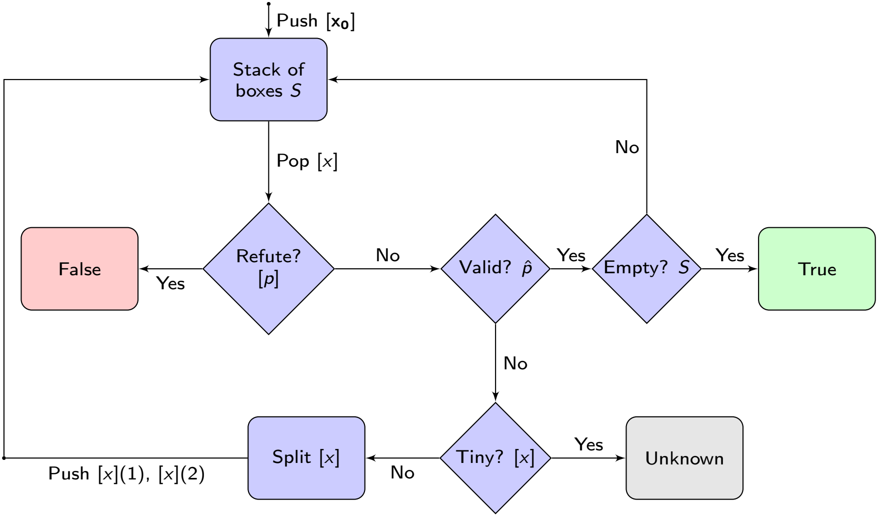
Algorithm: FPS

Input $[p]$ a box of parameter, f and a parametric function B



Algorithm: CSCInit case

Input $[p]$ a box of parameter and $[\mathbf{x}_0]$



Verification of a constraint – validation

For a given $\mathbf{p}$, validation of

$$\exists \mathbf{p} \in [\mathbf{p}], \forall \mathbf{x} \in [\mathbf{x}], \forall \mathbf{d} \in [\mathbf{d}], \quad (u(\mathbf{x},\mathbf{p}) \in \mathcal{A}) \vee (v(\mathbf{x},\mathbf{p},\mathbf{d}) \in \mathcal{B}).$$

One can outer-approximate for a given $\mathbf{p} \in [\mathbf{p}]$

$$u([\mathbf{x}], \mathbf{p}) = \{u(\mathbf{x},\mathbf{p}) \mid \mathbf{x} \in [\mathbf{x}]\}$$
$$v([\mathbf{x}], [\mathbf{d}], \mathbf{p}) = \{v(\mathbf{x},\mathbf{d},\mathbf{p}) \mid \mathbf{x} \in [\mathbf{x}], \mathbf{d} \in [\mathbf{d}]\}$$

using **inclusion functions** $[u]([\mathbf{x}], \mathbf{p})$ and $[v]([\mathbf{x}], [\mathbf{d}], \mathbf{p})$

Consequence

If $[u]([\mathbf{x}], \mathbf{p}) \subseteq \mathcal{A}$ or $[v]([\mathbf{x}], [\mathbf{d}], \mathbf{p}) \subseteq \mathcal{B}$ then

$u([\mathbf{x}], \mathbf{p}) \subseteq \mathcal{A}$ or $v([\mathbf{x}], [\mathbf{d}], \mathbf{p}) \subseteq \mathcal{B}$

and

$$\forall \mathbf{x} \in [\mathbf{x}], \forall \mathbf{d} \in [\mathbf{d}] \;\; u(\mathbf{x},\mathbf{p}) \in \mathcal{A} \vee u(\mathbf{x},\mathbf{d},\mathbf{p}) \in \mathcal{B}$$

is satisfied.

Verification of a constraint – refutation

- either using **Inclusion functions**, on the negation of the constraint

$$\forall \mathbf{p} \in [\mathbf{p}], \exists \mathbf{x} \in [\mathbf{x}], \exists \mathbf{d} \in [\mathbf{d}], \quad u(\mathbf{x}, \mathbf{p}) \subseteq \overline{\mathcal{A}} \wedge v(\mathbf{x}, \mathbf{p}, \mathbf{d}) \subseteq \overline{\mathcal{B}},$$

  for a given $\mathbf{x}$ and a given $\mathbf{d}$ (Note: we can try several random values)

- either using **Contractors**
  A contractor $\mathcal{C}_{g,[\mathbf{z}]}$ associated to $\{\mathbf{x} \in [\mathbf{x}] : g(\mathbf{x}) \in [\mathbf{z}]\}$ is s.t.
  - Reduction:
  $$\mathcal{C}_{g,[\mathbf{z}]}([\mathbf{x}]) \subseteq [\mathbf{x}]$$
  - Soundness:
  $$[g]([\mathbf{x}]) \cap [\mathbf{z}] = [g]\left(\mathcal{C}_{g,[\mathbf{z}]}([\mathbf{x}])\right) \cap [\mathbf{z}]$$

  They can be composed, with $c_i : \{\mathbf{x} \in [\mathbf{x}] : g_i(\mathbf{x}) \in [\mathbf{z}]_i\}$

$$\mathcal{C}_{c_1 \wedge c_2}([\mathbf{x}]) = \mathcal{C}_{c_1}([\mathbf{x}]) \cap \mathcal{C}_{c_2}([\mathbf{x}])$$
$$\mathcal{C}_{c_1 \wedge c_2}([\mathbf{x}]) = \mathcal{C}_{c_2}(\mathcal{C}_{c_1}([\mathbf{x}]))$$
$$\mathcal{C}_{c_1 \vee c_2}([\mathbf{x}]) = \square\{\mathcal{C}_{c_1}([\mathbf{x}]) \cup \mathcal{C}_{c_2}([\mathbf{x}])\}$$

Note: several contractor algorithms exist, *e.g.*, HC4Revise, 3BCID, etc.

Using contractors – 1

Proposition

Consider a box $[\mathbf{x}]$*, the constraint* $c : \{\mathbf{x} \in [\mathbf{x}] : g(\mathbf{x}) \in [\mathbf{z}]\}$*, and the contracted box* $\mathcal{C}_c\left([\mathbf{x}]\right) \subseteq [\mathbf{x}]$*. Then,*

$$\forall \mathbf{x} \in [\mathbf{x}] \setminus \mathcal{C}_c\left([\mathbf{x}]\right), \text{ one has } g(\mathbf{x}) \notin [\mathbf{z}], \tag{1}$$

where $[\mathbf{x}] \setminus \mathcal{C}_c\left([\mathbf{x}]\right)$ *denotes the box* $[\mathbf{x}]$ *deprived from* $\mathcal{C}_c\left([\mathbf{x}]\right)$*, which is not necessarily a box.*

Using contractors – 2

Consider the constraint

$$\tau : (u(\mathbf{x}, \mathbf{p}) \in \mathcal{A}) \vee (v(\mathbf{x}, \mathbf{p}, \mathbf{d}) \in \mathcal{B})$$

and a contractor $\mathcal{C}_\tau$ for this constraint.

For the boxes $[\mathbf{x}]$, $[\mathbf{p}]$, and $[\mathbf{d}]$, one gets

$$\left([\mathbf{x}]', \quad [\mathbf{p}]', \quad [\mathbf{d}]'\right) = \mathcal{C}_\tau\left([\mathbf{x}], \quad [\mathbf{p}], \quad [\mathbf{d}]\right)$$

We have different cases to consider in function of the values of the contracted boxes $[\mathbf{x}]'$, $[\mathbf{p}]'$, and $[\mathbf{d}]'$.

Note: we can do the same with $\overline{\tau}$

Using contractors – 3

3 cases are considered:

1. If $[\mathbf{p}] \setminus [\mathbf{p}]' \neq \emptyset$, then $\forall \mathbf{p} \in [\mathbf{p}] \setminus [\mathbf{p}]'$, $\forall \mathbf{x} \in [\mathbf{x}]$, $\forall \mathbf{d} \in [\mathbf{d}]$,

$$u(\mathbf{x}, \mathbf{p}) \notin \mathcal{A} \wedge v(\mathbf{x}, \mathbf{p}, \mathbf{d}) \notin \mathcal{B},$$

   $\Rightarrow$ the search space is reduced to $[\mathbf{p}]'$,

2. If $[\mathbf{x}] \setminus [\mathbf{x}]' \neq \emptyset$ then, one has $\forall \mathbf{p} \in [\mathbf{p}]$, $\forall \mathbf{x} \in [\mathbf{x}] \setminus [\mathbf{x}]'$, $\forall \mathbf{d} \in [\mathbf{d}]$,

$$u(\mathbf{x}, \mathbf{p}) \notin \mathcal{A} \wedge v(\mathbf{x}, \mathbf{p}, \mathbf{d}) \notin \mathcal{B}$$

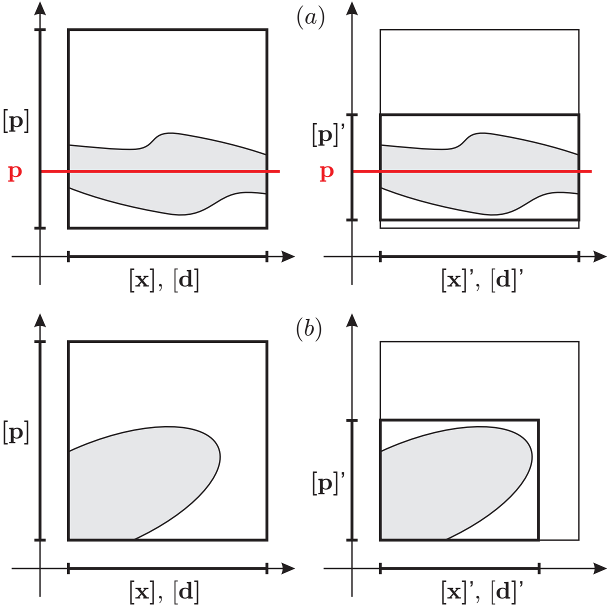
   and there is no $\mathbf{p} \in [\mathbf{p}]$ such that τ holds true for all $\mathbf{x} \in [\mathbf{x}]$,

3. If $[\mathbf{d}] \setminus [\mathbf{d}]' \neq \emptyset$, then $\forall \mathbf{p} \in [\mathbf{p}]$, $\forall \mathbf{x} \in [\mathbf{x}]$, $\forall \mathbf{d} \in [\mathbf{d}] \setminus [\mathbf{d}]'$,

$$u(\mathbf{x}, \mathbf{p}) \notin \mathcal{A} \wedge v(\mathbf{x}, \mathbf{p}, \mathbf{d}) \notin \mathcal{B}$$

   and there is no $\mathbf{p} \in [\mathbf{p}]$ such that τ holds true for all $\mathbf{d} \in [\mathbf{d}]$,

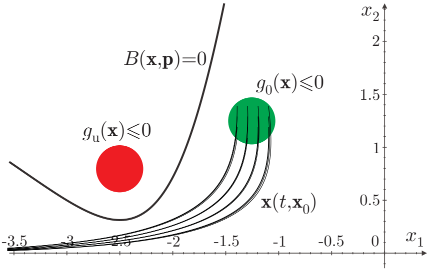
Using contractors – 3



Example: rational barrier function

Example

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 - 0.5 x_2^2 \end{pmatrix}$$

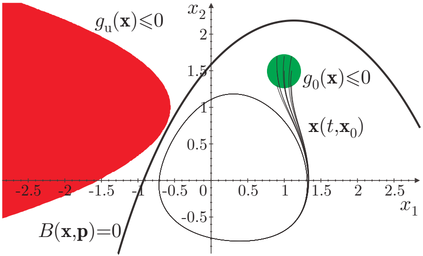


Parametric barrier function: $B(\mathbf{x}, \mathbf{p}) = \frac{p_1 p_2 (x_0 + p_3)}{(x_0 + p_3)^2 + p_2^2} + x_1 + p_4$

Example: system with limit cycle

Example

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 + (1 - x_1^2 - x_2^2)x_1 + \ln(x_1^2 + 1) \\ -x_1 + (1 - x_1^2 - x_2^2)x_2 + \ln(x_2^2 + 1) \end{pmatrix}$$



Parametric barrier function: $B(\mathbf{x}, \mathbf{p}) = \left(\frac{x_1 + p_1}{p_2}\right)^2 + \left(\frac{x_2 + p_3}{p_4}\right)^2 - 1$

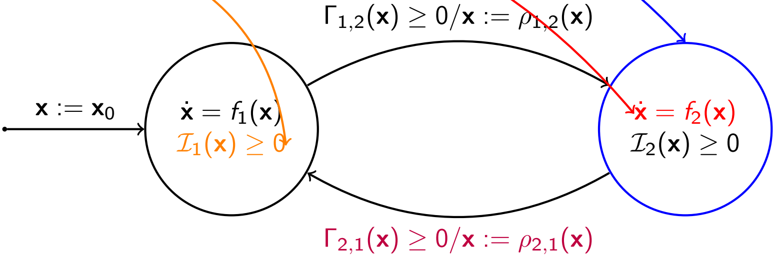
Results

			Without contr.		With contr.	
Example	n	m	time	bisect.	time	bisect.
1	2	4	36s	4520	16s	4553
2	2	3	T.O.	/	1s	159
3	2	6	1133s	20388	1s	6
4	2	6	253s	14733	7s	435
5	2	4	T.O.	/	98s	4072
6	3	4	167s	1753	21s	47
7	6	7	697s	67600	1s	261

- n the dimension of the dynamical system
- m the number of parameters of the template

Extension to hybrid automata

- **Location**
- **Flow**
- **Invariant**



- **Transtion** with **Guard** and **Reset**

Extension to Hybrid Automata

To be a valid barrier functions, [Prajna & Jadbabaie, HSCC04] shows that for an Hybrid Automaton $\mathcal{H} = (\mathcal{X}, \mathcal{L}, \mathcal{X}_0, \mathcal{I}, f, \Gamma, \rho)$

Theorem

Assume that there exist a family of differentiable functions $\beta_\ell(\mathbf{x})$, $\ell \in \mathcal{L}$ such that, for all pairs $(\ell, \ell') \in \mathcal{L}^2$ with $\ell \neq \ell'$, one has

$$
\begin{aligned}
\beta_\ell(\mathbf{x}) &\leqslant 0 \quad \forall \mathbf{x} \in \mathcal{X}_0(\ell) \\
\beta_\ell(\mathbf{x}) &> 0 \quad \forall \mathbf{x} \in \mathcal{X}_u(\ell) \\
\beta_\ell(\mathbf{x}) = 0 \implies \frac{\partial \beta_\ell(\mathbf{x})}{\partial \mathbf{x}} f_\ell(\mathbf{x}, \mathbf{d}) &< 0 \quad \forall \mathbf{x} \in \mathcal{I}(\ell), \forall \mathbf{d} \in \mathcal{D}_\ell \\
\beta_\ell(\mathbf{x}) \leqslant 0 \implies \beta_{\ell'}(\rho_{\ell,\ell'}(\mathbf{x})) &\leqslant 0 \quad \forall \mathbf{x} \in \Gamma(\ell, \ell')
\end{aligned}
$$

then the system $\mathcal{H}$ is safe.

Challenge: increasing number of constraints associated to the number of transitions

Interval analysis approach

Assume that there exists for each location $\ell \in \mathcal{L}$ some functions

- $g_0 : \mathcal{L} \times \mathcal{X} \to \mathbb{R}$,
- $g_u : \mathcal{L} \times \mathcal{X} \to \mathbb{R}$,
- $g_\Gamma : \mathcal{L} \times \mathcal{L} \times \mathcal{X} \to \mathbb{R}$
- $g_\mathcal{I} : \mathcal{L} \times \mathcal{X} \to \mathbb{R}$

such that

- $\mathcal{X}_0(\ell) = \{\mathbf{x} \in \mathcal{X} \mid g_0(\ell, \mathbf{x}) \leqslant 0\}$,
- $\mathcal{X}_u(\ell) = \{\mathbf{x} \in \mathcal{X} \mid g_u(\ell, \mathbf{x}) \leqslant 0\}$,
- $\Gamma(\ell, \ell') = \{\mathbf{x} \in \mathcal{X} \mid g_\Gamma(\ell, \ell', \mathbf{x}) \leqslant 0\}$
- $\mathcal{I}(\ell) = \{\mathbf{x} \in \mathcal{X} \mid g_\mathcal{I}(\ell, \mathbf{x}) \leqslant 0\}$.

Quanfitied Constraint Satisfaction Problem

Proposition

Consider a hybrid system described by $\mathcal{H} = (\mathcal{X}, \mathcal{L}, \mathcal{X}_0, \mathcal{I}, f, \Gamma, \rho)$*. Assume there exists a differentiable function* $\beta_\ell(\mathbf{x}, \mathbf{p})$ *which satisfies* $\forall \ell \in \mathcal{L},\ \exists \mathbf{p}_\ell \in [\mathbf{p}]_\ell, \forall \mathbf{x} \in [\mathbf{x}], \forall \mathbf{d} \in [\mathbf{d}]_\ell$

$$g_0(\ell, \mathbf{x}) > 0 \vee \beta_\ell(\mathbf{x}, \mathbf{p}_\ell) \leqslant 0, \tag{2}$$

$$g_u(\ell, \mathbf{x}) > 0 \vee \beta_\ell(\mathbf{x}, \mathbf{p}_\ell) > 0, \tag{3}$$

$$g_{\mathcal{I}}(\ell, \mathbf{x}) > 0 \vee \beta_\ell(\mathbf{x}, \mathbf{p}_\ell) \neq 0 \vee \frac{\partial \beta_\ell(\mathbf{x}, \mathbf{p}_\ell)}{\partial x} f_\ell(\mathbf{x}, \mathbf{d}) < 0, \tag{4}$$

and $\forall \ell' \in \mathcal{L}$*, with* $\ell' \neq \ell$*,*

$$g_\Gamma(\ell, \ell', \mathbf{x}) > 0 \vee \beta_\ell(\mathbf{x}, \mathbf{p}_\ell) > 0 \vee \beta_{\ell'}(\rho_{\ell,\ell'}(\mathbf{x}), \mathbf{p}_{\ell'}) \leqslant 0, \tag{5}$$

then the system $\mathcal{H}$ *is safe.*

Contribution generalization of the formalism used for continuous-time dynamical systems.

Incremental solution of QCSP

Many ways to solve this QCSP

Our approach, incremental algorithm. Main ideas:

- We add an order on the location value ranging from 1 to $|\mathcal{L}|$
- We associate a parametric barrier function to each location
- We consider all the location from 1 to $|\mathcal{L}|$
  - For location $\ell = 1$ we try to find $\mathbf{p}$ such that we have a barrier function
  - For location $\ell > 1$, we try to find $\mathbf{p}$ taking into account all the constraints associated to the transition involving ℓ
  - In case we cannot find $\mathbf{p}$ we go back to location $\ell - 1$ and retry

Results

Example	Dimension	#locations	computation time	#bisections
2-TANKS	2	2	1.7s	9488
ECO	2	2	0.082s	499
prajna	3	2	0.2s	111
CAR	6	3	0.021s	3
Collision	3	6	0.334s	1574

Conclusion and Future Work

Conclusion

- A new method to compute barrier certificate based on interval analysis.
- Can handle non-linear dynamic and non-linear barrier functions.
- **Extension of the state-of-the art** which is limited to:
  - polynomial dynamic and polynomial barrier function

Future work

- Automatically find template, see paper Goubault et al. ACC'14
- Adapt other work of Prajna et al. on reachability

Publications/Submissions

- a paper accepted at CDC'14
- a paper under review in Automatica (round 2)
- a paper in prepration