Geometry and Distributed Systems

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Based on work by M. Herlihy, S. Rajsbaum, N. Shavit...

Aim of the talk

Can we implement some functions on some distributed architecture, even if there are some crashes?

> Example: consensus on an asynchronous system
> NO: FLP'85!

- There is a nice "geometrization" of the problem
- We will solve easy problems to make you understand
- But it has also solved some new problems!
- ... and this is an active research area!

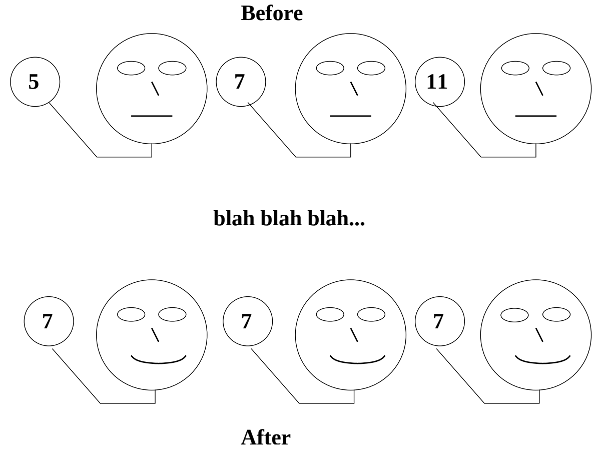
Decision tasks

Can we implement a function...given an "architecture" (faults? shared memory / message passing, synchronous / semi-synchronous / asynchronous etc.)?

Each problem is given by:

- For each processor $P_0, \ldots, P_{n-1}$ a set of possible initial values (in a domain $\mathcal{K} = \mathbb{N}$ or $\mathbf{R}$ etc.), i.e. a subset $\mathcal{I}$ of $\mathcal{K}^n$: "*input*"
- Similarly, we are given a set of possible final values $\mathcal{J}$ in $\mathcal{K}^n$: "*output*"
- Finally, we are given a map, the "*decision map*" $\delta : \mathcal{I} \to \wp(\mathcal{J})$ associating to each possible initial value, the set of authorized output values

Example: consensus



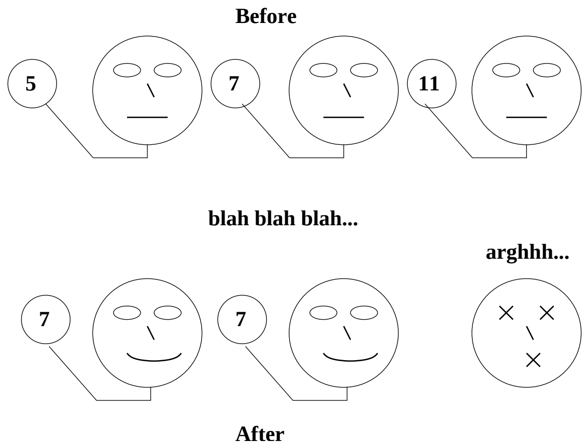
Main idea

- The *input* set and output sets have a geometrical structure (simplicial set)
- According to the architecture type, not all decision maps can be programmed
- There are geometrical constraints on the decision maps
- Very much like mainstream results in geometry, such as Brouwer's fixed point theorem...

Road map

- Input and output sets as simplicial sets (examples)
- Some basic algebraic topology
- The dynamics as sets of simplicial sets (protocol simplicial set, or complex)
- Some results and references

Even if...

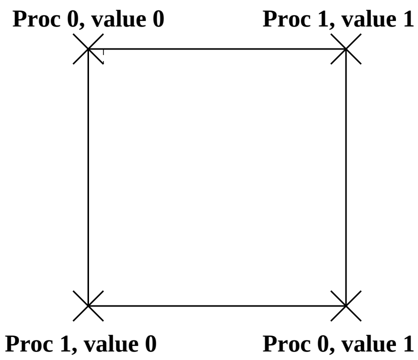


Example

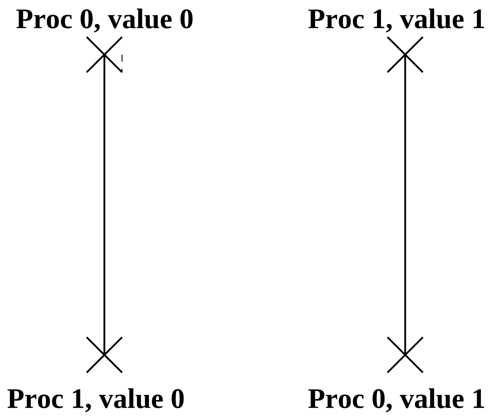
- $\mathcal{K} = \mathbb{N}, \mathcal{I} = \mathbb{N}^n$,
- $\mathcal{J} = \{(n, n, \ldots, n) \mid n \in \mathbb{N}\}$,
- $\delta(x_0, x_1, \ldots, x_{n-1}) = \left\{ \begin{array}{c} \{(x_0, x_0, \ldots, x_0), \\ (x_1, x_1, \ldots, x_1), \\ \ldots, \\ (x_{n-1}, x_{n-1}, \ldots, x_{n-1})\} \end{array} \right.$

Initial states for (binary) consensus

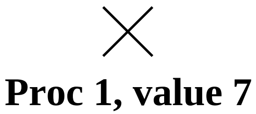
Here, 2 processors, i.e. dimension 2:



Final states for consensus

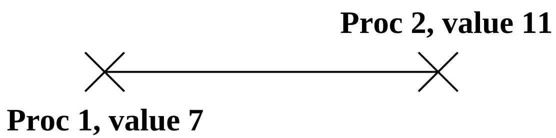


Simplicial model of states



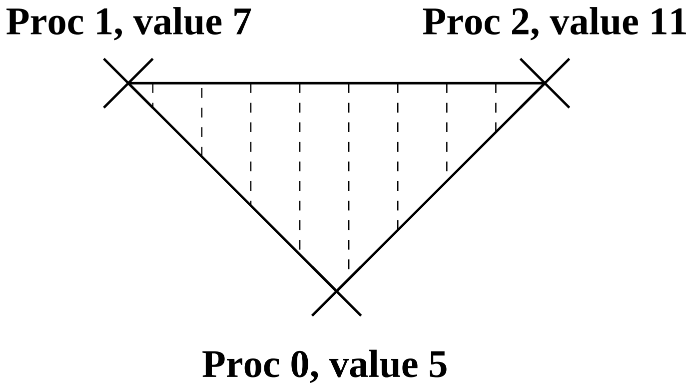
(local state)

Simplicial model of states



(compound state)

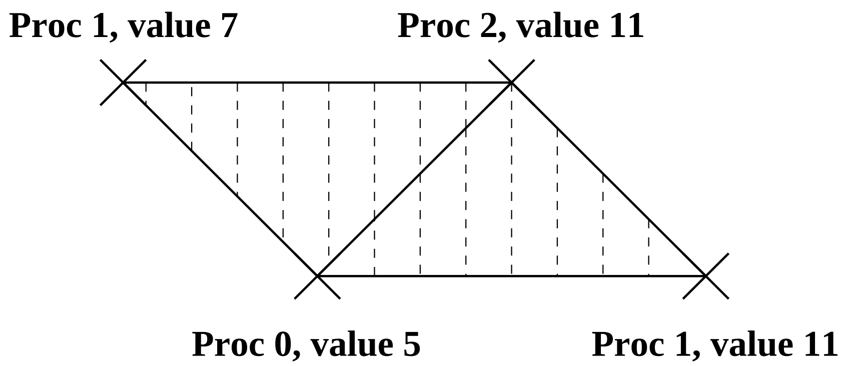
More generally: Simplicial model of states



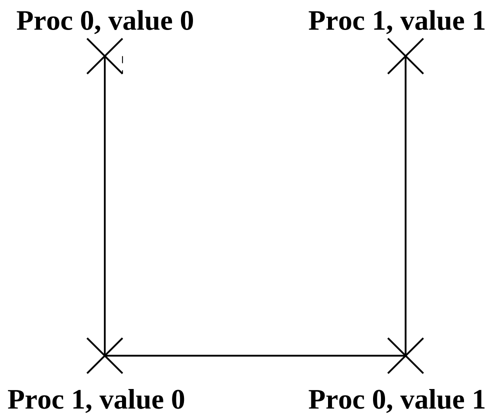
(More generally [than a graph]: global state)

Example

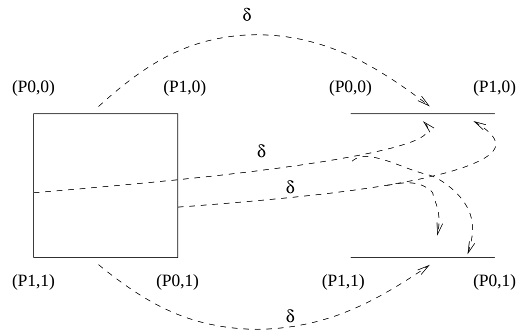
Simplicial set=set of global states (with some common local states)



Final states for pseudo-consensus



Example: Consensus specification



Example

Synchronous message passing; notion of round:

- at each round, every processor broadcasts its own value to the others
- in any order
- then every processor receives the broadcasted values and computes a new local value

Failure models

- crash (fail-stop),
- byzantine etc.

In what follows: crash failures only; can happen at any point of the broadcast, which can be done in any random order.

Back to *protocols*

- Finite program
- Starts with input values
- Fixed number of rounds
- Halts with decision value

The full-information protocol is the one where the local value is the full history of communications

Generic protocol

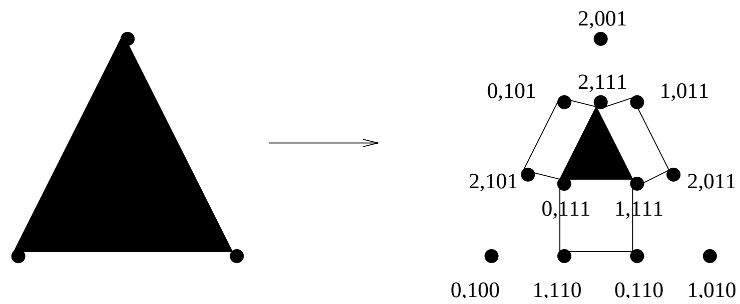
```
s = empty;
for (i=0; i<r; i++) {
  broadcast messages;
  s = s + messages received;
}
return delta(s);
```

Explanation

In the synchronous model, at round 1:

- no process has failed, hence everybody has received the message of the others (hence the central segment as global state)
- one process has failed, hence two points as possible states

Synchronous protocol complex



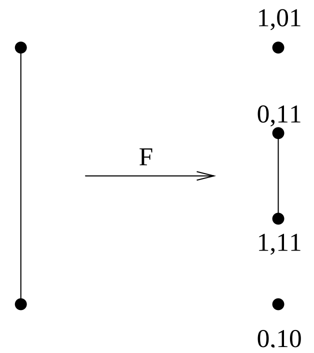
(wait-free - if up to 1 failure, forget the isolated points!)

Protocol complex

Each protocol on some architecture defines:

- a simplicial set (for all rounds r):
  - vertices: sequence of messages received at a given round r
  - simplices: compound states at round r
- This is an operator on an input simplex
- A choice of model of computation entails some geometrical properties of the protocol complex

Synchronous protocol complex



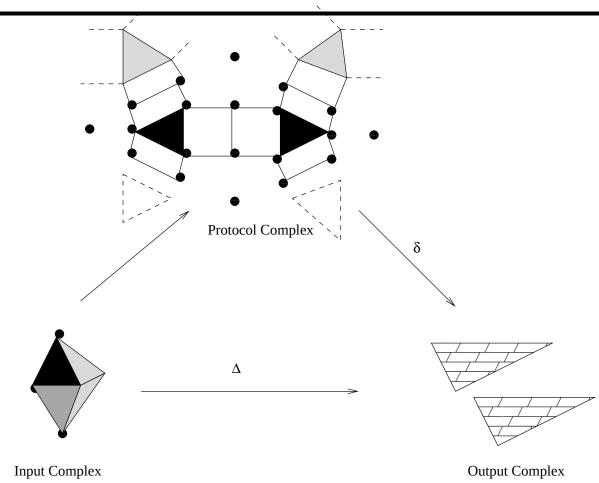
Decision map

The delta in the generic protocol is, mathematically speaking:

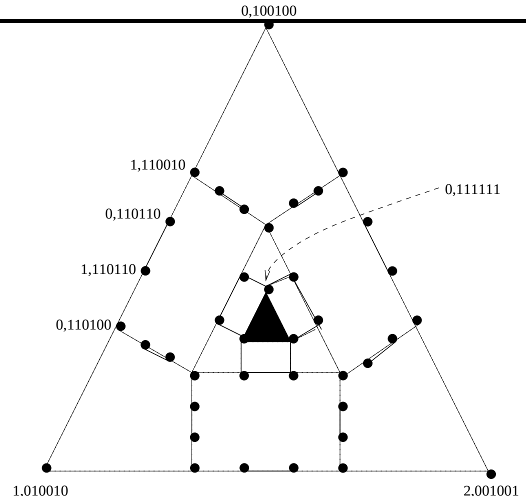
- is $\delta : P \rightarrow O$ (protocol to output complex)
- is a simplicial map (basically a function on vertices, extended on convex hulls)
- respects specification relation Δ, i.e. for all $x \in I$, for all $y \in P(I)$, $x\Delta(\delta(y))$

Proof strategy for impossibility/complexity results: find "topological obstruction" to the δ simplicial map (from protocol complex of any round/round up to k)

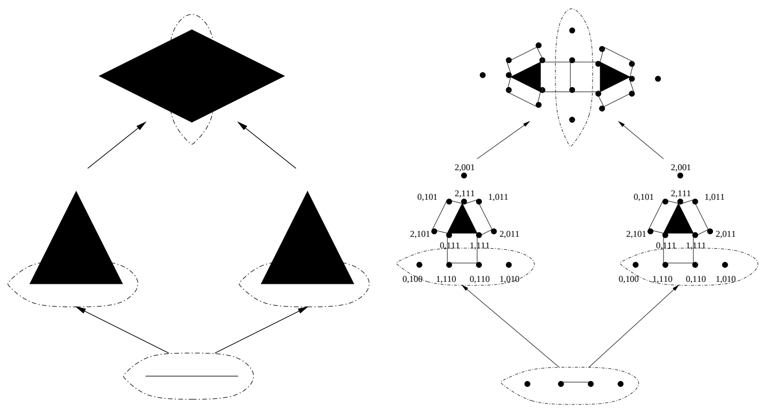
Main property



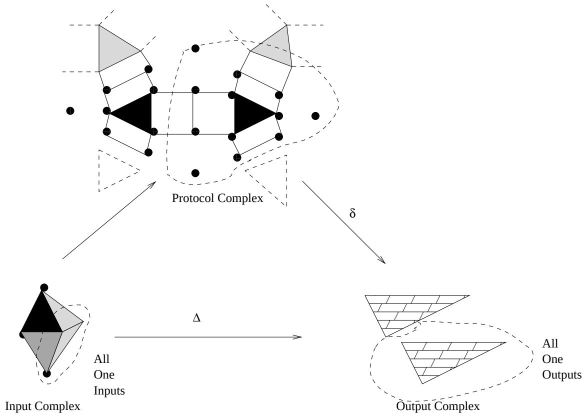
Synchronous protocol complex - round 2



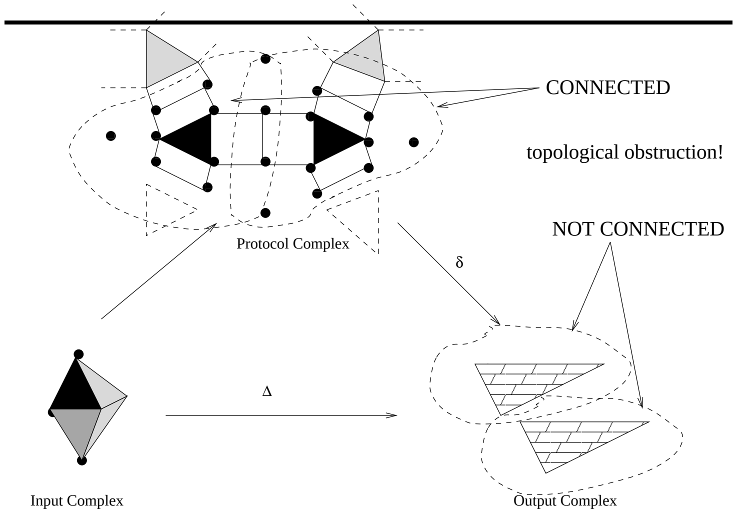
Synchronous protocol complex



Easy application



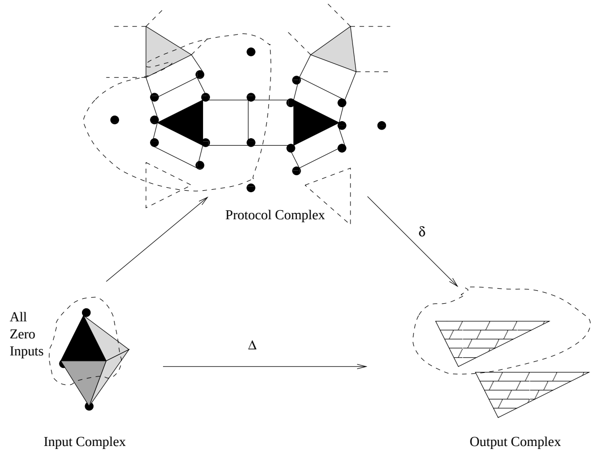
Easy application - for at most $n - 2$ failures only!



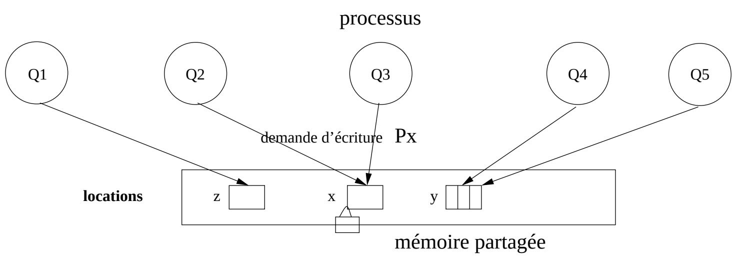
Easy application: consensus again...

- Binary consensus between 3 processes (synchronous message-passing model),
- Input complex is composed of 8 triangles: $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$, $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 0)$ and $(1, 1, 1)$,
- Input complex is homeomorphic to a sphere (one connected component); the first four determine a "north" hemisphere, the last four create a "south" hemisphere
- Output complex is composed of 2 triangles: $(0, 0, 0)$ and $(1, 1, 1)$ (hence two connected components),
- Here: just one round.

Easy application



Shared-memory model



Asynchronous wait-free protocols

- n processes share memory (unbounded size), partitioned: one private chunk for each process
- Each process can:
  - atomically write to its location (update)
  - atomically scan (read) all of the memory into its local memory
- Equivalent to the usual read/write models
- We want *wait-free* protocols, i.e. robust to up to $n-1$ crash failures

More generally

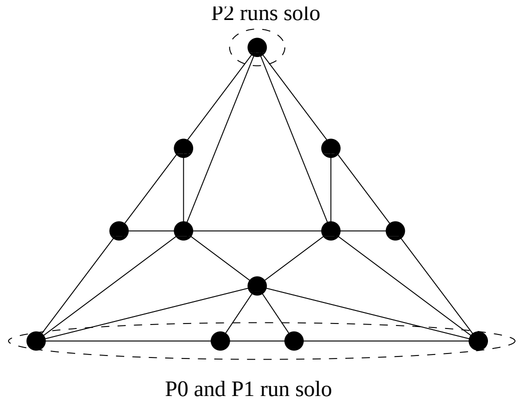
- In any such $(n-2)$-round protocol complex, the all-zero subcomplex and the all-one subcomplex are connected
- Corollary: no $(n-2)$-round consensus protocol

Easy and not new... but gives the idea...

Even more generally...

- Synchronous message-passing model with r rounds, and at most k failures
- $P(S^{n-1})$ is $(n-rk-2)$-connected: implies $(n-1)$-round consensus bound (for $k=1$).

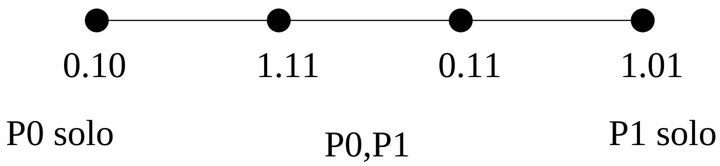
One-round protocol simplicial set (3D)



Theorem

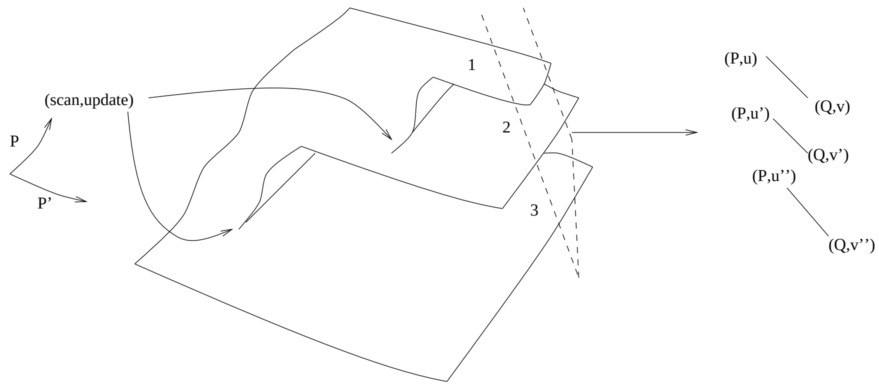
- Wait-free read/write protocol complexes are:
  - $(n-1)$-connected (no holes in any dimension)
  - no matter how long the protocol runs
- Application: k-set agreement

One-round protocol simplicial set (2D)



Semantics

Dynamics (and its cut up to time r=protocol complex):

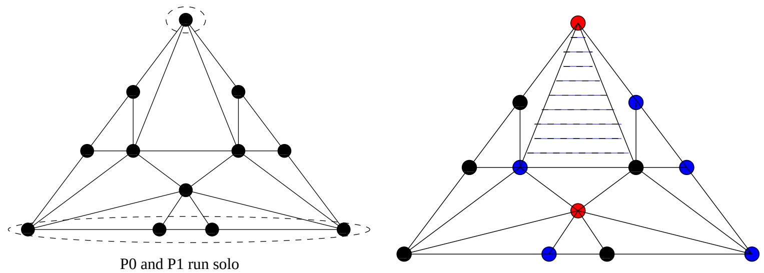


Sketch of a proof

A tool from algebraic topology (Sperner's lemma):

- Subdivide a simplex
- Give each "corner" a distinct "color"
- Give each vertex a corner color
- Giver interior vertices any corner color

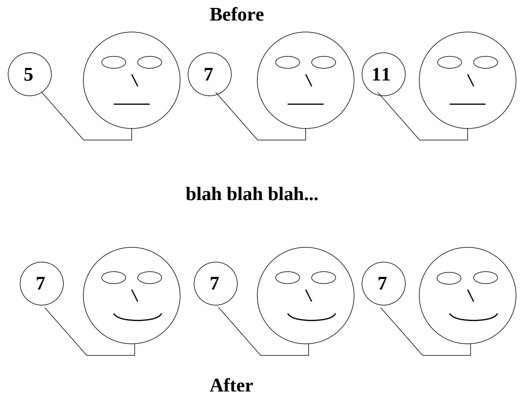
Sperner's lemma



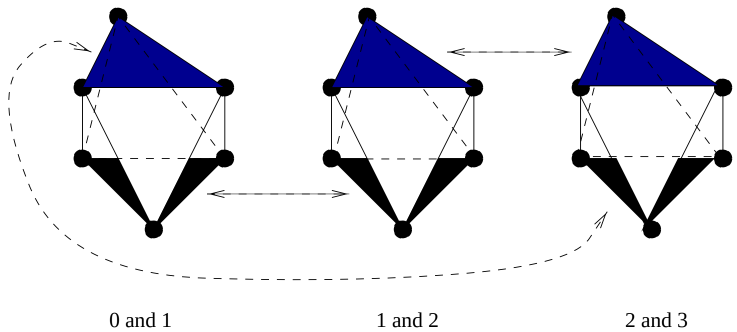
$\Rightarrow$ At least one simplex has all colors

k-set agreement task

Generalization of consensus; processes must end up with at most k different values (taken from the initial values):

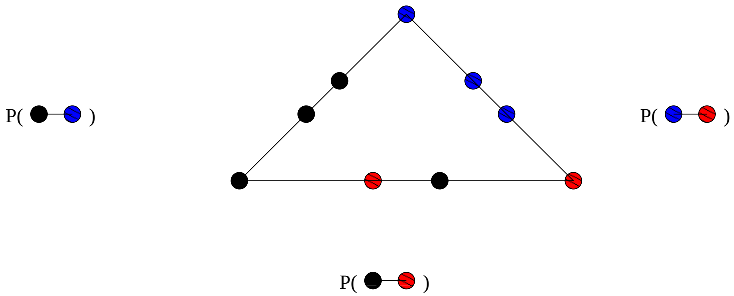


Output simplicial set ($n = 3$, $k = 2$)



3 spheres glued together minus the simplex formed of all 3 values: not 1-connected

Protocol complex - for all 2 process executions



Full protocol complex

- Because complex is simply-connected
- We can "fill-in" edge-paths
- Vertices colored with input colors

Input and protocol simplicial set

- Each process colored with distinct input
- Each vertex colored with decision

Protocol complex

- For a one-process execution: same vertex and same color (cannot decide anything else)
- For a two-process execution:
  - the protocol complex is connected
  - all vertices are of one of the two colors

Principle of the proof

$\Rightarrow$

- Protocol complex is $(n-1)$-connected (using Mayer-Vietoris)
- Exploit connectivity to
  - embed subdivided input complex into protocol complex
  - map protocol complex to output complex
  - just like k-set agreement proof

Principle of the proof

$\Leftarrow$

- We can reduce any task to "simplex agreement" [using the participating set algorithm of Borowsky and Gafni 1993]
- Start out at corners of subdivided simplex
- Must rendez-vous on vertices of single simplex in subdivision

End of proof

Apply Sperner's Lemma:

- Some simplex has all three colors
- That simplex is a protocol execution that decides three values!

Converse

- In fact, even more:
- A task has a wait-free read/write protocol if and only if there exists a simplicial map μ:
  - from subdivided input complex
  - to output complex
  - that respects Δ

Proof

Using the semantics, we have the following three possible 1-schedules (up to homotopy), since the only possible interactions are between the $scan$ and $update$ statements,

(i) Suppose the $scan$ operation of P is completed before the $update$ operation of P' is started: P does not know y so it chooses to write x. $Prog$ ends up with $((P, x), (P', y))$.

(ii) Symmetric case: $Prog$ ends up with $((P, x'), (P', y'))$.

(iii) The $scan$ operation of P is after the $update$ of P' and the $scan$ of P' is after the $update$ of P. $Prog$ ends up with $((P, x'), (P', y))$.

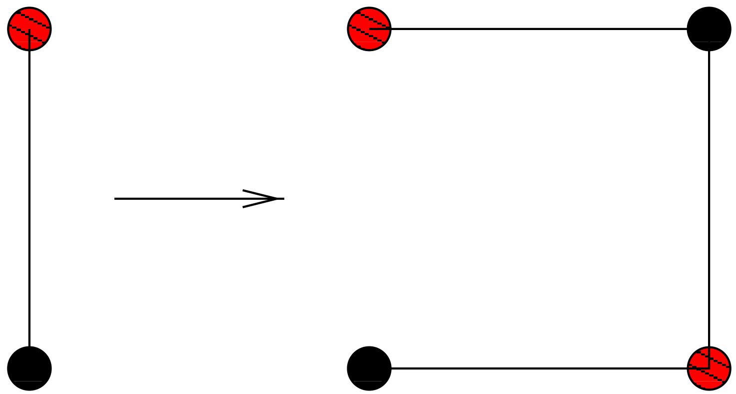
Other communication primitives

Real multiprocessors provide additional atomic synchronization:

- test&set
- fetch&add
- compare&swap
- queues...

Other protocol complexes...other results

Example



Subdivision of a segment into three segments

Protocol

$$
\begin{array}{rcl}
P & = & update; \\
 & & scan; \\
 & & case\ (u, v)\ of \\
 & & \quad (x, y') : \ u = x'; update; [] \\
 & & \quad default : \ update
\end{array}
\qquad
\begin{array}{rcl}
P' & = & update; \\
 & & scan; \\
 & & case\ (u, v)\ of \\
 & & \quad (x, y') : \ v = y; update; [] \\
 & & \quad default : \ update
\end{array}
$$

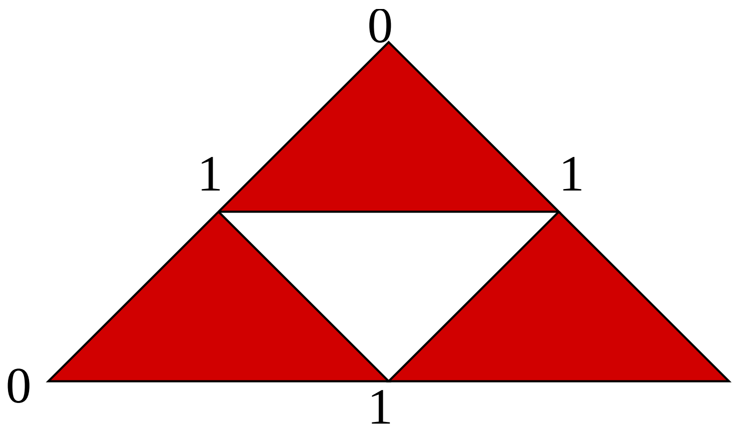
References and main results

- Begins with Fisher-Lynch-Patterson ("FLP") in 1985: there exists a simple task that cannot be solved in a (simple) message-passing system with at most one potential crash
- Created a very active research area, see for instance Nancy Lynch's book "Distributed Algorithms" (1996)

References and main results

- Later developed by Biran-Moran-Zaks in PoDC'88: characterization of the tasks that can be solved by a (simple) message-passing system in the presence of one failure
- The argument uses a "similarity chain", which could be seen as a 1-dimensional version of what we just developed
- Revealed to be difficult to extend to models with more failures

Example: test&set protocol complex



Test&Set

- Wait-free Test&Set protocol complexes
  - are all $(n-3)$-connected
  - more powerful than read/write (2-process consensus)
  - but still no 3-process consensus
- Similar results hold for other synchronization operations

References and main results

Later results, on the same line, include:

- Full characterization of wait-free asynchronous tasks with atomic read/writes on registers, see “The topological structure of asynchronous computability”, M. Herlihy and N. Shavit, J. of the ACM, jan. 2000
- Use of algebraic spans in “Algebraic Spans”, M. Herlihy and S. Rajsbaum as a unified methods for renaming, k-set agreement problems etc.
- Use of pseudo-spheres...

References and research directions

- Consensus numbers (see M. Herlihy and then E. Ruppert SIAM J. Comput. vol 30, No 4, 2000 for instance). Importance based on the remark (M. Herlihy): an object which solves the consensus problem for n processes can simulate in a wait-free manner (together with read/write registers) any object for n or fewer processes.
- Example: R/W registers have consensus number 1, test&set, queues, stacks, fetch and add have consensus number 2 etc.
- Example: There is no wait-free $(n+1, 2j)$-renaming protocol if processes share a read/write memory and $(n+1, j)$-consensus objects.

References and main results

Then, in PoDC’1993, independently,

- Borowsky-Gafni, Saks-Zaharoglou and Herlihy-Shavit derived lower bounds for the k-set agreement problem of Chaudhuri (proposed in 1990)

  [at least $\lfloor \frac{f}{k} \rfloor + 1$ steps in synchronous model]
- Saks-Zaharoglou and Herlihy-Shavit exploited topological properties to derive this lower bound

References and main results

- Renaming: Attiya-BarNoy-Dolev-Peleg JACM 1990,
- The $(n+1, K)$-renaming task starts with $n+1$ processes being given a unique input name in $0, \ldots, N$ and are required to choose unique output name in $0, \ldots, K$ with $n \leq K < N$ (independently of a “process id” - i.e. “anonymous renaming” in fact).
- Showed that (message-passing model) there is a wait-free solution for $K \geq 2n+1$, none when $K \leq n+2$
- Using these geometrical techniques: it has been shown that there is no renaming when $K \leq 2n$
- Herlihy and Shavit STOC’93: same result holds for the wait-free asynchronous model (using homology explicitely).

References and research directions

- Afek et Strup: characterization of the effect of the register size in the power of synchronization primitives
- Characterization of complexity and not only computability, see for instance "Towards a Topological Characterization of Asynchronous Complexity", G. Hoest and N. Shavit
- Links with (geometric) semantics [potential for more realistic models of distributed systems?], for instance my paper in CAAP'97 "Optimal Implementation of Wait-Free Binary Relations" ?
- Extension of this model for randomized algorithms etc.?