Cuts, cut elimination & cut-free proofs

The questions of cuts and cut-elimination are central in proof theory. In this year's course we chose to post-pone its study after the Curry-Howard isomorphism.

1.6 Logical cuts

Roughly, a cut in a proof can be understood as a way to prove a general statement only once. For instance when proving $(2+x)^2 = x^2 + 4x + 4$ we can

- either use the result $\forall a\, b, (a+b)^2 = a^2 + b^2 + 2ab$ and instantiate a and b by x and 2,
- or do a proof from scratch, which will have the same structure as the generic one, but only considers 2 and x.

The first option is a proof with a *cut*. In practice, being able to use such cuts is essential for making mathematics tractable. In theory however, cuts are redundant and can be eliminated. The corresponding cut-elimination theorems are important for various results like consistency or the completeness of automated deduction procedures.

To make things simple, we can say that:

- Proofs with cuts can be shorter, because some (parts of the) proof(s) can be reused and shared.
- But proofs without cuts have some interesting structural properties.

Definition 1.6.1 (Cut in Natural Deduction)**.** In the context of natural deduction, a *logical cut* is a proof that contains an elimination rule whose first premise in a introduction rule of the same connector. Figure 1.3 gives an extensive listing of possible cuts. A proof that does not contain any cut is said to be *cut-free*.

1.7 Properties of cut-free proofs

Lemma 1.7.1. *A cut-free proof of* $[] \vdash A$ *ends with an introduction rule.*

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \psi} \qquad \dfrac{\vdots}{\Gamma \vdash \neg\psi}}{\Gamma \vdash \bot}\,(\bot\text{-I})}{\Gamma \vdash \phi}\,(\bot\text{-E}) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \phi \vdash \psi}}{\Gamma \vdash \phi \implies \psi}\,(\Rightarrow\text{-I}) \qquad \dfrac{\vdots}{\Gamma, \phi \vdash \psi}}{\Gamma \vdash \psi}\,(\Rightarrow\text{-E})$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \phi}}{\Gamma \vdash \phi \vee \psi}\,(\vee\text{-I}_1) \qquad \dfrac{\vdots}{\Gamma, \phi \vdash \eta} \qquad \dfrac{\vdots}{\Gamma, \psi \vdash \eta}}{\Gamma \vdash \eta}\,(\vee\text{-E})$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \psi}}{\Gamma \vdash \phi \vee \psi}\,(\vee\text{-I}_2) \qquad \dfrac{\vdots}{\Gamma, \phi \vdash \eta} \qquad \dfrac{\vdots}{\Gamma, \psi \vdash \eta}}{\Gamma \vdash \eta}\,(\vee\text{-E})$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \phi} \qquad \dfrac{\vdots}{\Gamma \vdash \psi}}{\Gamma \vdash \phi \wedge \psi}\,(\wedge\text{-I})}{\Gamma \vdash \phi}\,(\wedge\text{-E}_1) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \phi} \qquad \dfrac{\vdots}{\Gamma \vdash \psi}}{\Gamma \vdash \phi \wedge \psi}\,(\wedge\text{-I})}{\Gamma \vdash \psi}\,(\wedge\text{-E}_2)$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \phi} \qquad x \notin \mathrm{FV}(\phi)}{\Gamma \vdash \forall x.\, \phi}\,(\forall\text{-I})}{\Gamma \vdash \phi[x \mapsto t]}\,(\forall\text{-E})$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \phi[x \mapsto t]}}{\Gamma \vdash \exists x.\, \phi}\,(\exists\text{-E}) \qquad \dfrac{\vdots}{\Gamma, \phi \vdash \psi} \qquad x \notin \mathrm{FV}(\Gamma, \psi)}{\Gamma \vdash \psi}\,(\exists\text{-E})$$

Figure 1.3: Listing of Cuts in Natural Deduction

Proof. By induction over the structure of the proof. Exercise : do the details. □

Corolary 1.7.2. *There is no cut-free proof of* $[] \vdash \bot$.

1.8 Cut elimination steps

It is possible to transform the proofs to get rid of the cuts. The first thing is to see that one can perform substitutions over natural deduction proofs.

Lemma 1.8.1 (weakening)**.** *Given a proof of* $\Gamma \vdash A$ *and a proposition* B *we have a proof of* $\Gamma; B \vdash A$.

Proof. The proof derivation is exactly the same, just keeping the extended context everywhere. □

Lemma 1.8.2. *Given a proof of* σ *of* $\Gamma; A \vdash B$ *and a proof* τ *of* $\Gamma \vdash A$ *we can construct a proof* $\sigma[A\backslash\tau]$ *of* $\Gamma \vdash B$ *which follows the structure of* σ *but replaces the uses of the axiom rule for* A *by copies of* τ.

Consider a proof ending with a cut:

$$\dfrac{\dfrac{\dfrac{\sigma}{\Gamma, A \vdash B}}{\Gamma \vdash A \implies B}\ (\implies\text{-I}) \quad \dfrac{\tau}{\Gamma \vdash A}}{\Gamma \vdash B}\ (\implies\text{-E})$$

We can erase this cut by rewriting the proof to:

$$\dfrac{\sigma[A\backslash\tau]}{\Gamma \vdash B}$$

Exercise

Describe the corresponding transformations erasing the other logical cuts (conjunction, disjunction, etc…)

Question

Why is it *not* obvious that the process of applying these transformation terminates ending with a cut-free proof ?

How can we show it terminates ?