2.7.1 — Foundations of Proof Systems

Classroom exercises (TD)

Oct. 2^{nd} 2023

1 Impredicative Definitions

Question 1 Give a definition of the property "being a power of 2" in HOL (that is that a number is some 2^i). ◇

Solution.

$$pow2 \equiv \lambda n^{\iota}.\forall X^{\iota\to o}.X\ (S\ 0) \Rightarrow (\forall x^{\iota}.X\ x \Rightarrow X\ (S\ x)) \Rightarrow X\ n.$$

2 Cuts in HOL

2.1 Logical Cuts

Remember that in First-Order Logic (FOL) we a notion of logical cut for each connector or quantifier, and each time a corresponding cut-elimination step. For instance

$$\dfrac{\dfrac{\dfrac{\sigma_A}{\vdash A} \qquad \dfrac{\sigma_B}{\vdash B}}{\vdash A \wedge B}}{\vdash A} \qquad \text{is simplified to} \qquad \dfrac{\sigma_A}{\vdash A}.$$

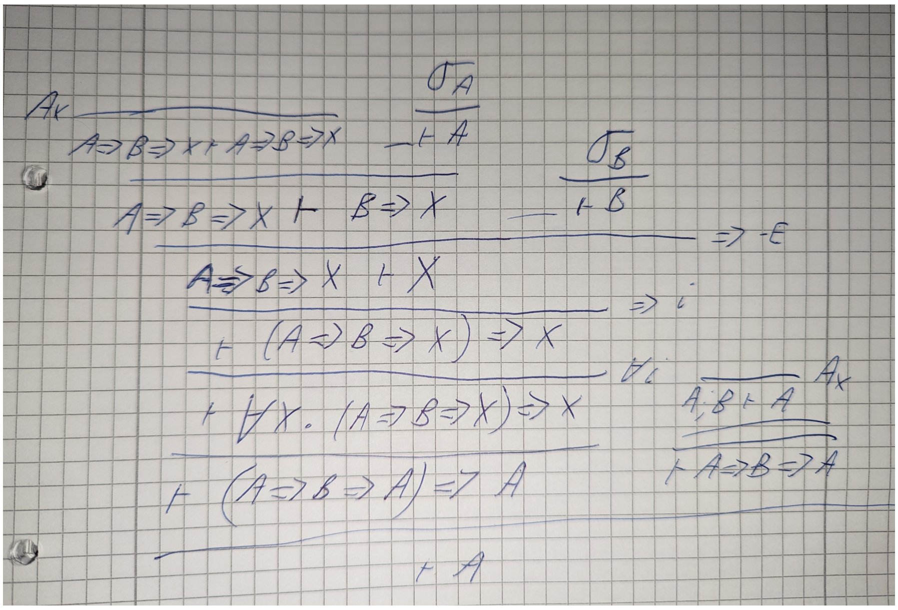
In HOL however, the conjunction connector is not primitive but defined by :

$$A \wedge B \equiv \forall_o\ \lambda X^o.(A \Rightarrow B \Rightarrow X) \Rightarrow X.$$

Remark : In the following, I will sometimes write $\forall x^T.\dots$ for $\forall_T\ \lambda X^T.\dots$

Question 2 Write the proof derivation corresponding to the FOL logical cut above, when in HOL. ◇

Solution.



□

Question 3 Check whether this cut can be eliminated. That is whether the elimination of $\forall$ and $\Rightarrow$ cuts in HOL are sufficient for eliminating the $\wedge$ cuts. ⋄

Solution. Yes, it works!

But it is tedious to write the steps down here... □

Question 4 Do the same for the disjunction in HOL. ⋄

2.2 Axiomatic Cuts

Question 5 Do the same for the axiomatic equality cut. Or, in other words, check whether the logical $\forall$ and $\Rightarrow$ cuts in HOL are sufficient to eliminate the axiomatic = cuts. ⋄

At this stage, it is probably more useful to look at the Coq exercises; even for better understanding the next exercise. However, if you are not afraid you can look at the induction cuts below.

We now want to look at the axiomatic induction cuts. If we state the induction principle as an axiom, we will not be able to view the induction cuts as logical cuts. We can however slightly change the approach.

We first define a property over objects of type ι stating the they verify the induction principle :

$$\mathsf{Nat} \equiv \lambda x^{\iota}.\forall_{\iota\to o}\,\lambda P^{\iota\to o}.(P\;0) \implies (\forall n^{\iota}.(P\;n) \implies (P\;(S\;n))) \implies (P\;x).$$

We have $\mathsf{Nat} : \iota \to o$. We can understand $(\mathsf{Nat}\;n)$ as "n is a natural number".

Then, each time we quantify over a natural number, we add an assumption that it verifies Nat. So we can prove :

$$(1) \quad \forall n^{\iota}.(\mathsf{Nat}\;n^{\iota}) \Rightarrow \exists p^{\iota}.n^{\iota} = r^{\iota} + r^{\iota} \vee n^{\iota} = S\;(r^{\iota} + r^{\iota}).$$

Question 6 Describe roughly how you prove, say, (Nat 5). $\diamond$

Solution. First prove :

— Nat 0, that is $\forall_{\iota\to o}P^{\iota\to o}.P^{\iota\to o}\ 0 \Rightarrow (\dots) \Rightarrow P^{\iota\to o}\ 0$.

— then $\forall n^{\iota}.\mathsf{Nat}\ n \Rightarrow \mathsf{Nat}\ (S\ n)$

Then use the second lemma 5 times and the first once. $\square$

Question 7 Can you seen what happens when you combine (1) with the result of the previous question to obtain a proof of $\exists p^{\iota}.5 = r^{\iota} + r^{\iota} \vee 5 = S\ (r^{\iota} + r^{\iota})$ and then eliminate the cuts ? $\diamond$