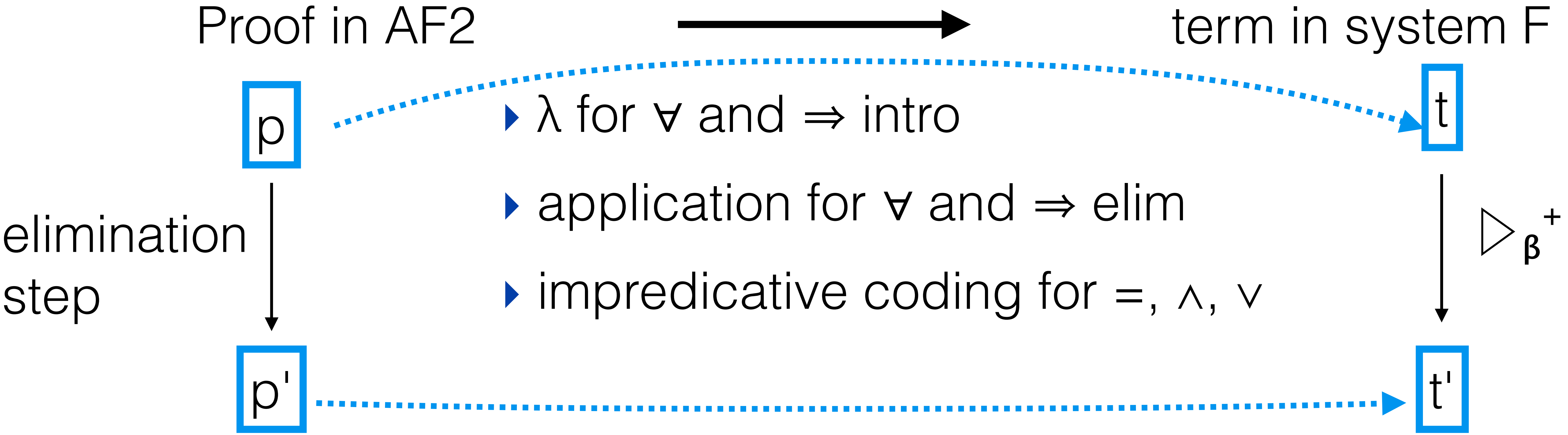
Fω,
The Calculus of Constructions,
Barendregt's Cube
&
Pure Type Systems

Why was System F invented ?

In parallel:

- by John Reynolds (computer science)
- Jean-Yves Girard (logic)

Girard's aim: termination of cut-elimination for second-order arithmetic



Going further: cut elimination for HOL

AF2: $\forall X.\ X \Rightarrow X$, $\forall X.\ X$, $\forall X.\ A \Rightarrow B \Rightarrow X$ are propositions

HOL: $X{:}\iota \rightarrow o$ is mapped to $X : o$ OK

But what about $X : o \rightarrow o$ or $(X\ P)$??

We need to extend System F: F_ω

Kind level: K ::= $\mathrm{Type} \mid K \rightarrow K$ $\quad T_1 \rightarrow T_2$

Type level: $T ::= \alpha \mid T\ T \mid \Lambda\ \alpha{:}K.T \mid \Pi\ \alpha{:}K.T$

Term level: $t ::= x \mid \lambda\ x{:}T.t \mid t\ t \mid \Lambda\ \alpha{:}K.t \mid t\ T$

} simply typed λ-calcu

System F, but for the K

F_ω Rules

$$\frac{\Gamma \vdash T{:}\mathrm{Type}}{\Gamma\ (x{:}T)\ \mathrm{wf}} \qquad \frac{\Gamma\ \mathrm{wf}}{\Gamma\ (\alpha{:}K)\ \mathrm{wf}} \qquad \frac{}{[]\ \mathrm{wf}} \qquad \frac{\Gamma\ \mathrm{wf}}{\Gamma \vdash x{:}T} \qquad \frac{\Gamma\ \mathrm{wf}}{\Gamma \vdash \alpha{:}K}$$

$$\frac{\Gamma \vdash T : K_1 \to K_2 \qquad \Gamma \vdash U : K_1}{\Gamma \vdash T\ U : K_2} \qquad \frac{\Gamma(\alpha{:}K_1) \vdash T : K_2}{\Gamma \vdash \Lambda\ \alpha{:}K_1.T : K_1 \to K_2}$$

$$\frac{\Gamma \vdash t : T_1 \to T_2 \ \Gamma \vdash u : T_1}{\Gamma \vdash t\ u : T_2} \qquad \frac{\Gamma(x{:}T_1) \vdash t : T_2}{\Gamma \vdash \lambda\ x{:}T_1.t : T_1 \to T_2}$$

$$\frac{\Gamma(\alpha{:}K) \vdash t : T}{\Gamma \vdash \Lambda\ \alpha{:}K.t : \forall\ \alpha{:}K.T} \qquad \frac{\Gamma \vdash t : \forall\ \alpha{:}K.T \ \Gamma \vdash U{:}K}{\Gamma \vdash t\ U : T[\alpha \setminus U]} \qquad \frac{\Gamma \vdash U{:}\mathrm{Type} \quad \Gamma \vdash t : T}{\Gamma \vdash t : U}\ \text{if } T =_\beta U$$

Main theorem: terms well-typed in $F\omega$ are strongly normalizable

Corollary: Cut elimination in HOL terminates

Proof of the theorem: basically like in System F

$\alpha : \mathrm{Type}$ interpreted by a reducibility candidate

$\beta : \mathrm{Type} \to \mathrm{Type}$ interpreted by a function between reducibility candidates,

$\gamma : (\mathrm{Type} \to \mathrm{Type}) \to \mathrm{Type}$ etc…

Another presentation of Fω

merge λ and Λ, x and α

$$t \quad ::= \quad x \mid \lambda\, x : t.t \mid t\ t \mid \Pi\ x{:}t.t \mid \text{Type} \mid \text{Kind}$$

$$\frac{\Gamma \vdash T{:}s}{\Gamma\ (x{:}T)\ \text{wf}} \qquad \frac{}{[]\ \text{wf}} \qquad \frac{\Gamma\ \text{wf}}{\Gamma \vdash x{:}T}\ (x{:}T)\in\Gamma \qquad \frac{\Gamma\vdash\ t : \Pi\ x{:}T_1\ .\ T_2 \quad \Gamma\vdash\ u : T_1}{\Gamma\vdash\ t\ u : T_2[x \setminus u]}$$

$$\frac{\Gamma(x{:}T_1) \vdash t : T_2 \quad \Gamma\vdash\ \Pi\ x{:}T_1.T_2 : s}{\Gamma\vdash\ \lambda\ x{:}T_1.t : \Pi\ x{:}T_1.T_2}$$

$$\frac{\Gamma(x{:}T_1) \vdash T : T_2 \quad \Gamma(x{:}T_1) \vdash T_2 : \text{Type}}{\Gamma\vdash\ \Pi\ x : T_1.\ T_2 : \text{Type}}$$

$$\frac{\Gamma(x{:}K_1) \vdash T : K_2 \quad \Gamma\vdash\ K_1 : \text{Kind} \quad \Gamma(x{:}K_1) \vdash K_2 : \text{Kind}}{\Gamma\vdash\ \Pi\ x{:}K_1.\ K_2 : \text{Kind}}$$

Pure Type Systems

s := Type | Kind …

$\mathcal{S}$ set of sorts

$$t \;::=\; x \mid \lambda x : t.t \mid t\ t \mid \Pi\ x{:}t.t \mid s$$

$$\mathcal{A}, \mathcal{R} \in \mathcal{S}\times\mathcal{S}$$

$$\frac{\Gamma\vdash T{:}s}{\Gamma(x{:}T)\ \mathrm{wf}} \qquad \frac{\Gamma\ \mathrm{wf}}{\Gamma\vdash x{:}T}\ (x{:}T)\in\Gamma \qquad [\,]\ \mathrm{wf}$$

$$\frac{\Gamma\ \mathrm{wf}}{\Gamma\vdash s1{:}s2}\ \ s_1{:}s_2 \in \mathcal{A} \qquad \frac{\Gamma\vdash A{:}s_1 \quad \Gamma(x{:}A)\vdash B{:}s_2}{\Gamma\vdash \Pi\ x{:}A.B : s_2}\ (s_1,s_2) \in \mathcal{R}$$

$$\frac{\Gamma\vdash \Pi\ x{:}A.B : s \quad \Gamma(x{:}A)\vdash t{:}B}{\Gamma\vdash \lambda\ x{:}A.t : \Pi\ x{:}A.B} \qquad \frac{\Gamma\vdash t : \Pi\ x{:}A.B \quad \Gamma\vdash u{:}A}{\Gamma\vdash t\ u : A\,[x \setminus u]}$$

$$\frac{\Gamma\vdash t{:}T \quad \Gamma\vdash U{:}s}{\Gamma\vdash t{:}U}\ T =_\beta U$$

Already known PTSs

$\mathcal{S}$ = { Type; Kind} $\mathcal{A}$ = {(Type : Kind)}

$\mathcal{R}$ = {(Type, Type)} simply typed calculus

$\mathcal{R}$ = {(Type, Type); (Kind, Type)} system F

$\mathcal{R}$ = {(Type, Type); (Type, Kind)} LF

$\mathcal{R}$ = {(Type, Type); (Kind, Type); (Kind,Kind)} system Fω

$\mathcal{R}$ = $\mathcal{S} \times \mathcal{S}$ Calculus of Constructions (1985)

$\mathcal{A}$ = {(Type:Type)} Martin-Löf's paradoxical system

Metatheory of PTSs

Basic lemmas work generically:

- inversion lemma
- subject reduction

But normalization does not always work, depends of the system, etc

Normalization of CoC : map to Fω