INF562 – Géométrie Algorithmique et Applications

Curve and surface reconstruction

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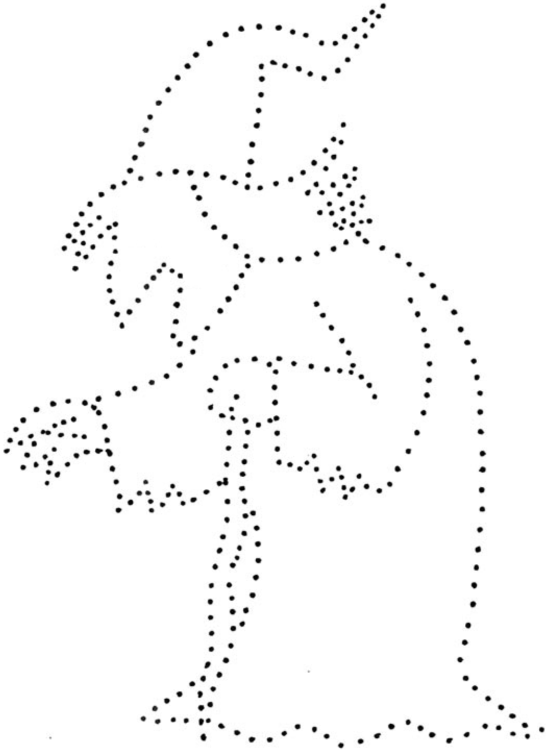
Reconstruction Paradigm

Q What do you see?

Why?



http://www.pointapoint.com

Reconstruction Paradigm

Q What do you see?

Why?

http://jolicoloriage.free.fr

Reconstruction Paradigm

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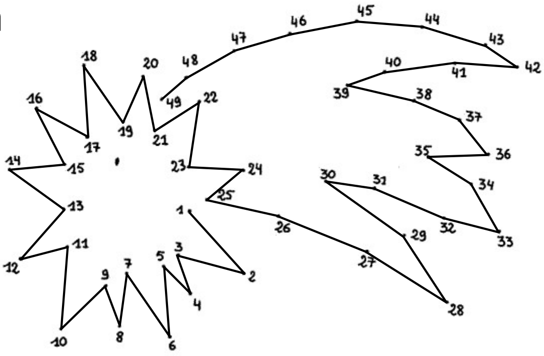

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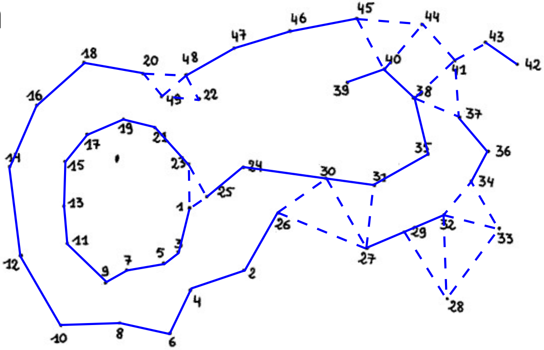
Reconstruction Paradigm

Q What do you see?

Why?

without the numbers...



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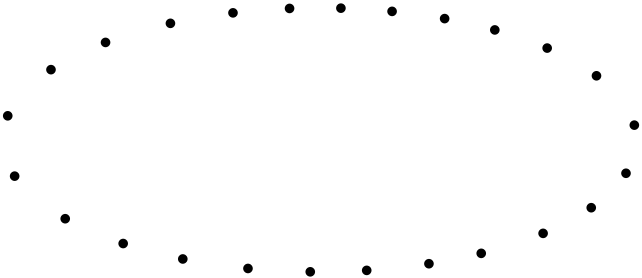
Reconstruction Paradigm (Cont'd)



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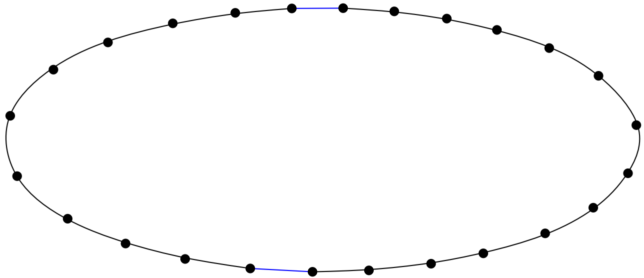
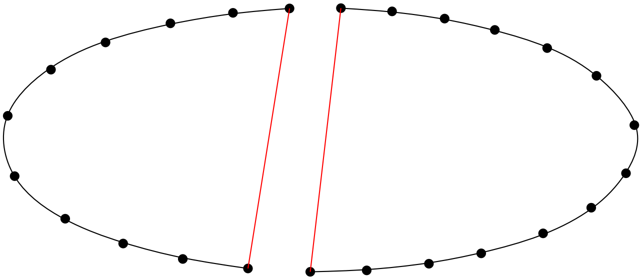


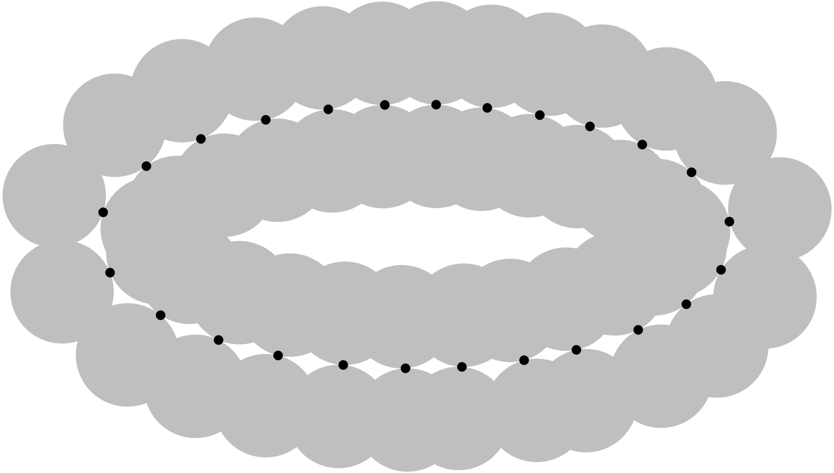
Q Given a point cloud, build a *faithful* (implicit, PL, ...) approximation of the shape underlying the data.

Reconstruction Paradigm (Cont'd)



Reconstruction problem is ill-posed by nature.

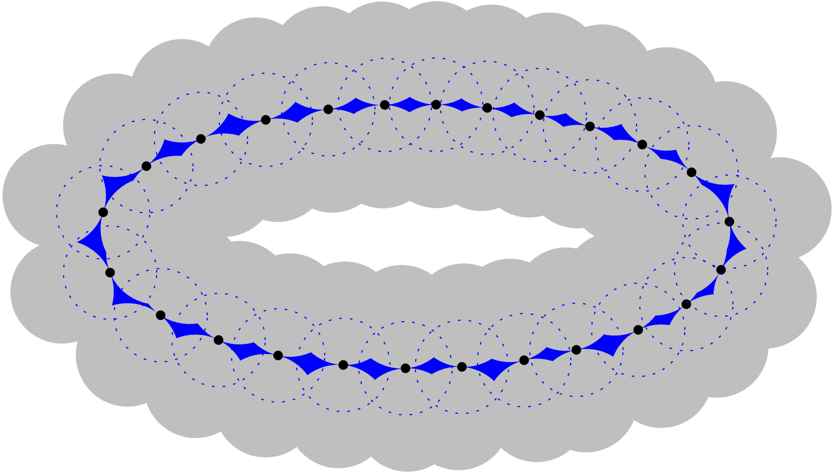
Reconstruction Paradigm (Cont'd)



Reconstruction problem is ill-posed by nature.

→ make assumptions on the underlying shape, *e.g.*: fix dimension, topological type, regularity (differentiability), Hausdorff distance to input...

Reconstruction Paradigm (Cont'd)



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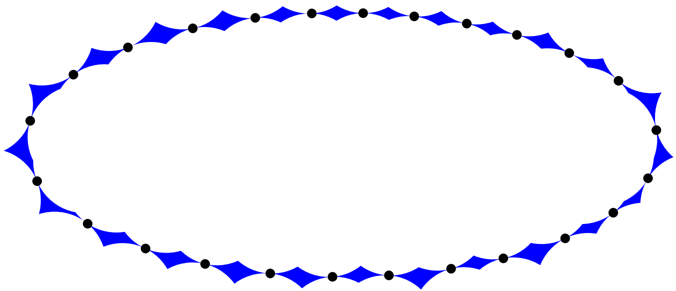
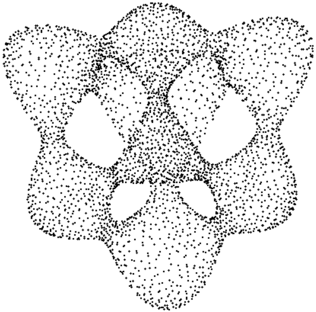


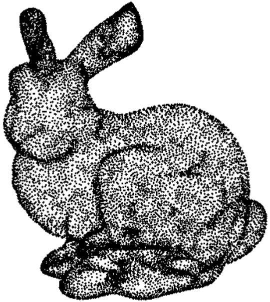
Reconstruction problem is ill-posed by nature.

→ make assumptions on the underlying shape, *e.g.*: fix dimension, topological type, regularity (differentiability), Hausdorff distance to input...

→ for a suitable choice of hypotheses, the solution becomes unique **up to a set of local regular deformations** (solution never unique!)

Other (weaker) forms of reconstruction





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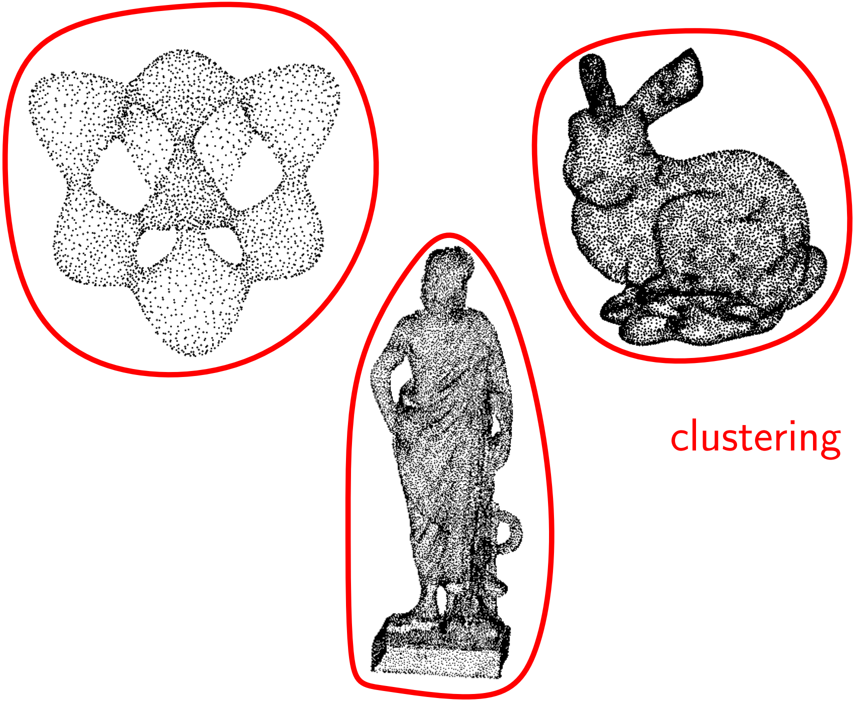
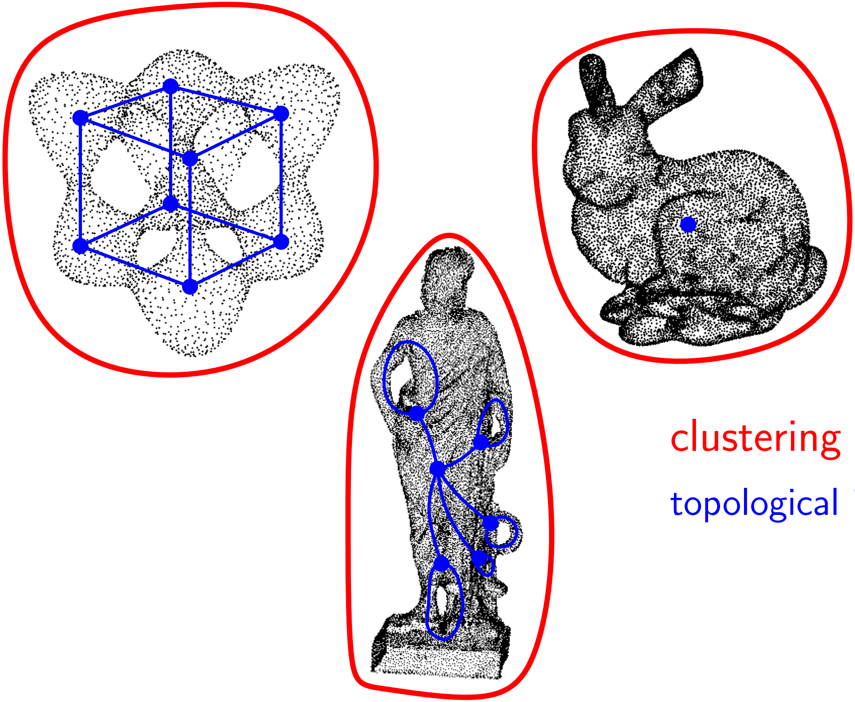


clustering

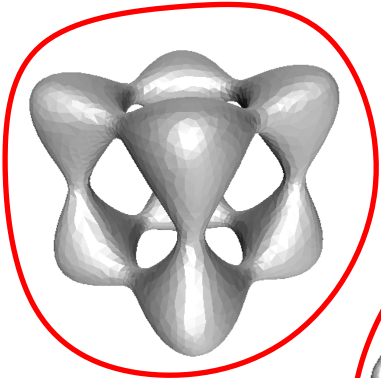
Other (weaker) forms of reconstruction

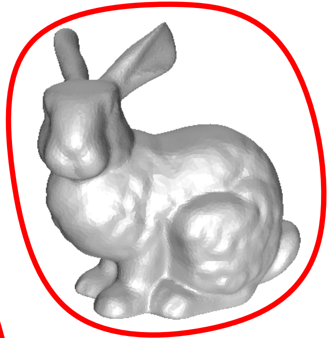


clustering

topological inference

Other (weaker) forms of reconstruction





clustering

topological inference

reconstruction

Where do the data come from?

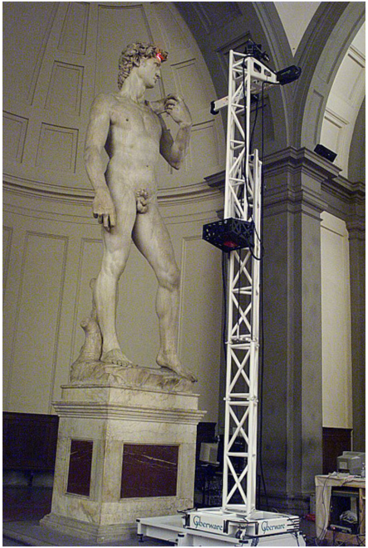
3D scans

Sources

LASER
stereo vision
mechanical sensor

Applications

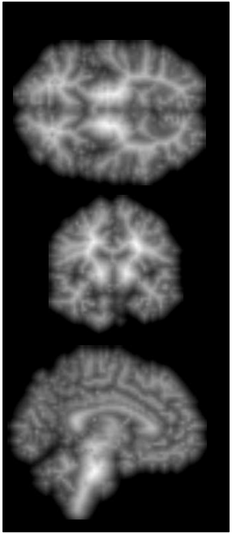
Reverse engineering
Prototyping
Quality control



Stanford Michelangelo Project

Where does the data come from?

Medical Imaging



Sources

- MRI scan
- echograph
- . . .

Applications

- Diagnostic
- Endoscopy simulation
- Chirurgical intervention planning

Where does the data come from?

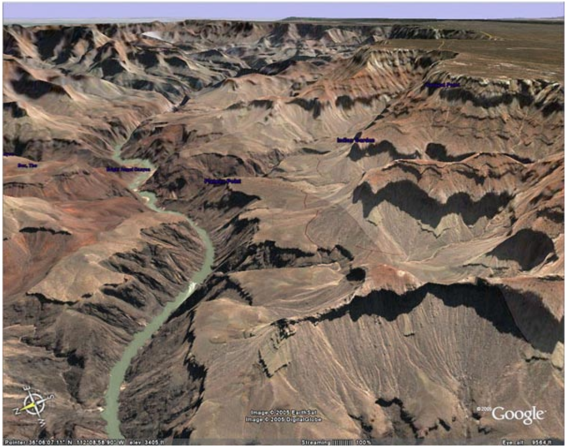
Geography, Geology

Sources

satellite/aerial images

ground probing

seismograph



Applications

Maps making / Terrain modeling

Prospection (tunnels, oil)

Where does the data come from?

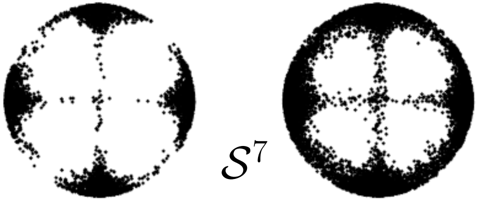
Higher-Dimensions

Sources

Data bases
Simulations

Applications

Machine Learning
Path planning
Pattern recognition
Image processing
...



$\mathcal{S}^7$

Various reconstruction techniques

Delaunay-based

- Crust / Power Crust
- Cocone
- Gabriel / α-shape / β-skeleton
- flow complex

Implicitization

- Local polynomial fitting
- Natural Neighbors (Voronoi-based)
- Radial Basis Functions

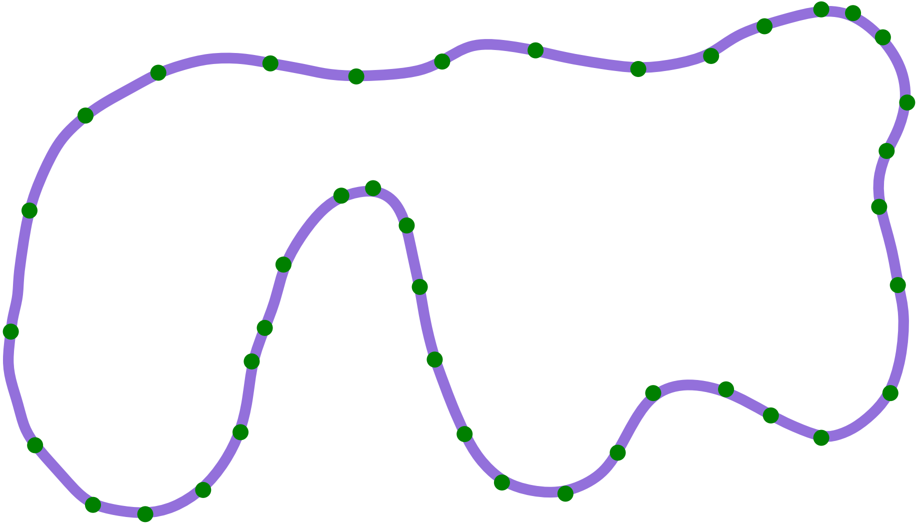
Projection operators

- Moving Least Squares
- Extremal surfaces

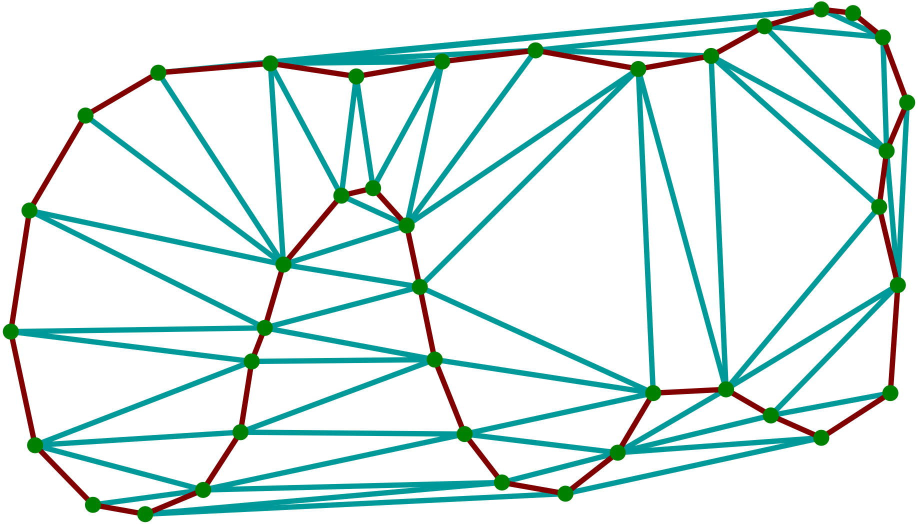
For arbitrary dimensions and co-dimensions

- Unions of balls / nerves
- Witness Complex

What Delaunay has to do with reconstruction



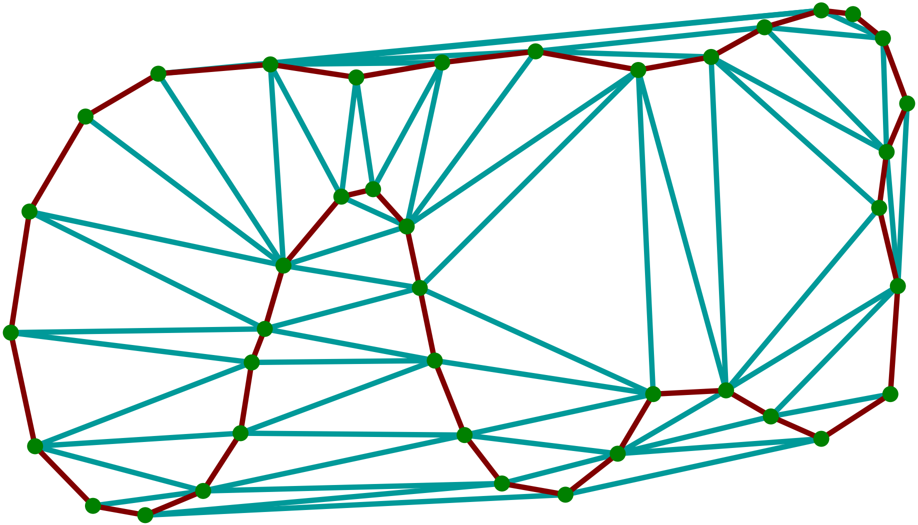
What Delaunay has to do with reconstruction



→ a faithful approximation of the curve appears as a subcomplex of the Delaunay

→ this should hold whenever the point cloud is sufficiently densely sampled along the curve

What Delaunay has to do with reconstruction

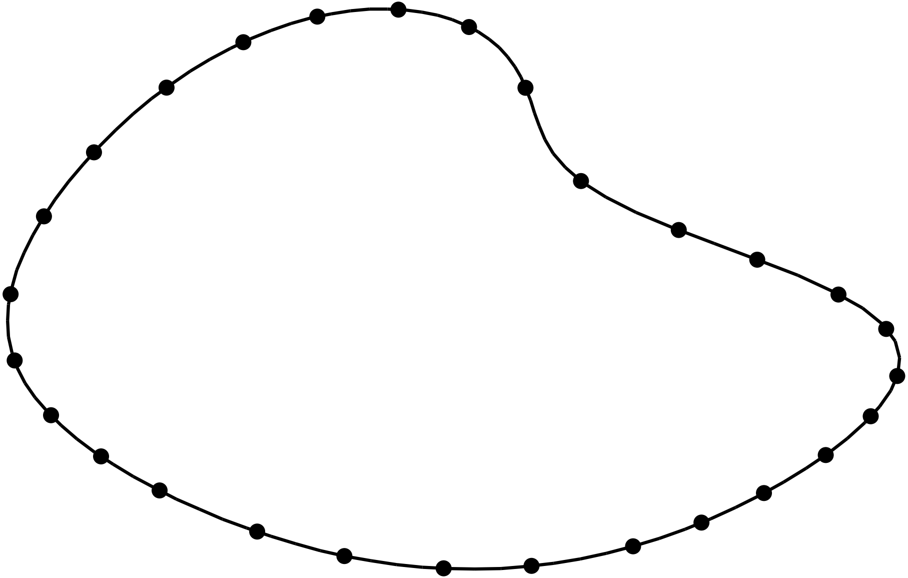


→ a faithful approximation of the curve appears as a subcomplex of the Delaunay

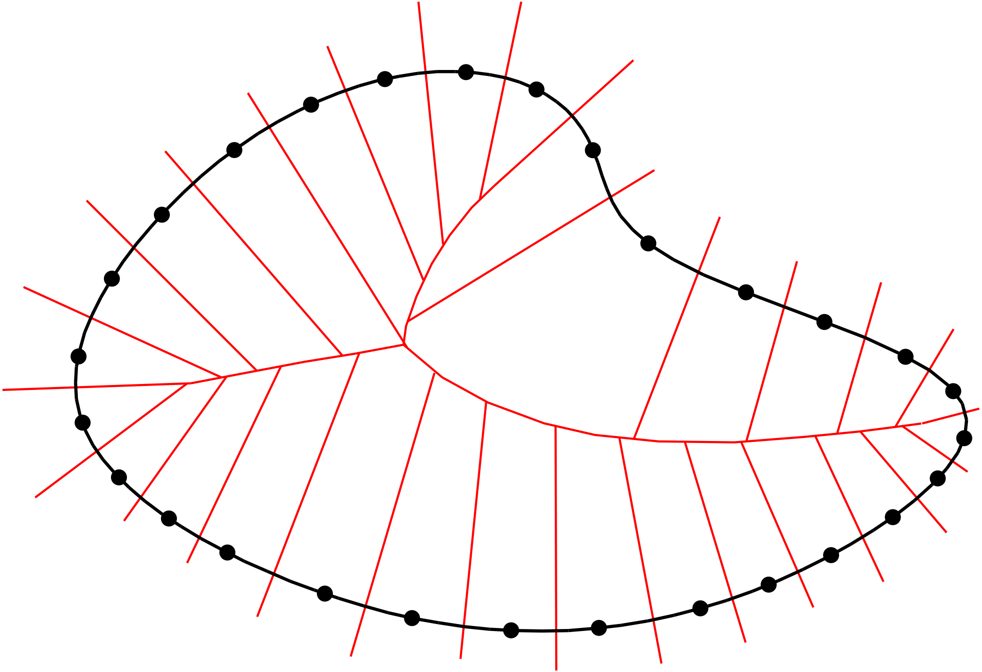
→ this should hold whenever the point cloud is sufficiently densely sampled along the curve

Q What is this *good* subcomplex? Can it be defined in some canonical way?

Restricted Delaunay triangulation

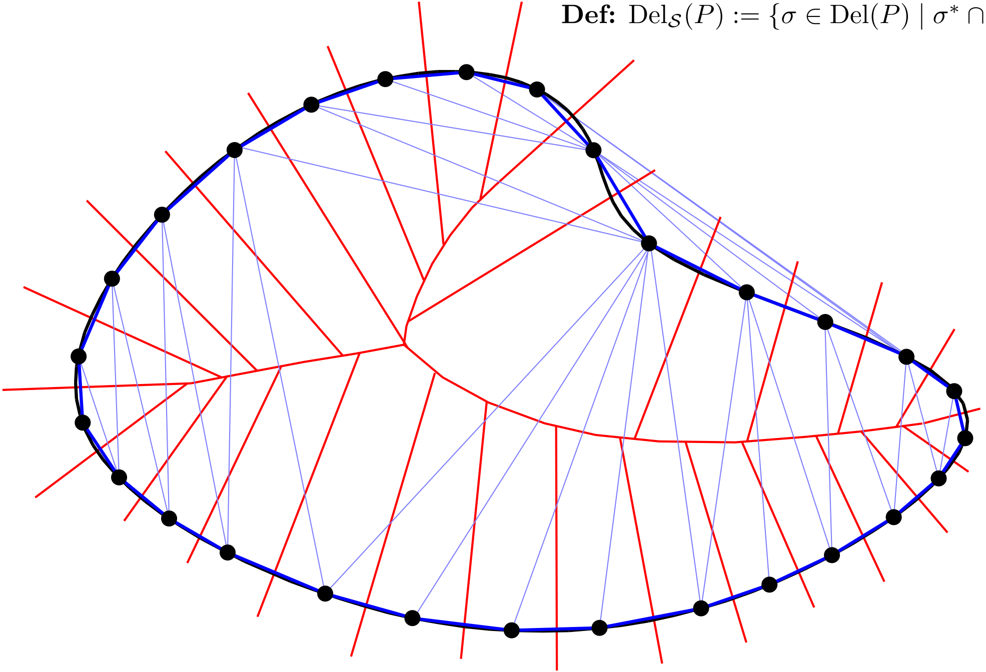


Restricted Delaunay triangulation



Restricted Delaunay triangulation

Def: $\mathrm{Del}_{\mathcal{S}}(P) := \{\sigma \in \mathrm{Del}(P) \mid \sigma^* \cap \mathcal{S} \neq \emptyset\}$



Approximation power of the restricted Delaunay

→ Our assumptions:

1. the underlying shape $\mathcal{S}$ is a closed curve or surface with positive *reach* $\varrho_{\mathcal{S}}$

2. the point cloud P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon \in O(\varrho_{\mathcal{S}})$.

Approximation power of the restricted Delaunay

→ Our assumptions:

→ analogy with 1-d signal theory (Shannon's reconstruction theorem):

1. the underlying shape $\mathcal{S}$ is a closed curve or surface with positive *reach* $\varrho_{\mathcal{S}}$

1'. the underlying signal is a weighted sum of sinusoids

2. the point cloud P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon \in O(\varrho_{\mathcal{S}})$.

2'. the sampling has ≥ 2 samples per period (signal has bounded bandwidth)

Approximation power of the restricted Delaunay

Theorem: [Amenta et al. 1998-99]
If $\mathcal{S}$ is a curve or surface with positive *reach*, and if P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon < \varrho_{\mathcal{S}}$ (curve) or $\varepsilon < 0.1\varrho_{\mathcal{S}}$ (surface), then:

- $\mathrm{Del}_{\mathcal{S}}(P)$ is homeomorphic to $\mathcal{S}$,
- $\mathrm{d_H}(\mathrm{Del}_{\mathcal{S}}(P), \mathcal{S}) \in O(\varepsilon^2)$,
- $\forall f \in \mathrm{Del}_{\mathcal{S}}(P)$, $\forall v \in f$, $\angle n_f n_v \mathcal{S} \in O(\varepsilon)$,
- $\cdots$ (similar areas, curvature estimation, etc.)

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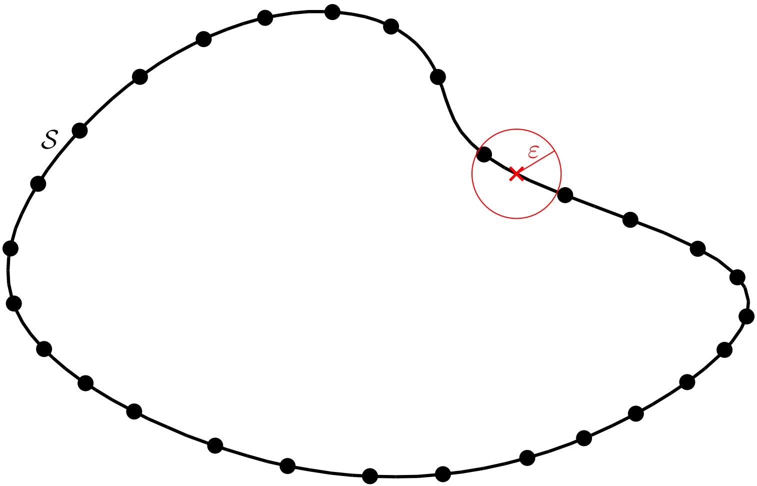
$\rightarrow$ to be explicited: ε-sampling, reach

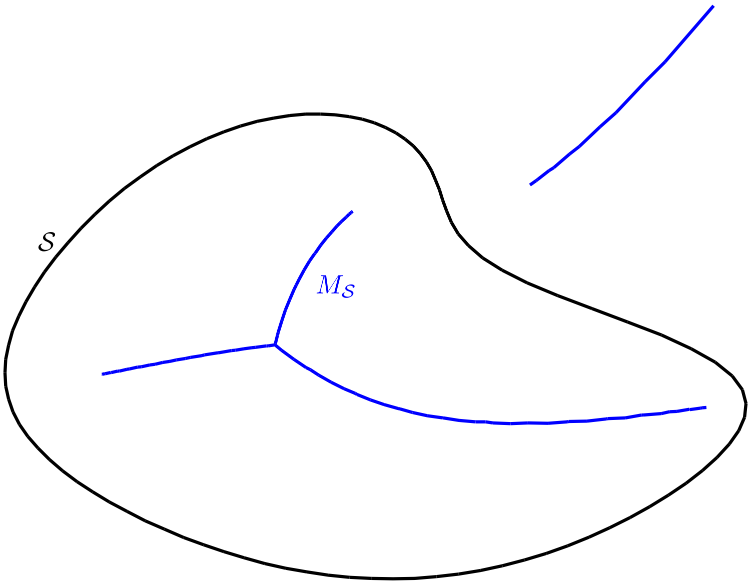
ε-samples

Def: P is an ε-sample of $\mathcal{S}$ if $\forall x \in \mathcal{S}$, $\min\{\|x - p\| \mid p \in P\} \leq \varepsilon$.



Shapes with positive reach [Federer 1958]

Def: $M_{\mathcal{S}}$ is the closure of the set of points of $\mathbb{R}^d$ that have ≥ 2 nearest neighbors on $\mathcal{S}$.



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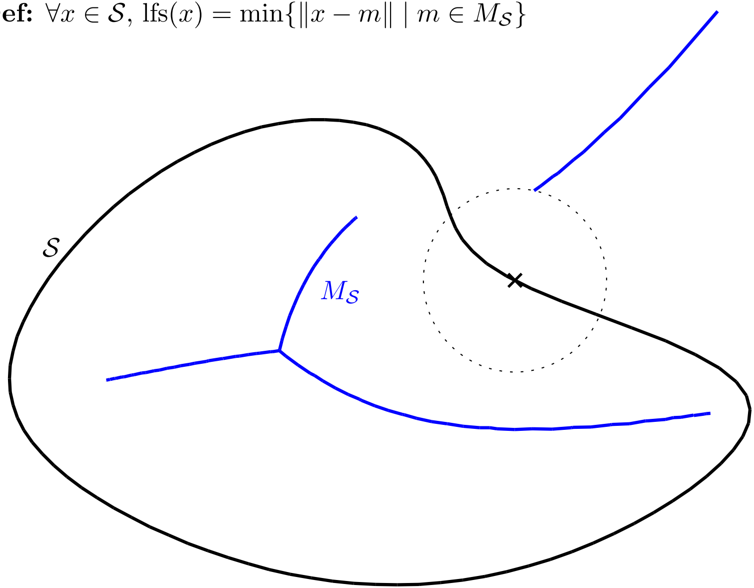

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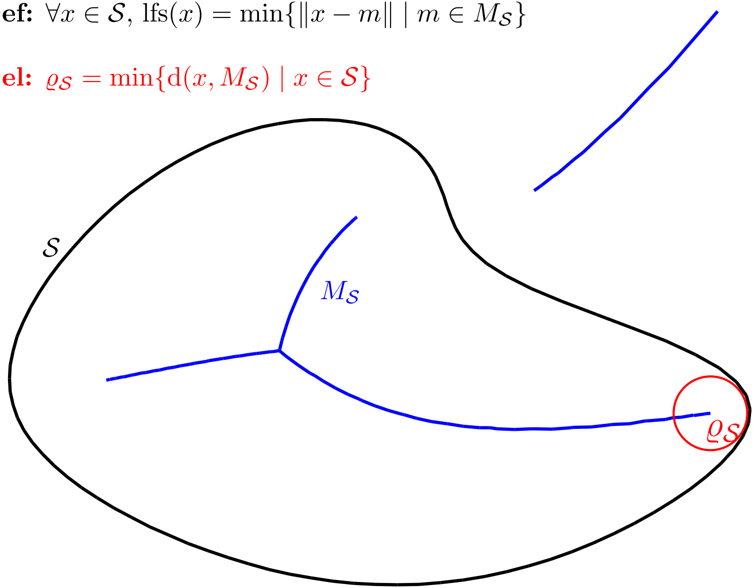
Del: $\varrho_{\mathcal{S}} = \min\{\mathrm{d}(x, M_{\mathcal{S}}) \mid x \in \mathcal{S}\}$



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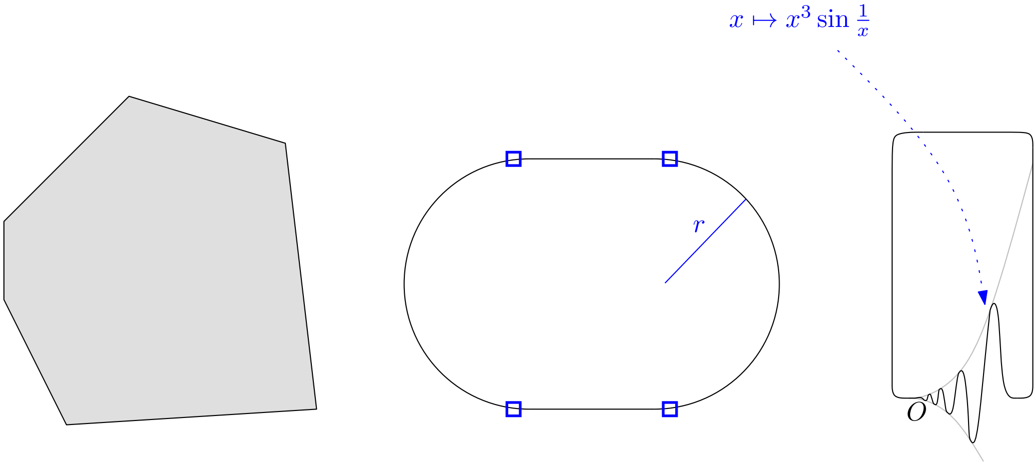

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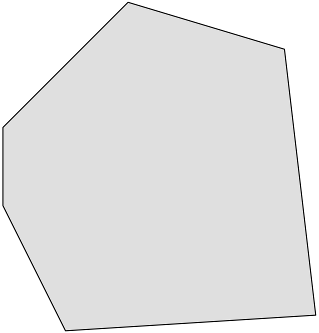
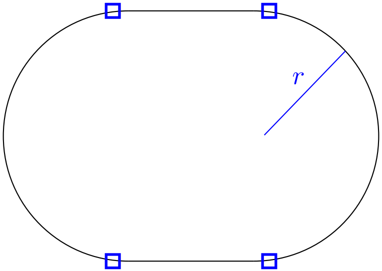


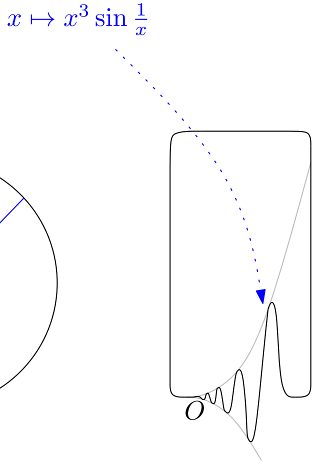
$\varrho_{\mathcal{S}} = +\infty$

(convex)



$\varrho_{\mathcal{S}} = r$

$C^{1,1}$ but not C^2)



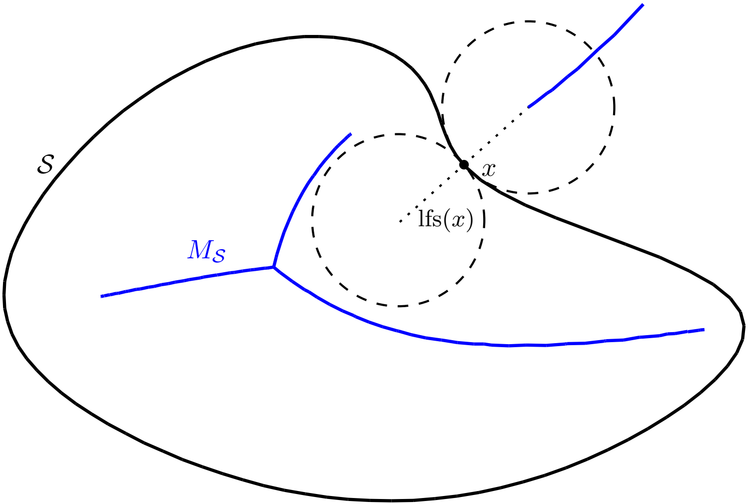
$\varrho_{\mathcal{S}} = 0$

(C^1 but not $C^{1,1}$)

Shapes with positive reach (Cont'd)

→ Fundamental properties: (see [Federer 1958])

Tangent Ball Lemma: $\forall x \in \mathcal{S}$, $\forall c \in n_x\mathcal{S}$, $\|x - c\| < \text{lfs}(x) \Rightarrow B(c, \|x - c\|) \cap \mathcal{S} = \emptyset$.



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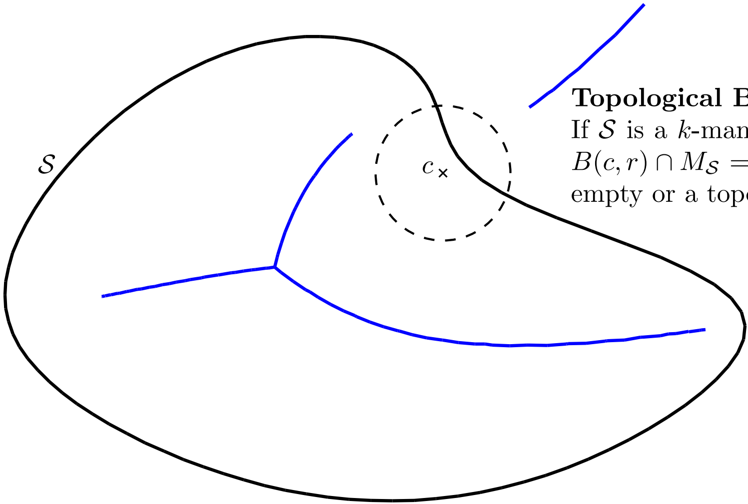

Topological Ball Lemma:
If $\mathcal{S}$ is a k-manifold, then $\forall B(c, r)$ s.t. $B(c, r) \cap M_{\mathcal{S}} = \emptyset$, $B(c, r) \cap \mathcal{S}$ is either empty or a topological k-ball.

Approximation power of the restricted Delaunay

Theorem: [Amenta et al. 1998-99]
If $\mathcal{S}$ is a curve or surface with positive *reach*, and if P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon < \varrho_{\mathcal{S}}$ (curve) or $\varepsilon < 0.1\varrho_{\mathcal{S}}$ (surface), then:

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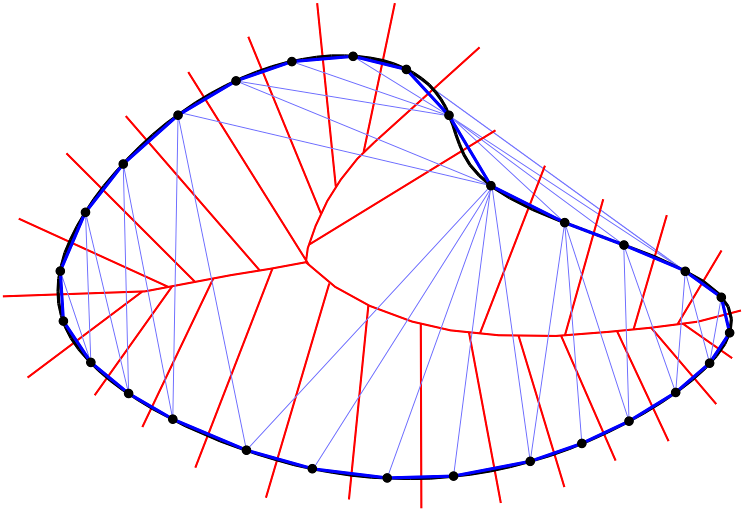
Approximation power of the restricted Delaunay

Proof for curves:

$\rightarrow$ show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa



Approximation power of the restricted Delaunay

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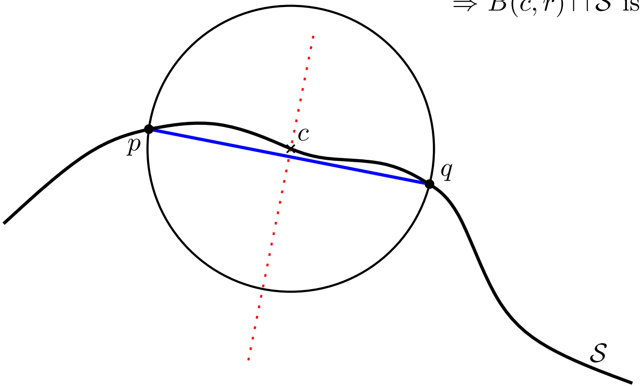
Let $c \in pq^* \cap \mathcal{S}$.

$r = \|c - p\| = \|c - q\| = \mathrm{d}(c, P) \leq \varepsilon < \varrho_{\mathcal{S}} \leq \mathrm{lfs}(c)$

$\Rightarrow B(c, r) \cap \mathcal{S}$ is a topological arc



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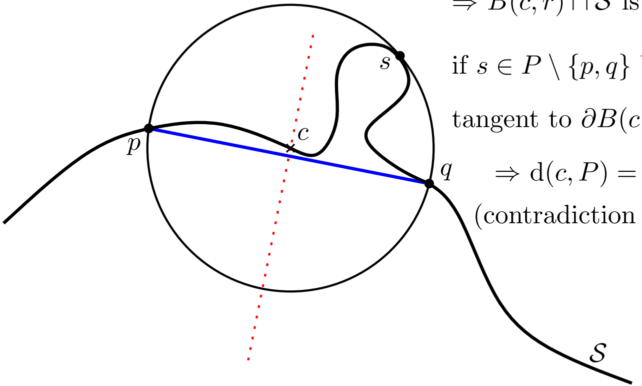

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$\Rightarrow B(c,r) \cap \mathcal{S}$ is a topological arc

if $s \in P \setminus \{p,q\}$ belongs to this arc, then the arc is tangent to $\partial B(c,r)$ in p, q or s (say s)

$\Rightarrow \mathrm{d}(c,P) = r = \|c-s\| \geq \mathrm{lfs}(s) > \varepsilon$.

(contradiction with the hypothesis of the theorem)

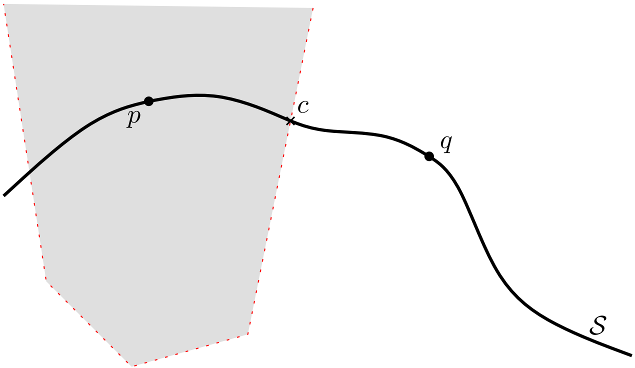
Approximation power of the restricted Delaunay

Proof for curves:

$\rightarrow$ show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa

Let $c \in \mathrm{arc}_{\mathcal{S}}(pq) \cap \partial p^*$.

$c \in ps^*$ for some $s \in P \setminus \{p\}$



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Let $c \in \mathrm{arc}_{\mathcal{S}}(pq) \cap \partial p^*$.

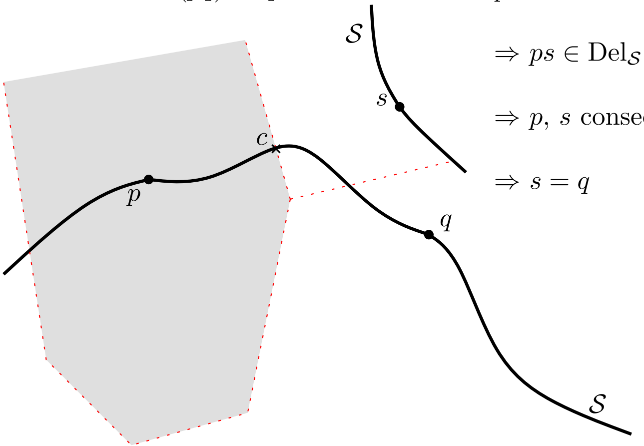
$c \in ps^*$ for some $s \in P \setminus \{p\}$

$\Rightarrow ps \in \mathrm{Del}_{\mathcal{S}}(P)$

$\Rightarrow p$, s consecutive along $\mathcal{S}$, with $c \in \mathrm{arc}_{\mathcal{S}}(ps)$

(by previous part of the proof)

$\Rightarrow s = q$

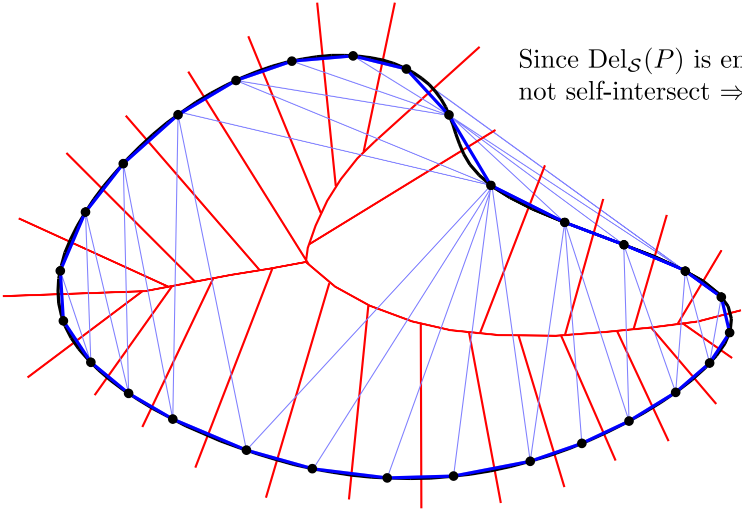


Approximation power of the restricted Delaunay

Proof for curves:

$\rightarrow$ show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa

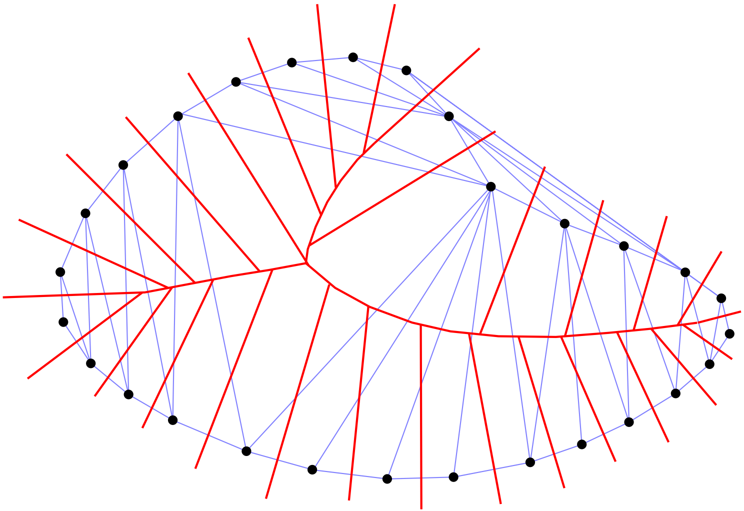
$\Rightarrow$ $\mathrm{Del}_{\mathcal{S}}(P)$ is homeomorphic to $\mathcal{S}$ between each pair of consecutive points of P



Since $\mathrm{Del}_{\mathcal{S}}(P)$ is embedded in $\mathrm{Del}(P)$, it does not self-intersect $\Rightarrow$ global homeomorphism

Computing the restricted Delaunay

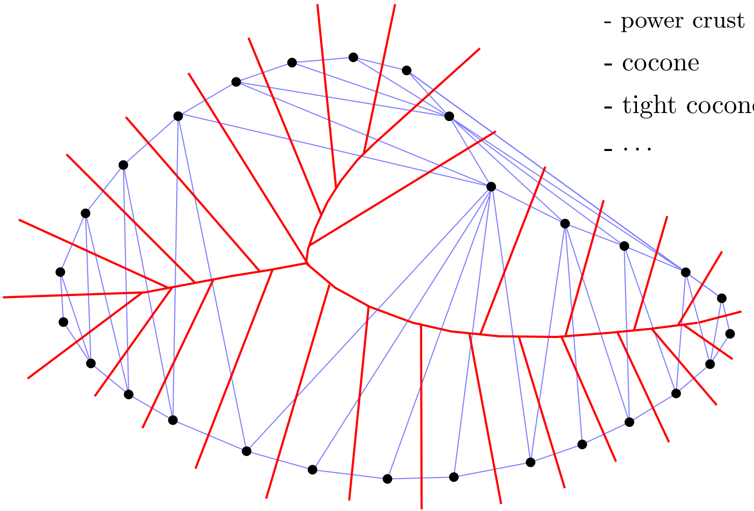
Q How to compute $\mathrm{Del}_{\mathcal{S}}(P)$ when $\mathcal{S}$ is unknown?



Computing the restricted Delaunay

Q How to compute $\mathrm{Del}_{\mathcal{S}}(P)$ when $\mathcal{S}$ is unknown?

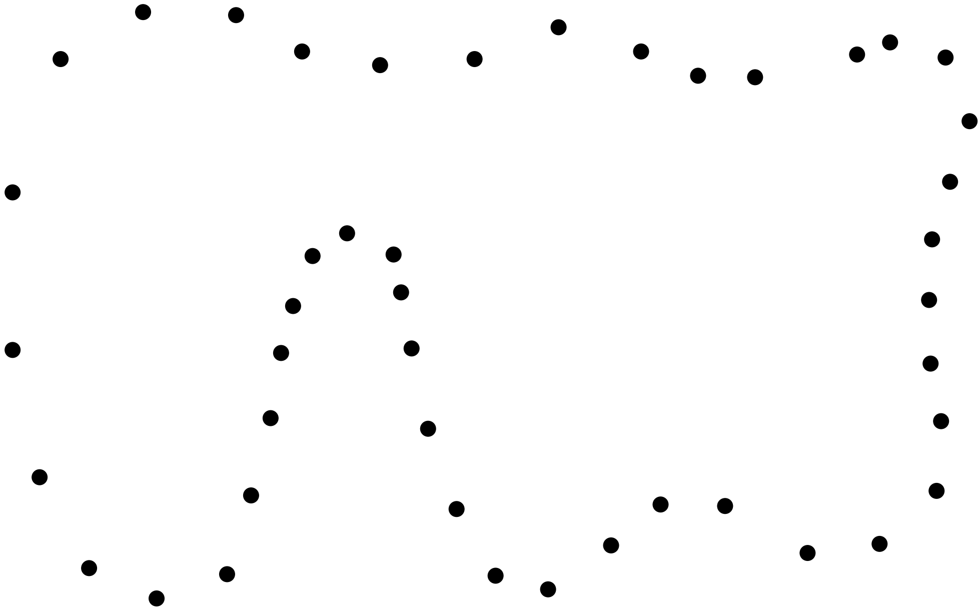
→ a whole family of algorithms use various Delaunay extraction criteria:



- crust
- power crust
- cocone
- tight cocone
- $\cdots$

Crust algorithm

[Amenta et al. 1997-98]



Crust algorithm

[Amenta et al. 1997-98]

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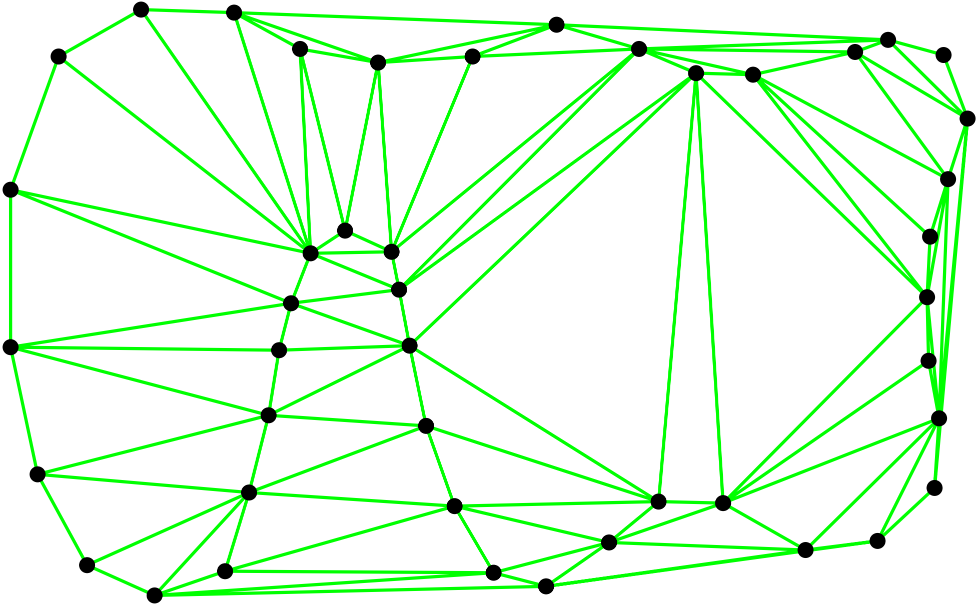

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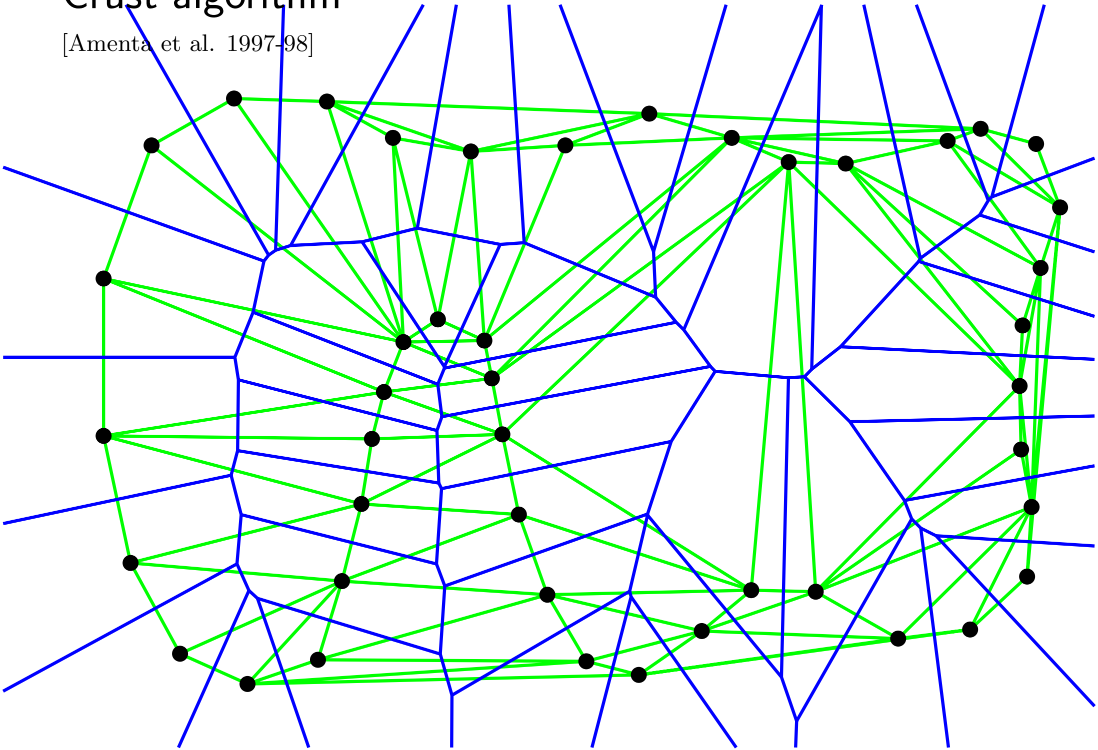
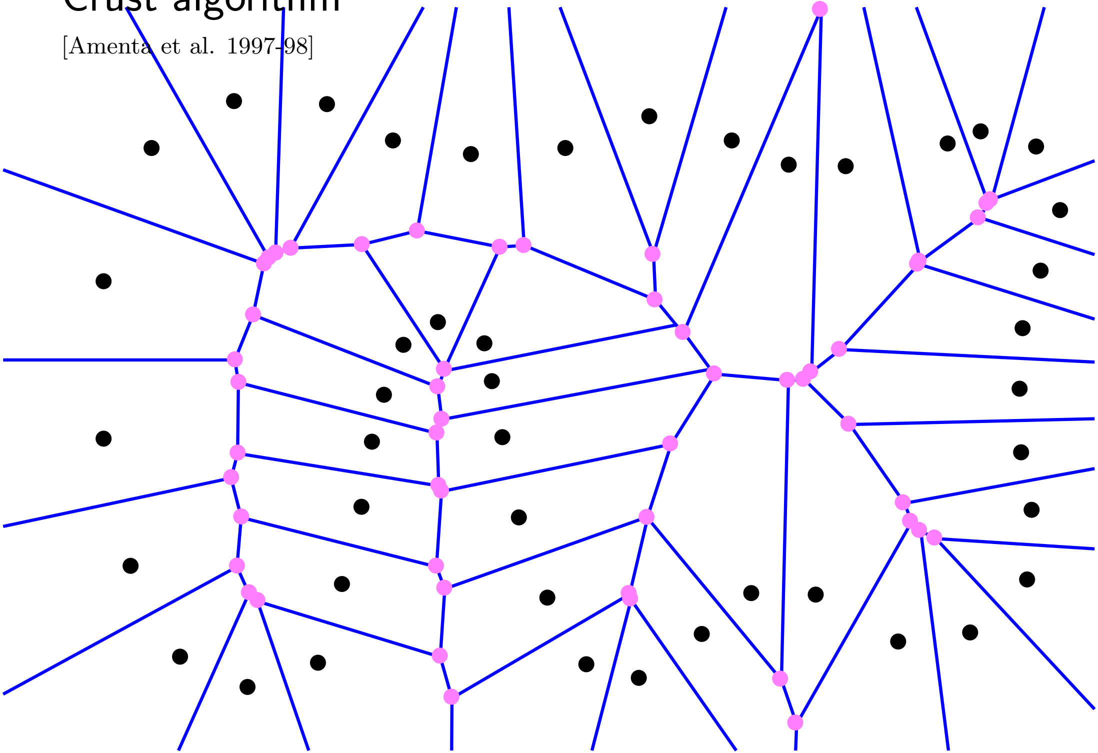


Crust algorithm

[Amenta et al. 1997-98]

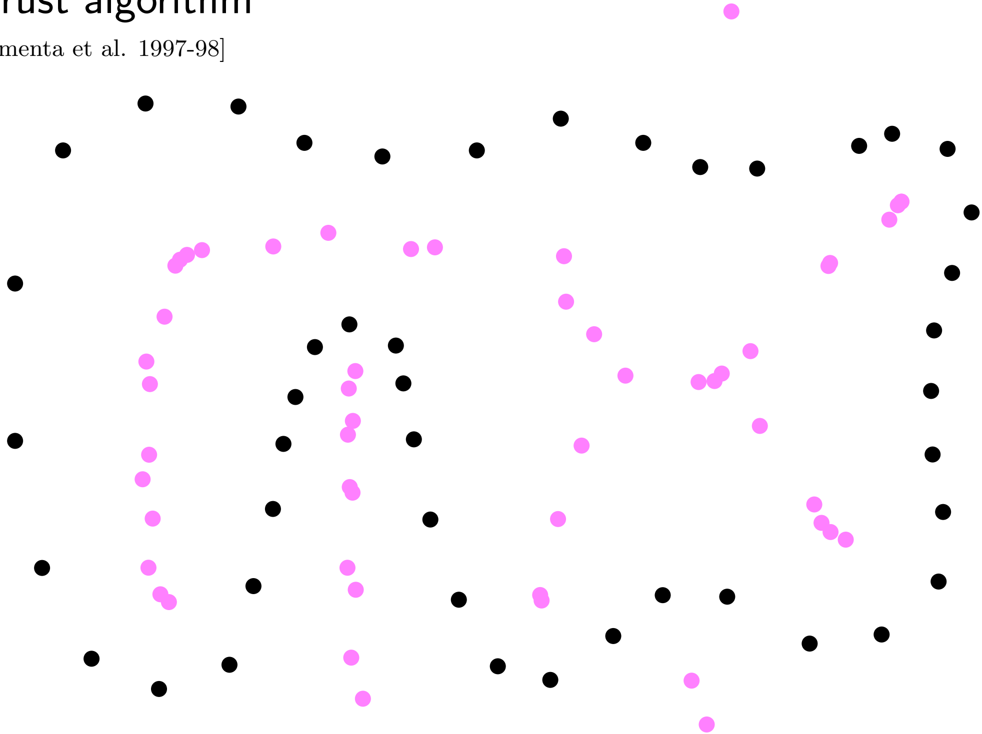
2. Compute *poles* (furthest Voronoi vertices)



Crust algorithm

[Amenta et al. 1997-98]

3. Add poles to the set of vertices



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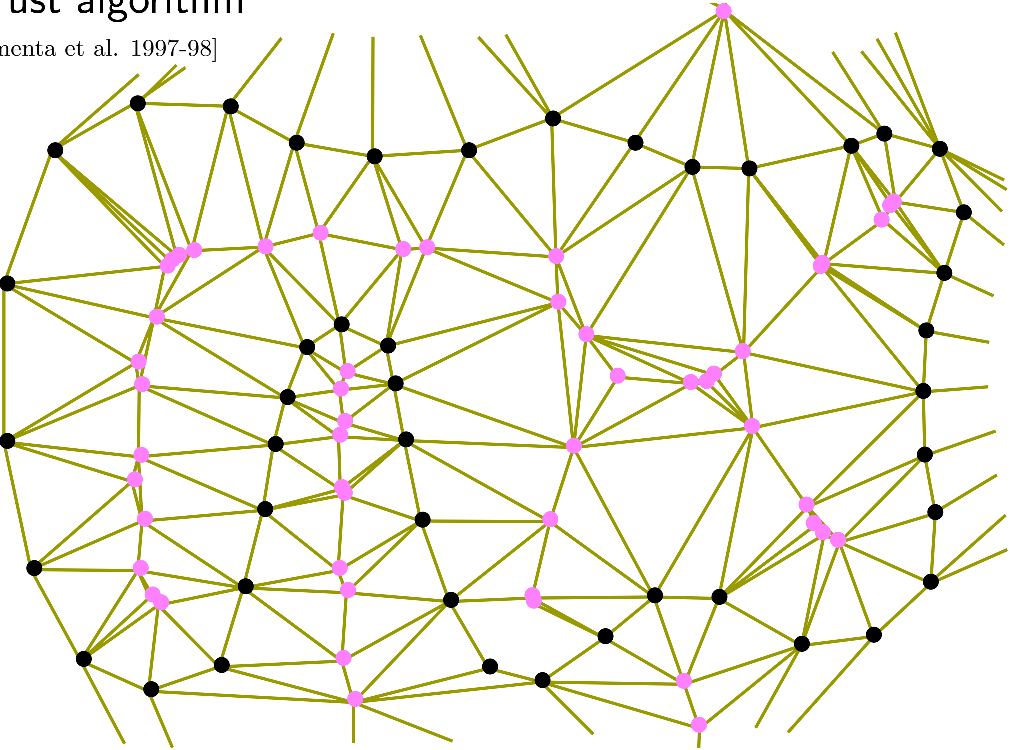
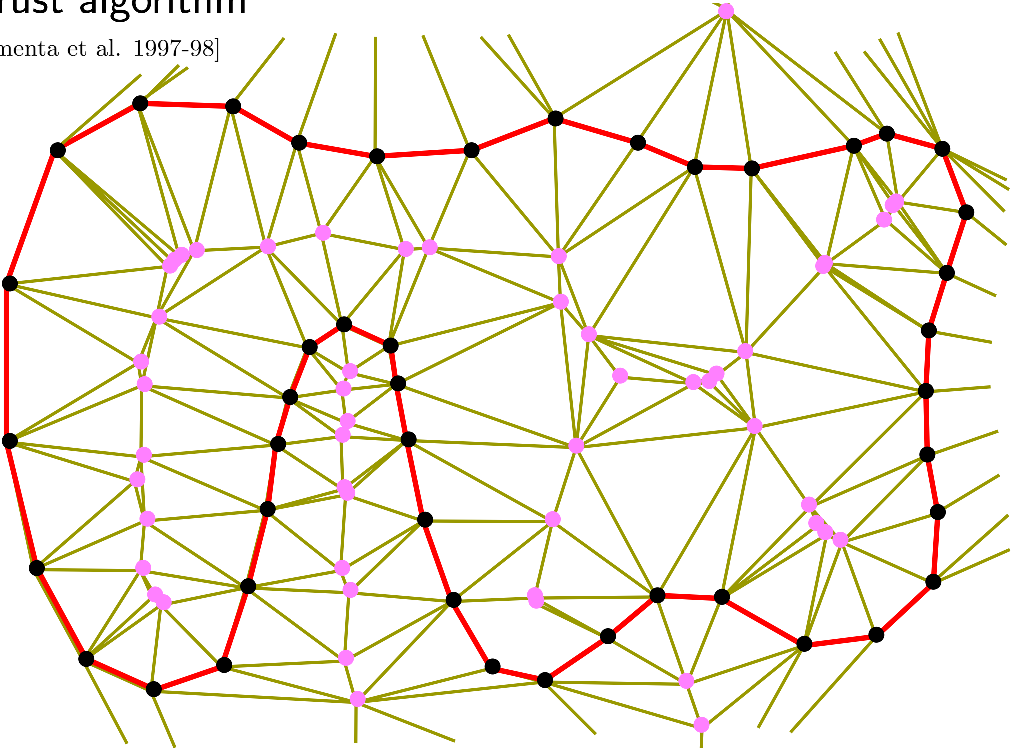


Crust algorithm

[Amenta et al. 1997-98]

4. Keep Delaunay simplices whose vertices are in P



Crust algorithm

$\rightarrow$ in 2-d, crust = $\mathrm{Del}_{\mathcal{S}}(P) \approx \mathcal{S}$

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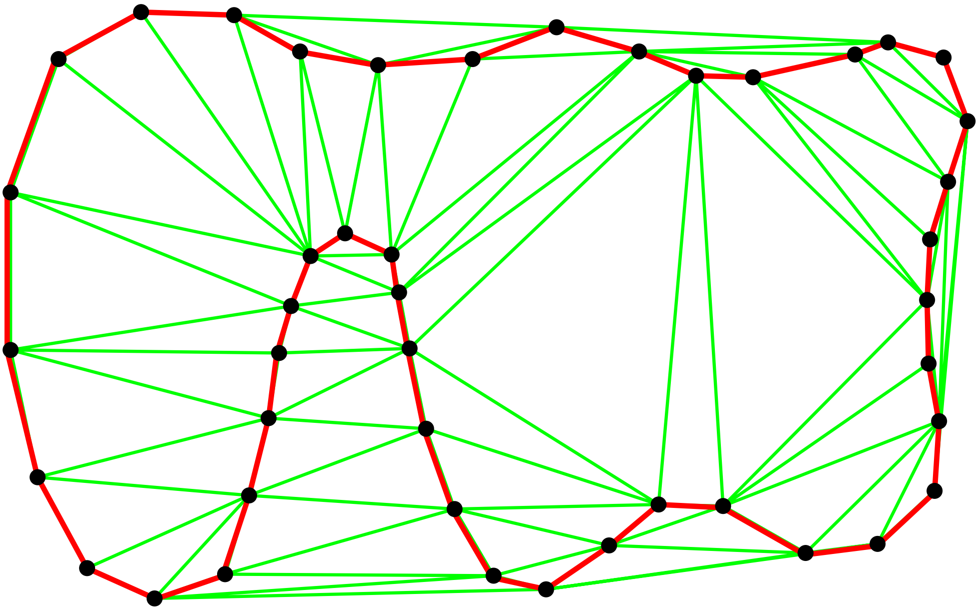

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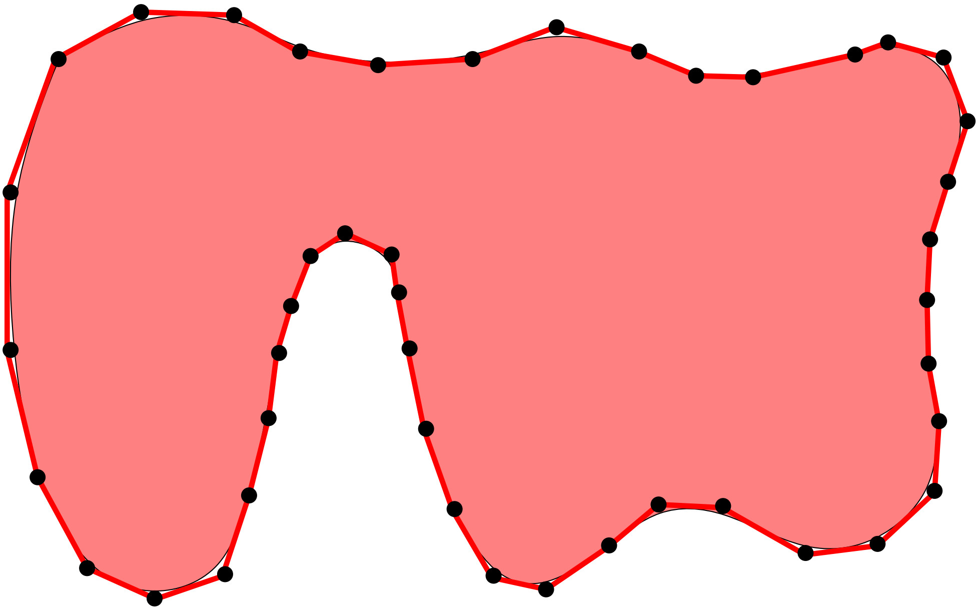

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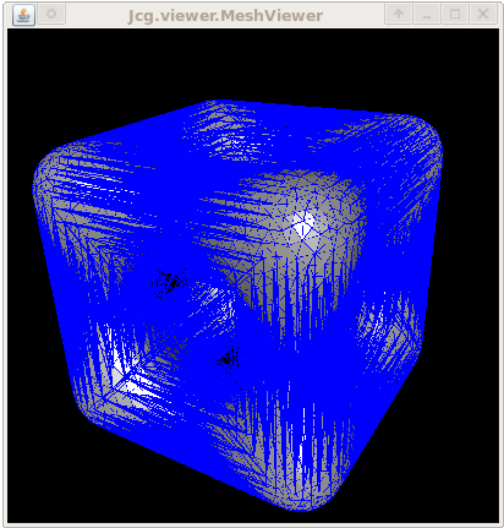

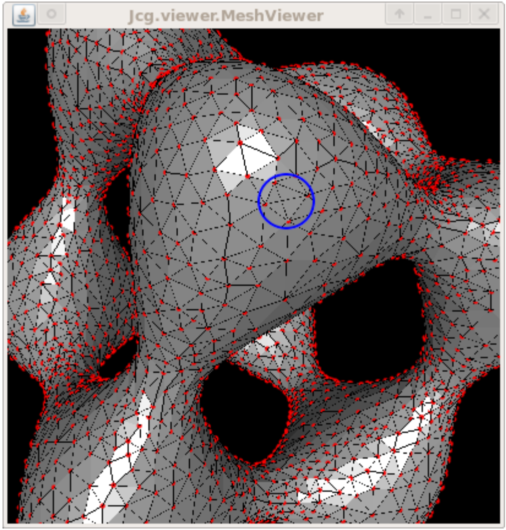

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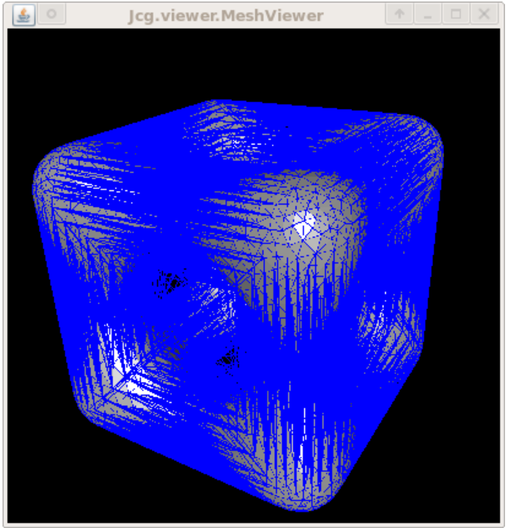
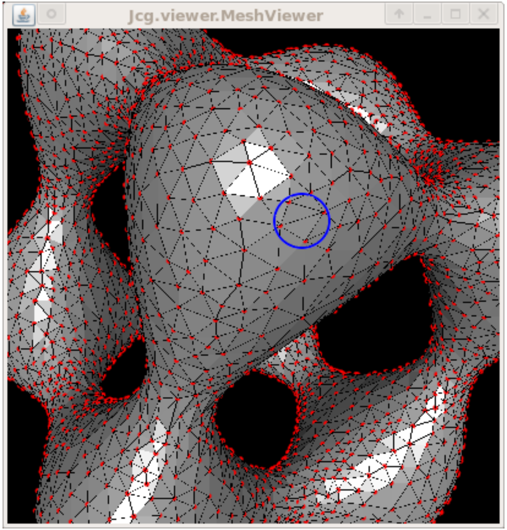
$\Rightarrow$ manifold extraction step in post-processing

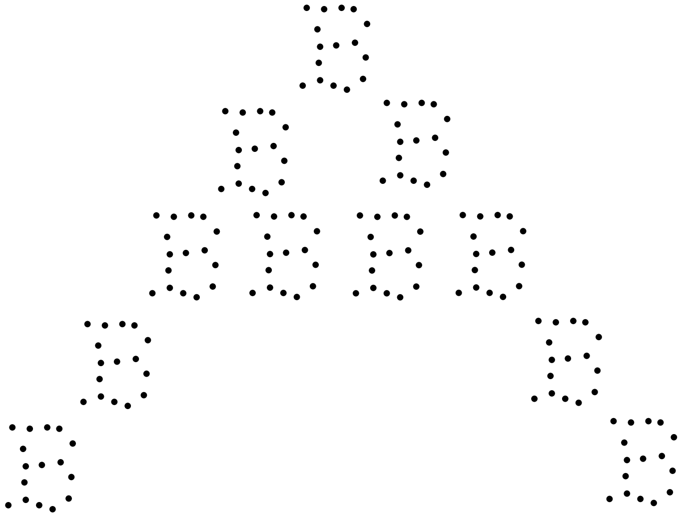




Back to the reconstruction paradigm

Q What do you see?

Why?



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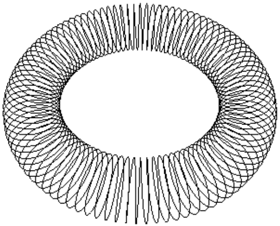
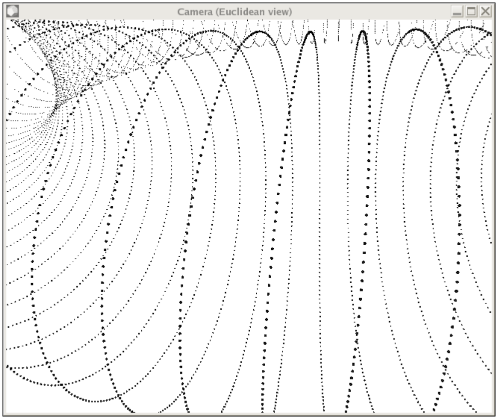
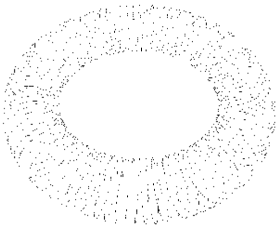






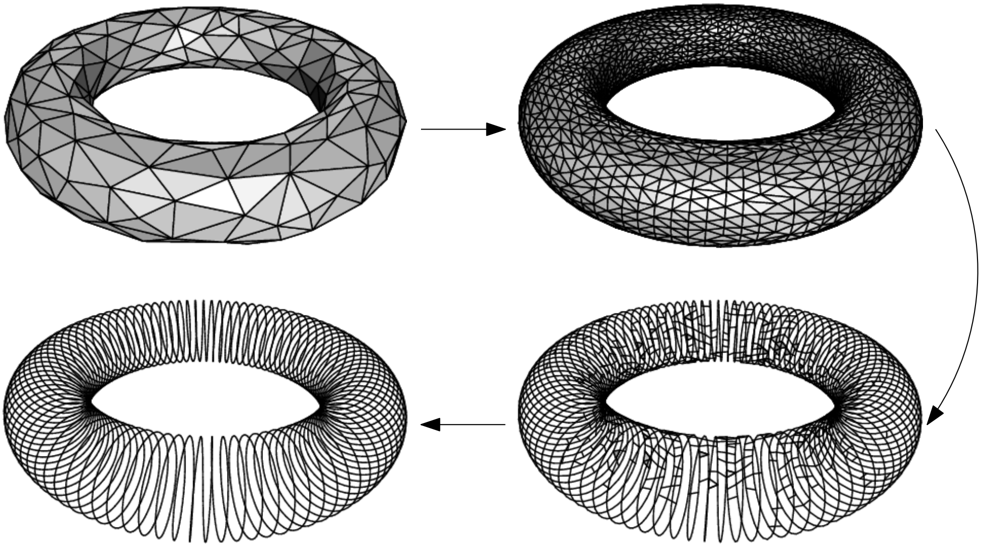
Back to the reconstruction paradigm

$\rightarrow$ **When the dimensionality of the data is unknown or there is noise, the reconstruction result depends on the scale at which the data is looked at.**

$\rightarrow$ need for multi-scale reconstruction techniques

Multi-scale approach in a nutshell

→ build a one-parameter family of complexes approximating the input at various scales



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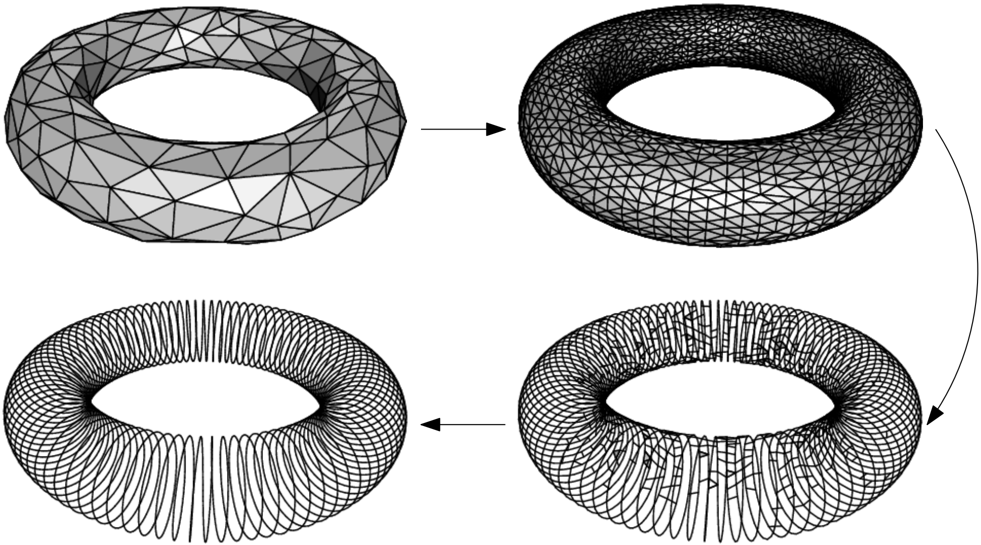
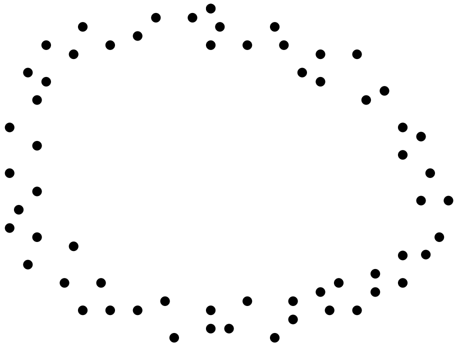


→ connections with manifold learning and topological persistence

Multi-scale algorithm [Guibas, Oudot 2007]

Input: a finite point set $W \subset \mathbb{R}^n$

$\rightarrow$ resample W iteratively, and maintain a simplicial complex:



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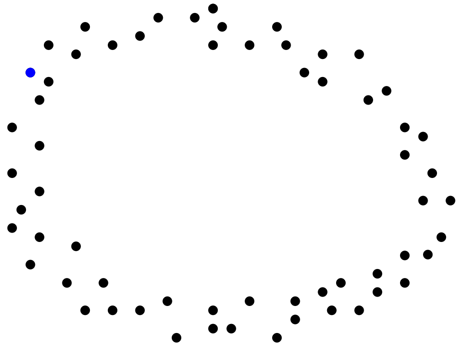

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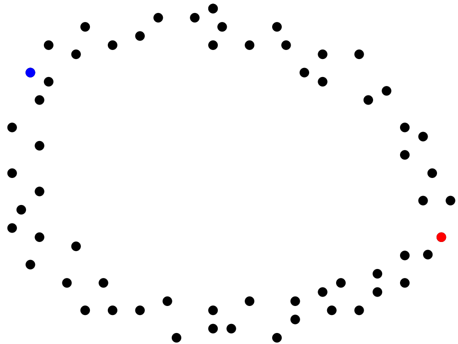

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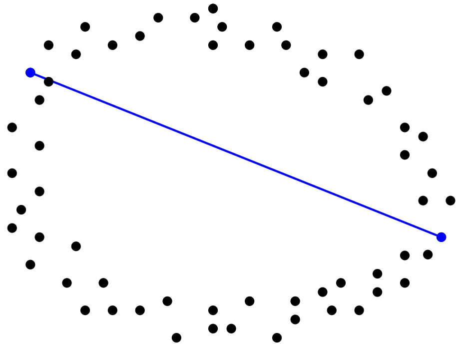

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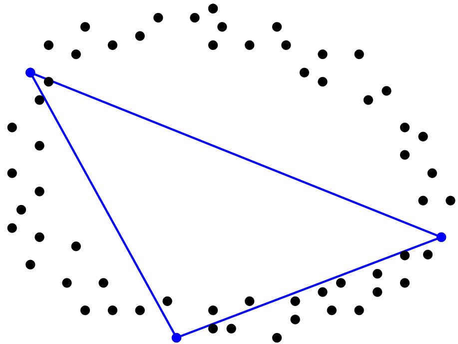

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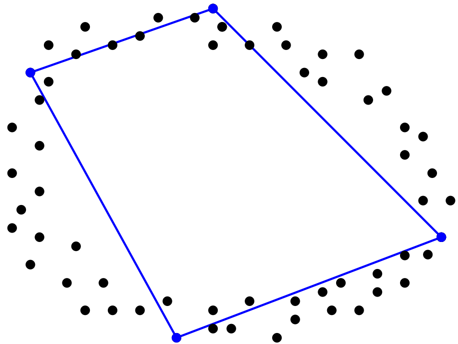

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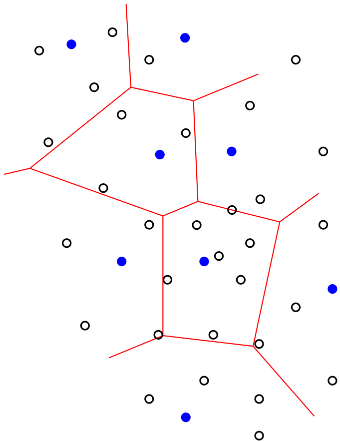
END_WHILE

Output: the sequence of simplicial complexes

The simplicial complex to maintain

→ maintain the **witness complex** $\mathrm{C}^W(L)$ [de Silva 2003]:

Let $L \subseteq \mathbb{R}^d$ (landmarks) s.t. $|L| < +\infty$ and $W \subseteq \mathbb{R}^d$ (witnesses)



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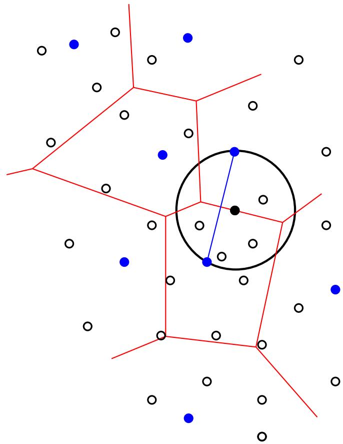
Def. $w \in W$ *strongly witnesses* $[v_0, \cdots, v_k]$ if $\|w - v_i\| = \|w - v_j\| \leq \|w - u\|$ for all $i, j = 0, \cdots, k$ and all $u \in L \setminus \{v_0, \cdots, v_k\}$ (Delaunay test)



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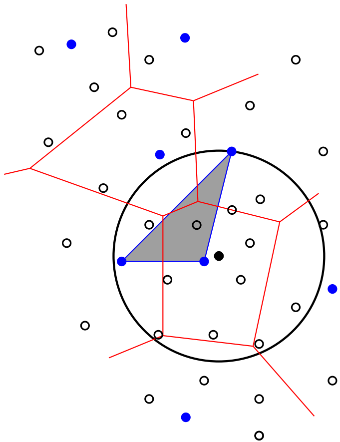

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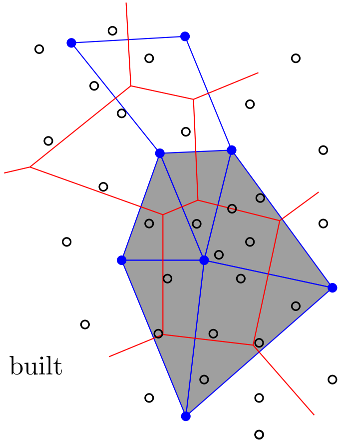

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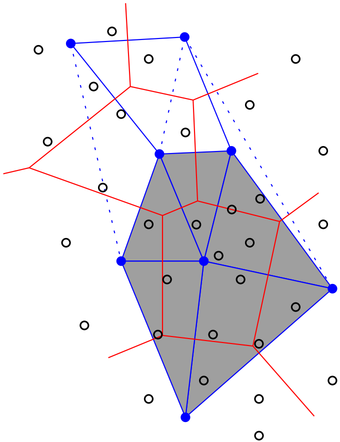
Def. $C^W(L)$ is the largest abstract simplicial complex built over L, whose faces are weakly witnessed by points of W.

The witness complex (properties)

Thm. 1 [de Silva 2003] $\forall W, L, \forall \sigma \in \mathrm{C}^W(L), \exists c \in \mathbb{R}^d$ that strongly witnesses σ.

$\Rightarrow \mathrm{C}^W(L)$ is a subcomplex of $\mathrm{Del}(L)$

$\Rightarrow \mathrm{C}^W(L)$ is embedded in $\mathbb{R}^d$



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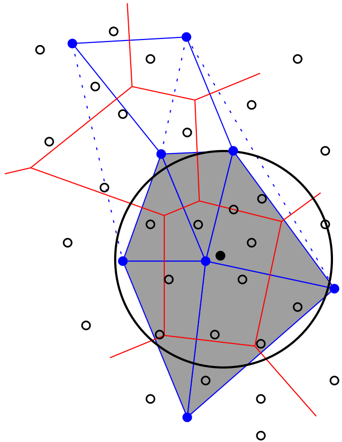

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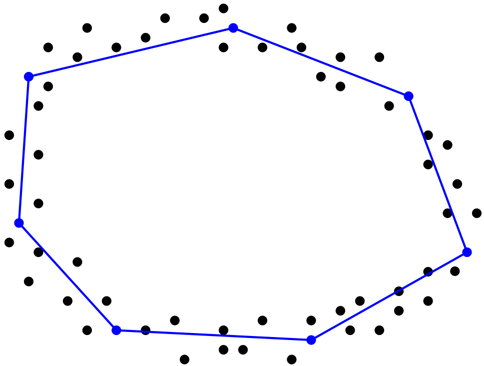

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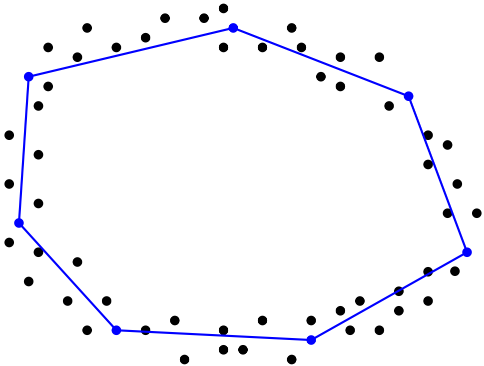
Thm. 3 [Guibas, Oudot 2007]
[Attali, Edelsbrunner, Mileyko 2007]
Under some conditions, $\mathrm{C}^W(L) = \mathrm{Del}_{\mathcal{S}}(L) \approx \mathcal{S}$

The witness complex (properties)

→ connection with reconstruction:

- $W \subset \mathbb{R}^d$ is given as input
- $L \subseteq W$ is generated
- underlying manifold $\mathcal{S}$ unknown
- only distance comparisons

⇒ algorithm is applicable in any metric space



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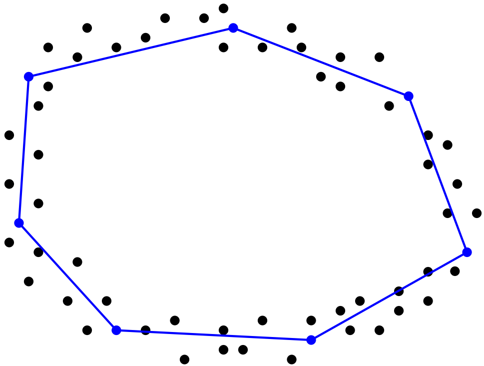
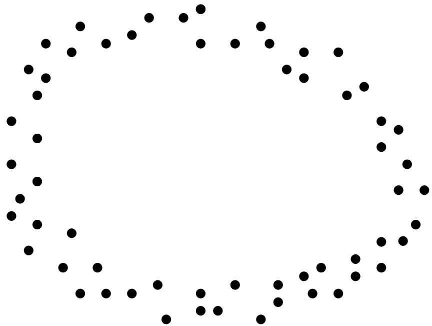


• In $\mathbb{R}^d$, $\mathrm{C}^W(L)$ can be maintained by updating, for each witness w, the list of $d+1$ nearest landmarks of w.

$$\Rightarrow \begin{array}{lcl} \text{space} & \leq & O\left(d|W|\right) \\ \text{time} & \leq & O\left(d|W|^2\right) \end{array}$$

The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.



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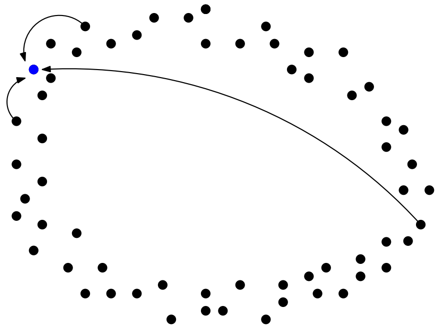

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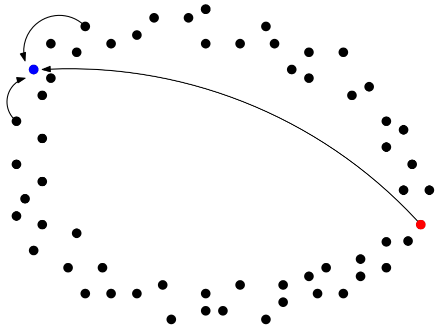

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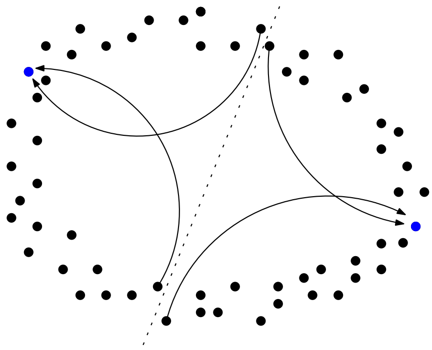

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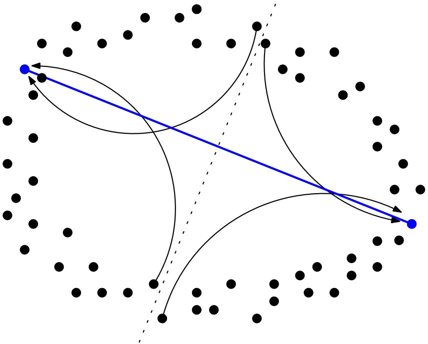

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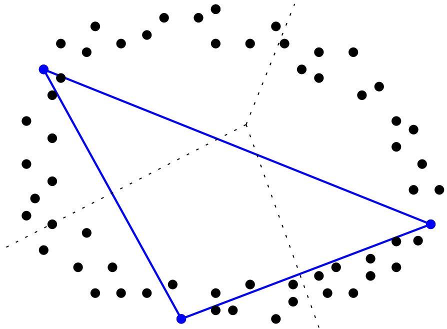

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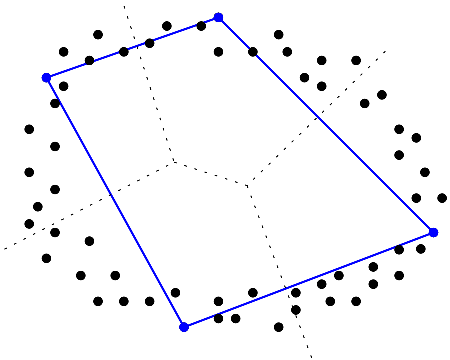
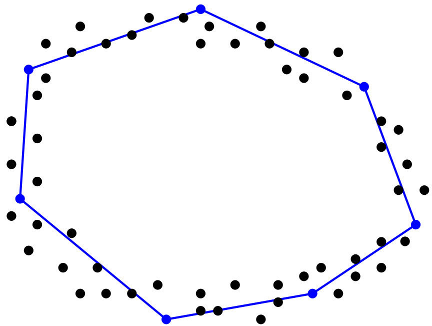


Output: the sequence of complexes $\mathrm{C}^W(L)$

Theoretical guarantees

$\rightarrow$ case of curves:

Conjecture [Carlsson, de Silva 2004]: $\mathrm{C}^W(L)$ coincides with $\mathrm{Del}_{\mathcal{S}}(L)$...



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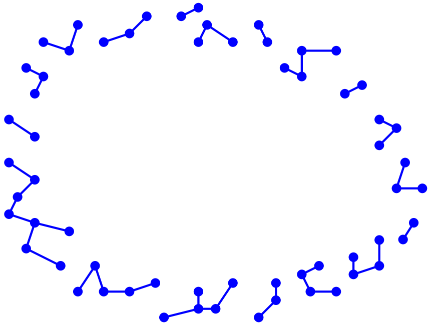

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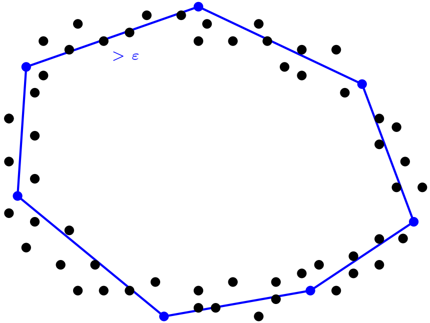
Thm. 3 If $\mathcal{S}$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $\mathrm{d_H}(W, \mathcal{S}) \leq \delta$, $L \subseteq W$ ε-sparse ε-sample of W with $\delta \ll \varepsilon \ll \varrho_{\mathcal{S}}$, then $\mathrm{C}^W(L) = \mathrm{Del}_{\mathcal{S}}(L) \approx \mathcal{S}$.



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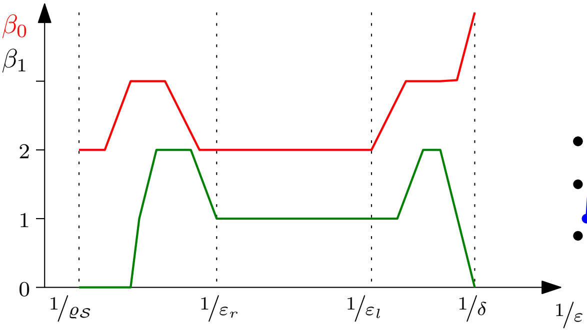
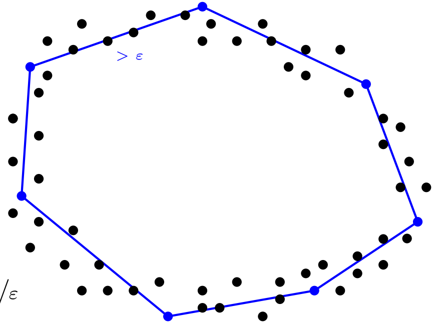
ε_l ε_r





$\rightarrow$ There is a plateau in the diagram of Betti numbers of $C^W(L)$.

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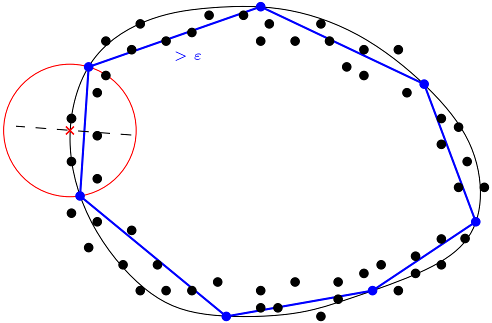

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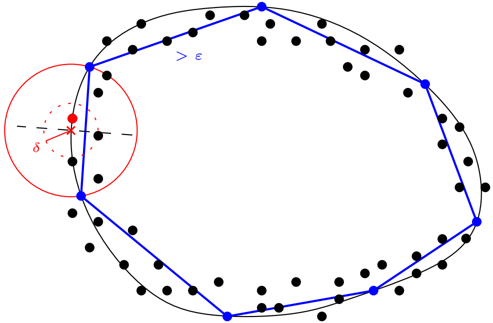

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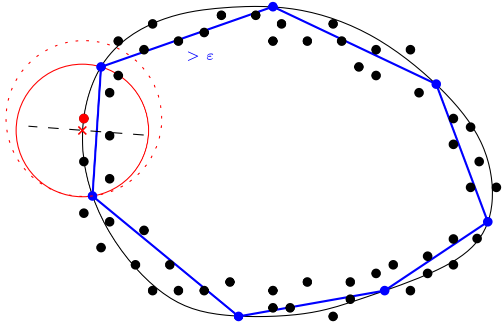

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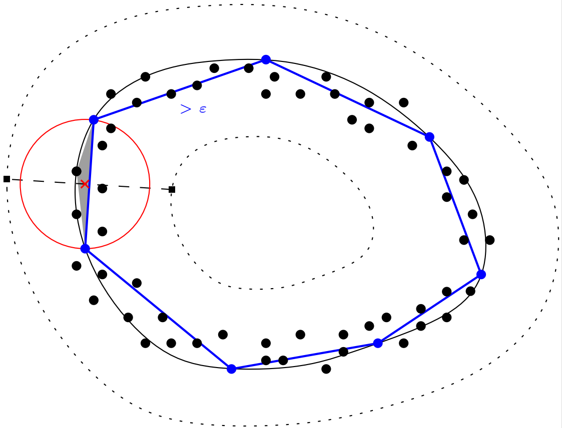

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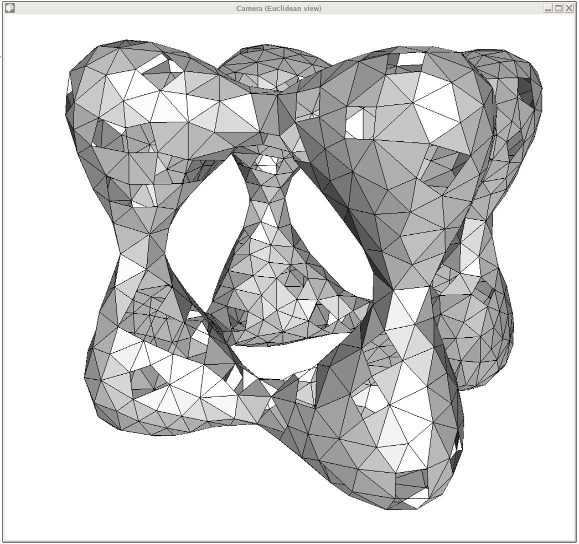

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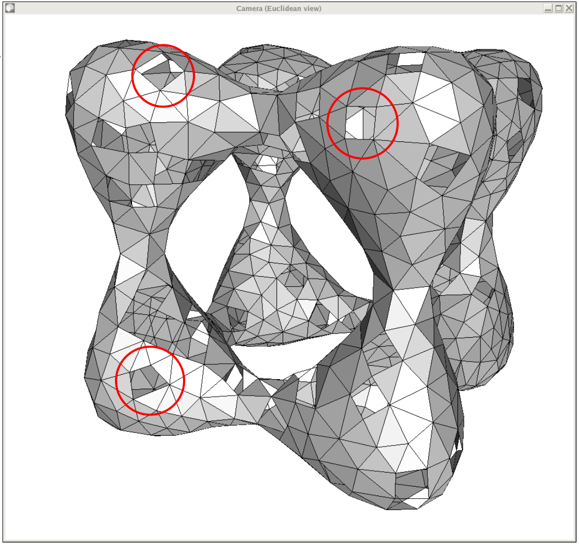

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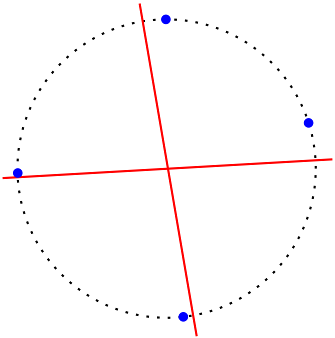
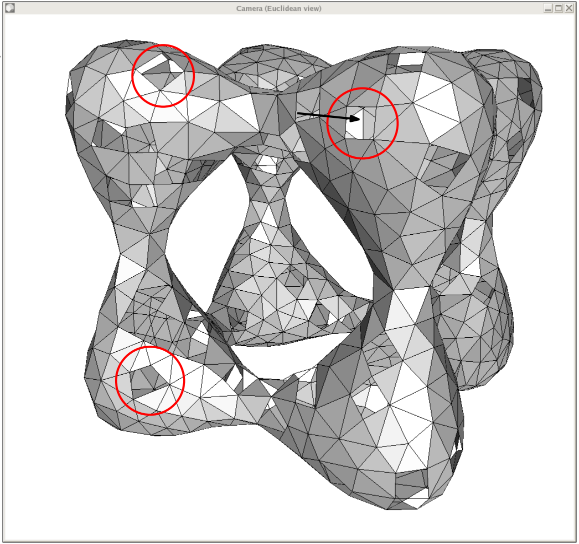


order-2 Voronoi diagram



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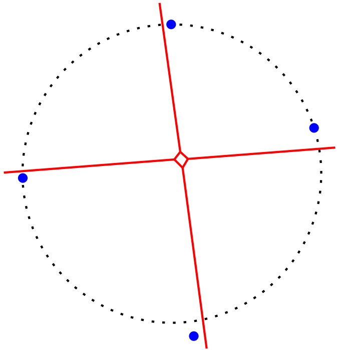

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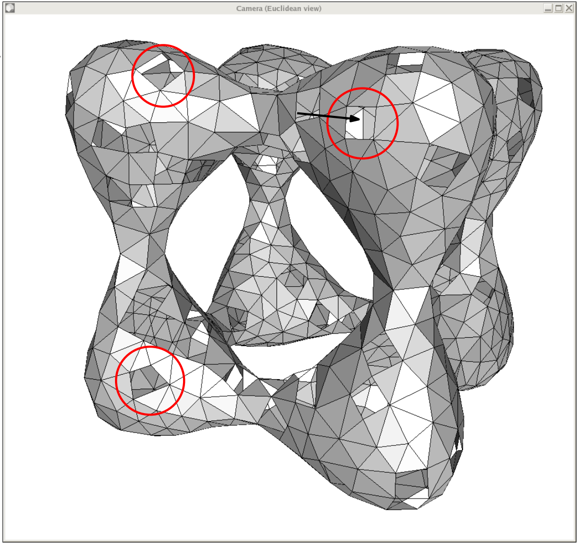

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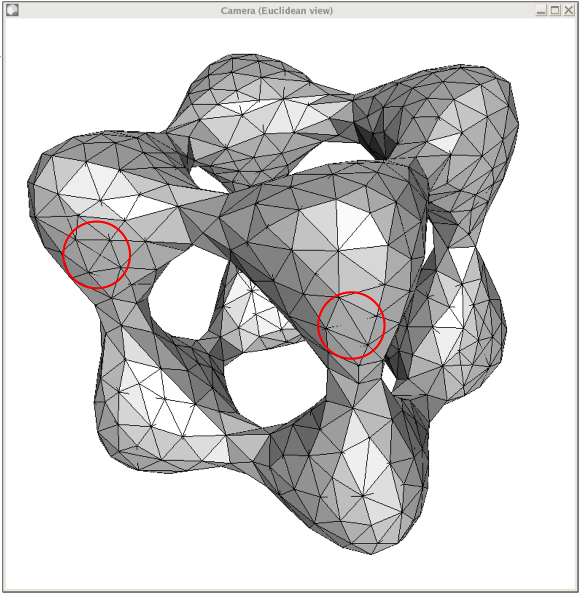
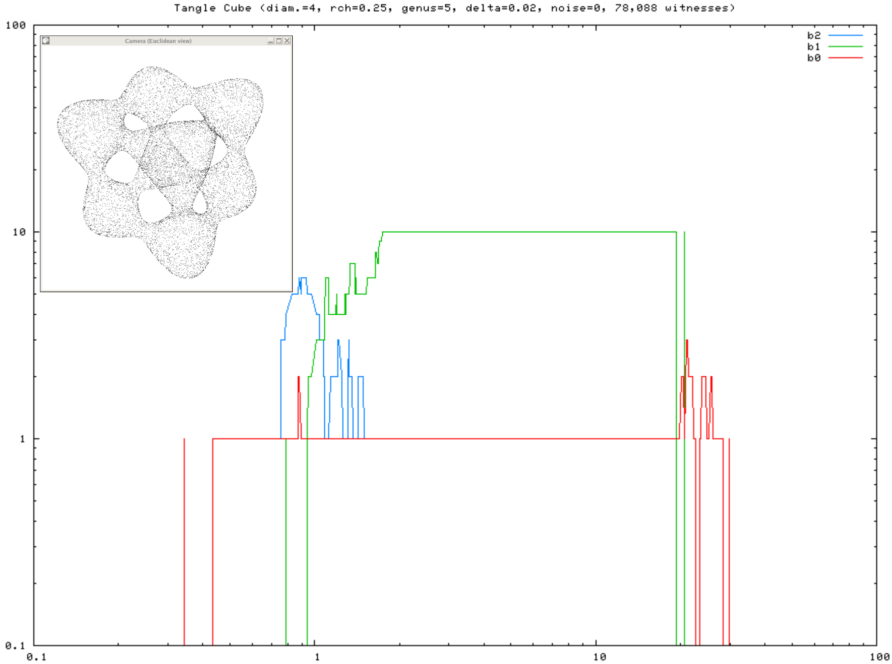
Solution relax witness test
[Guibas, Oudot]

$\Rightarrow \mathrm{C}_\nu^W(L) = \mathrm{Del}_{\mathcal{S}}(L)+$ slivers
$\Rightarrow \mathrm{C}_\nu^W(L) \nsubseteq \mathrm{Del}(L)$
$\Rightarrow \mathrm{C}_\nu^W(L)$ not embedded.

Post-process extract manifold M from $\mathrm{C}_\nu^W(L) \cap \mathrm{Del}(L)$
[Amenta, Choi, Dey, Leekha]



Some results



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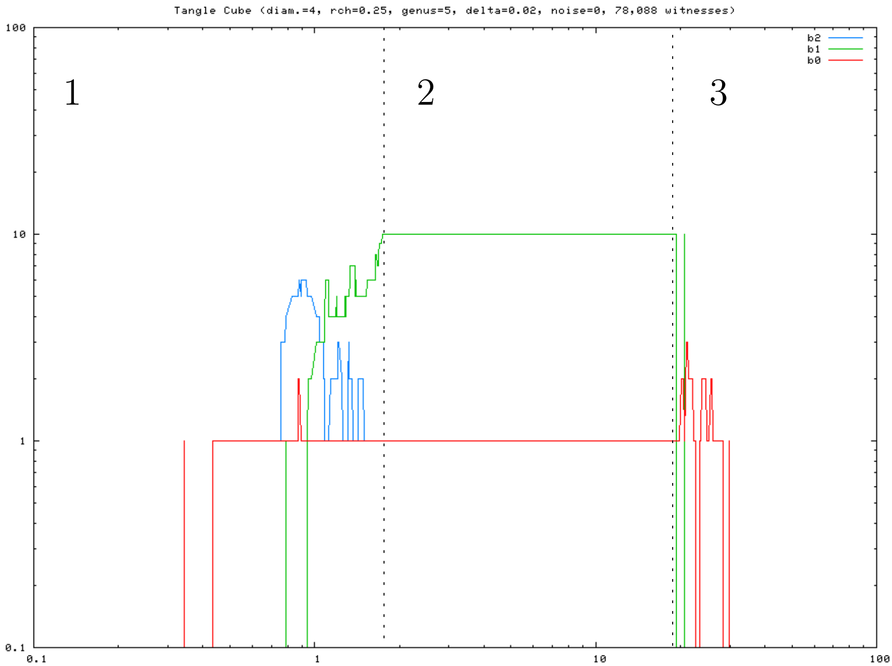

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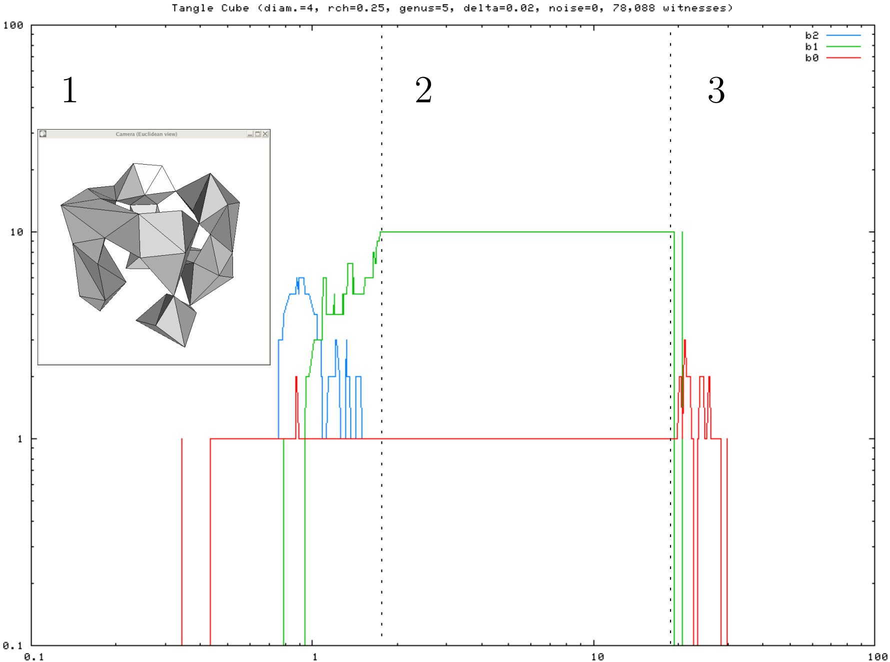

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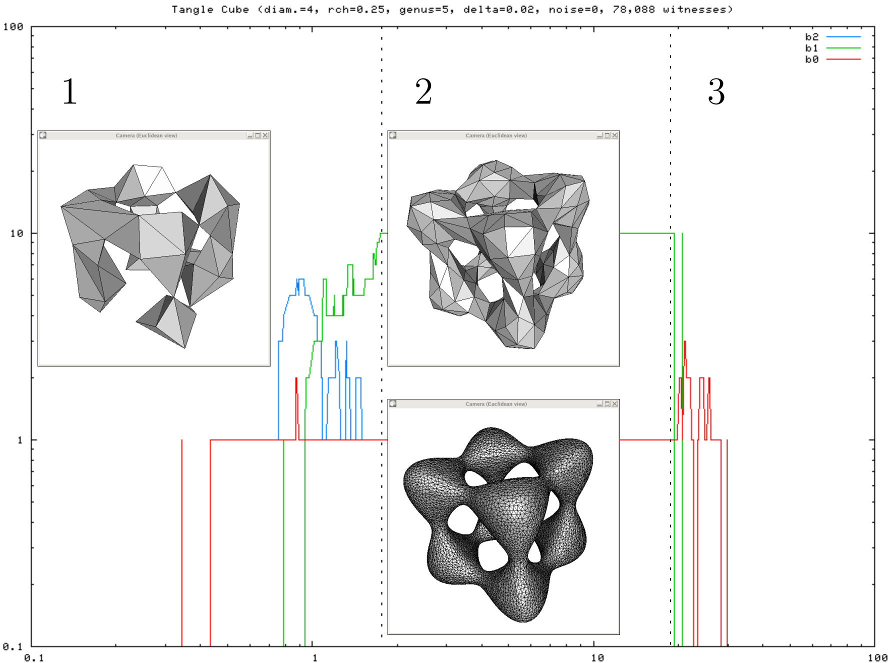

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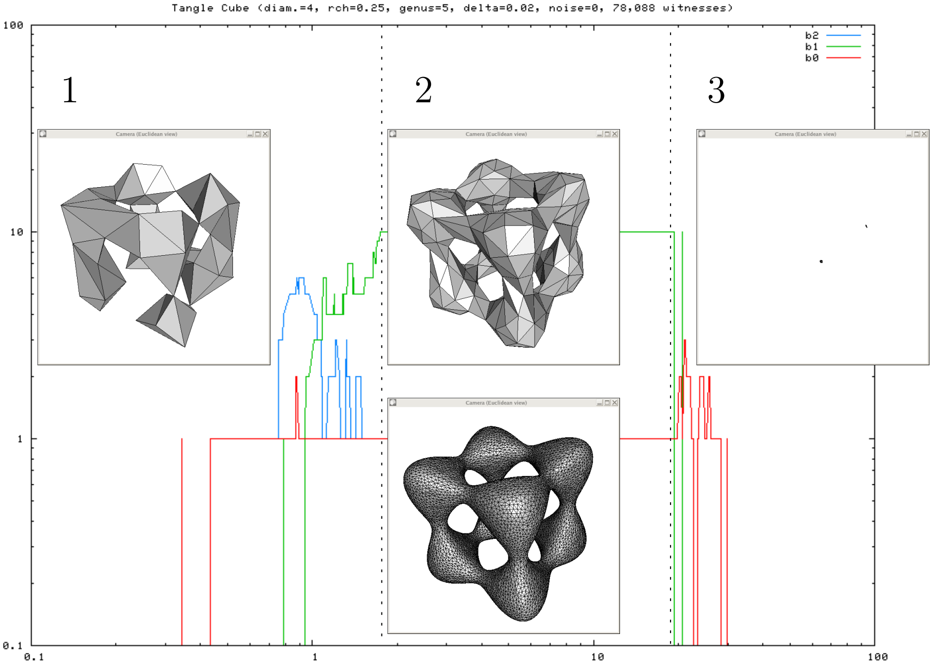
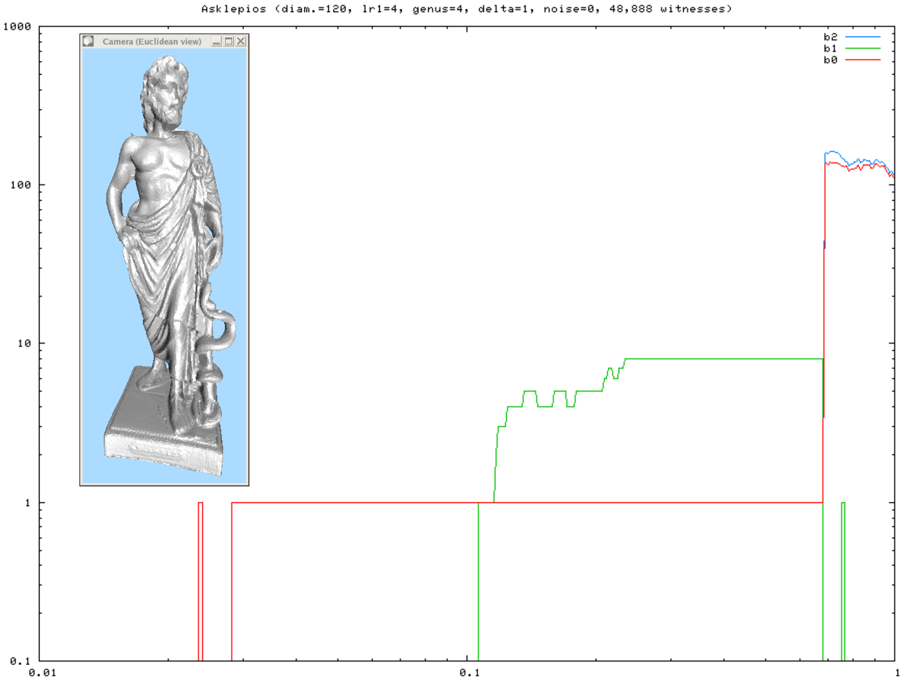
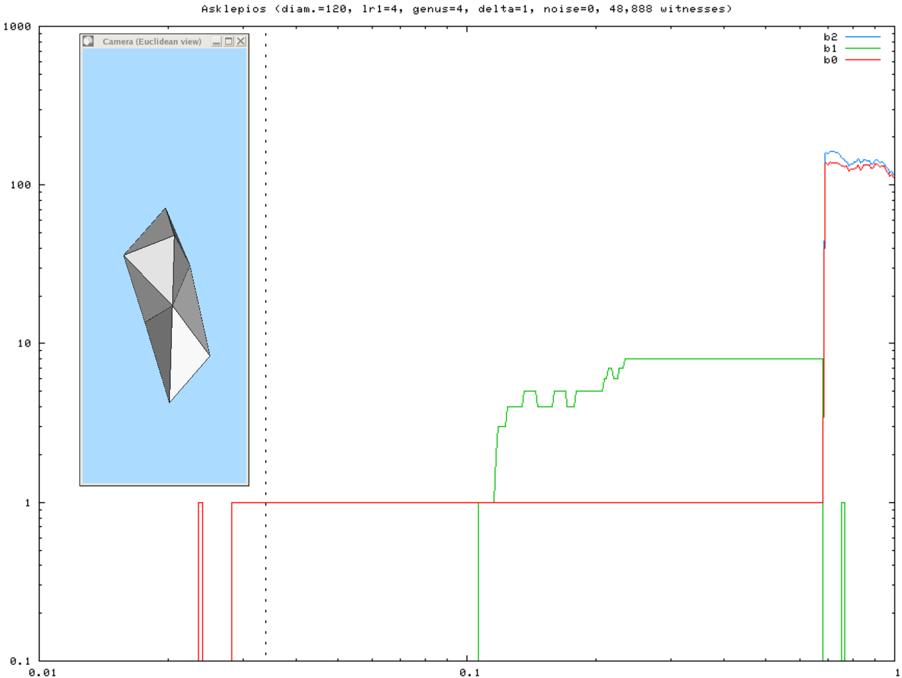


Some results (cont'd)



input model provided courtesy of IMATI by the Aim@Shape repository

Some results (cont'd)



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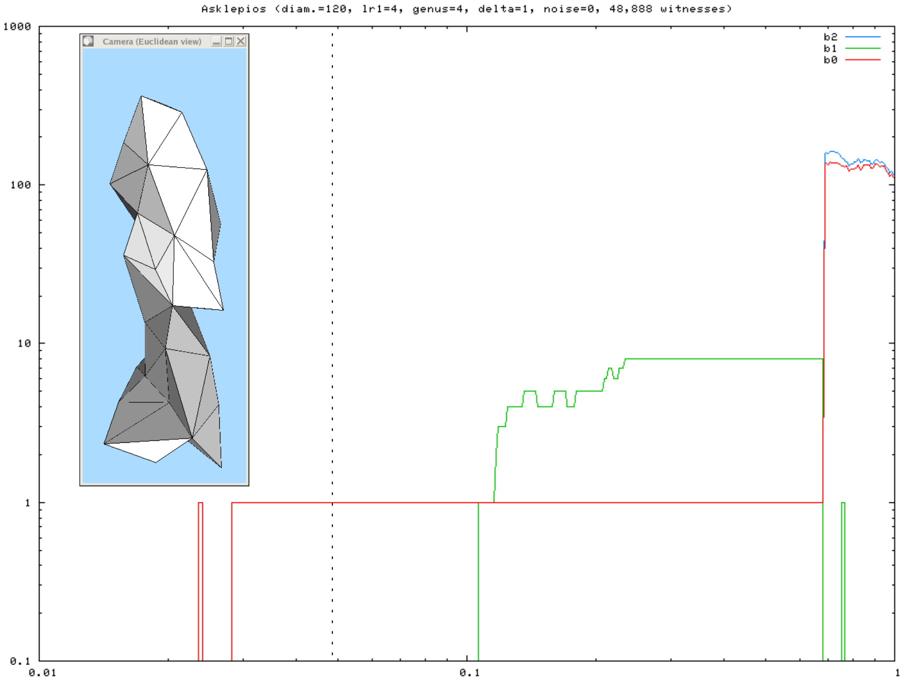

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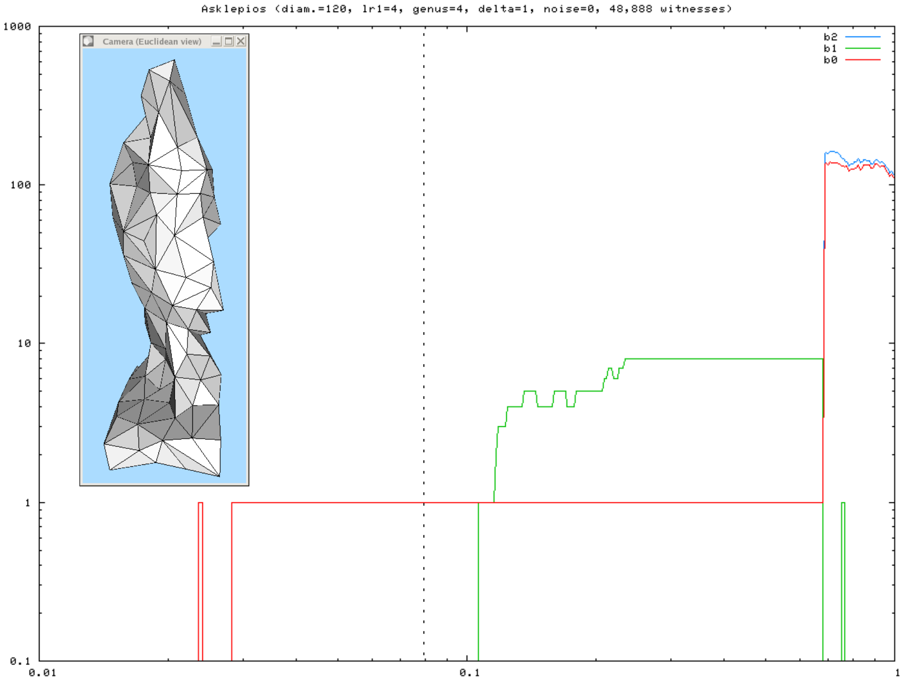

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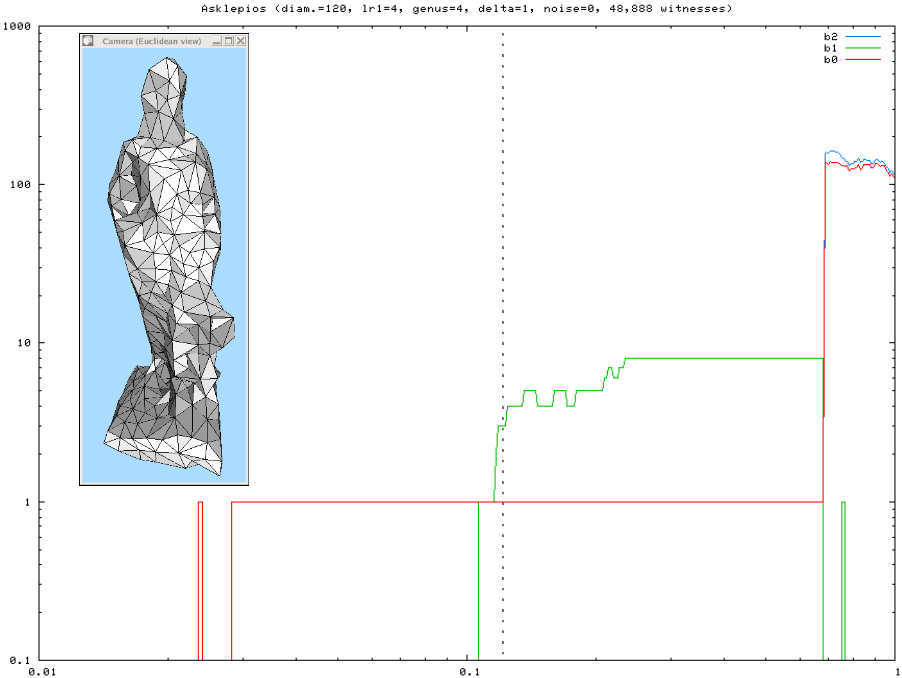

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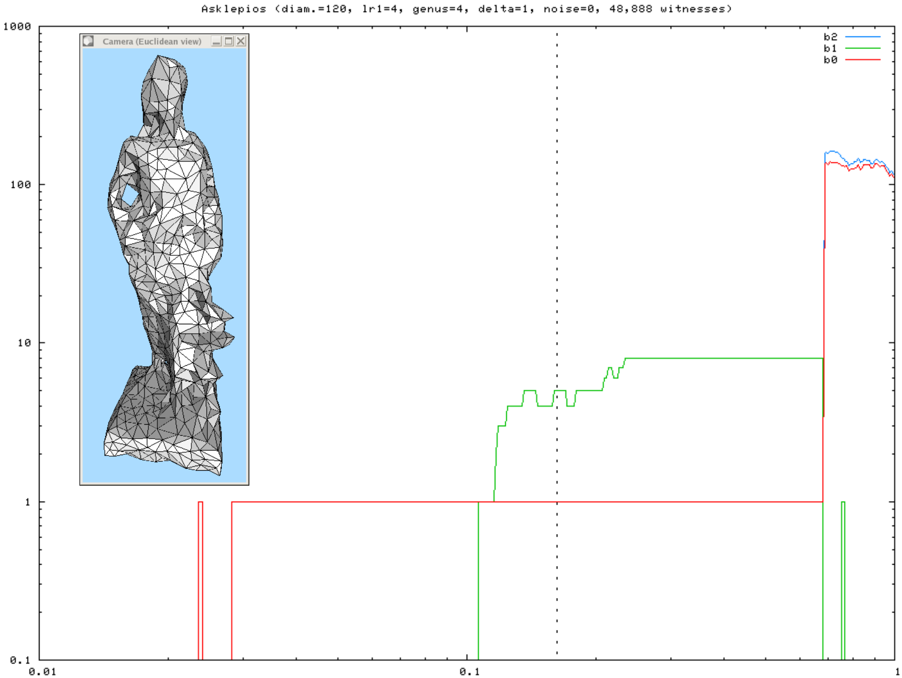

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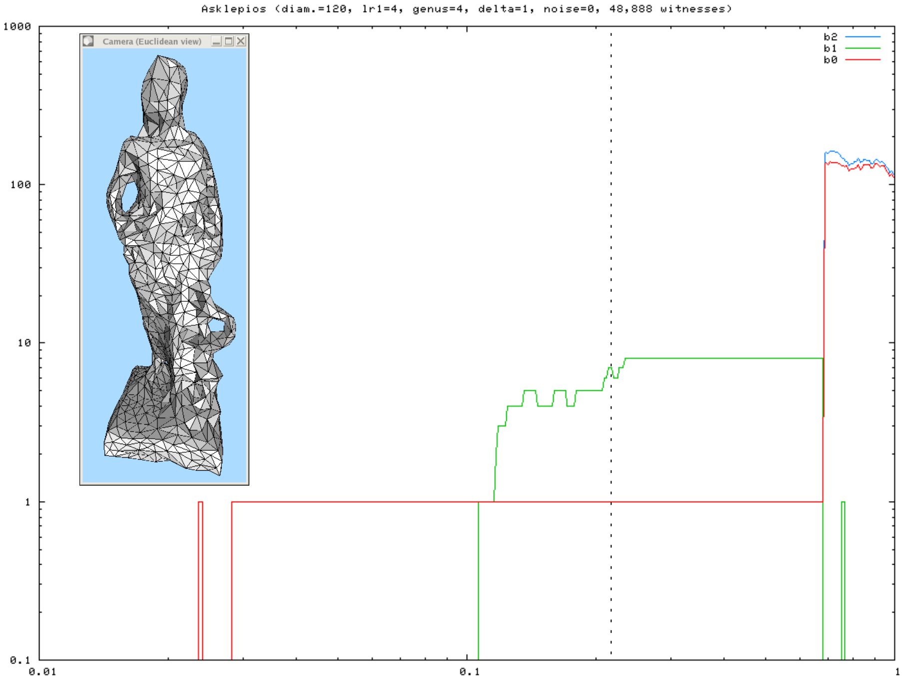

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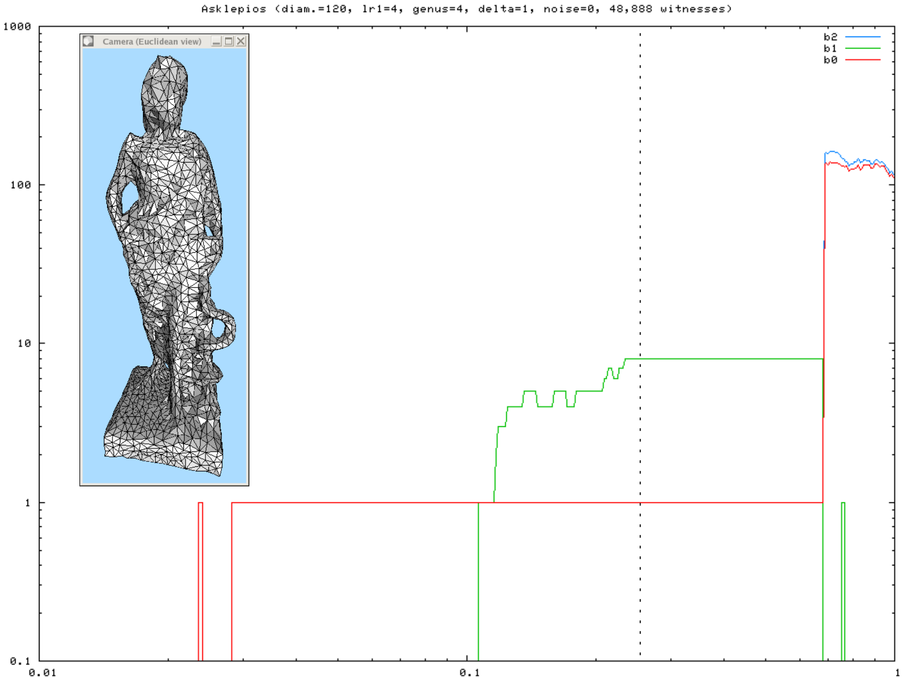

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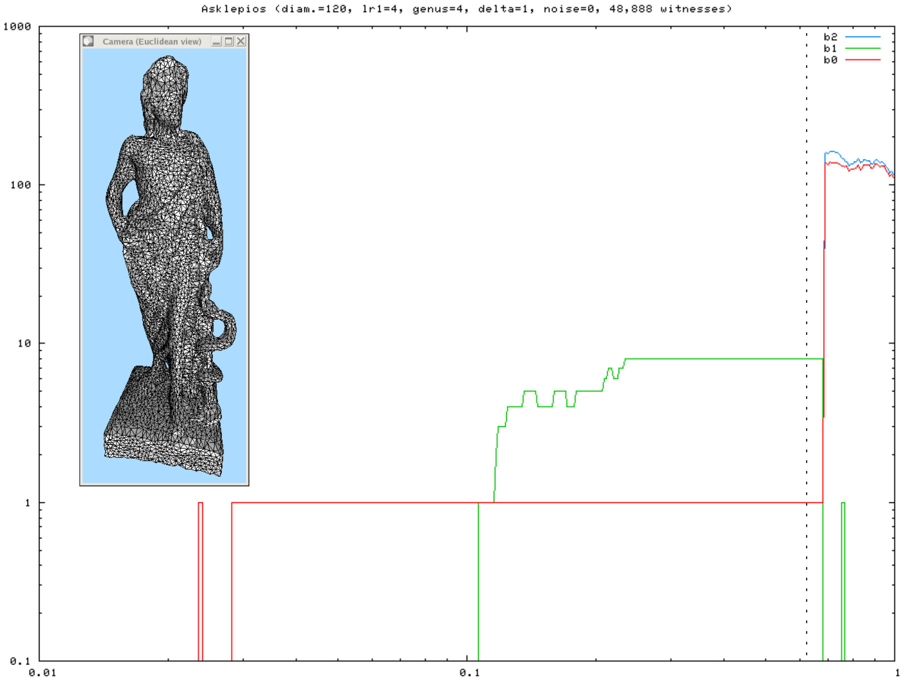

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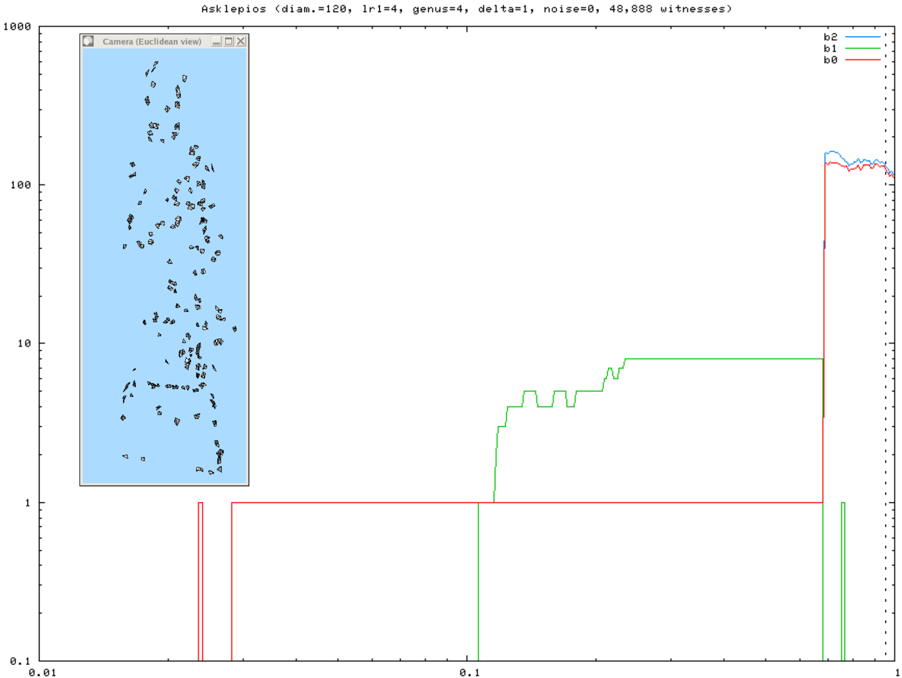

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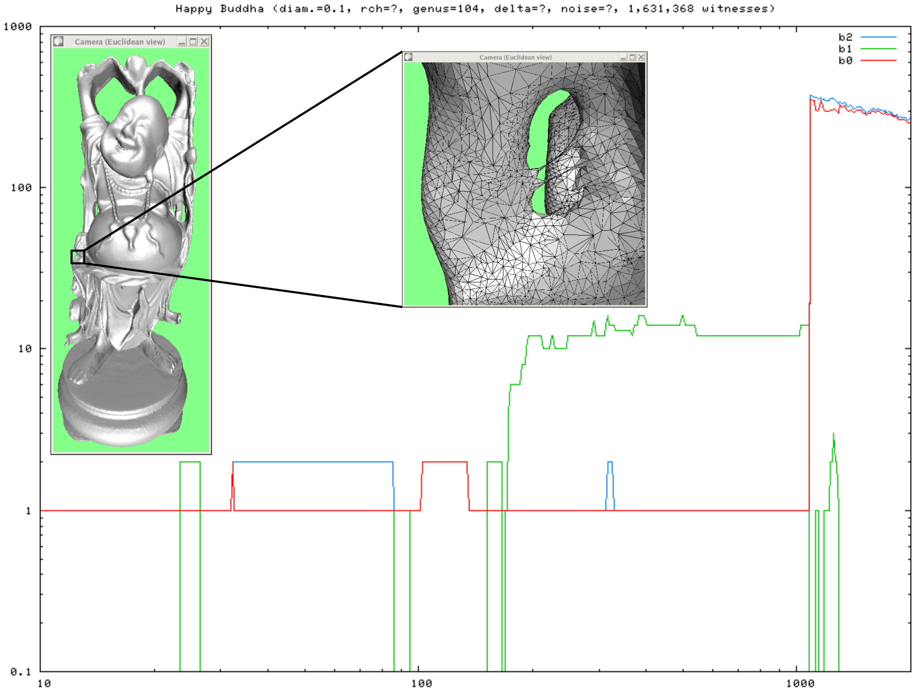

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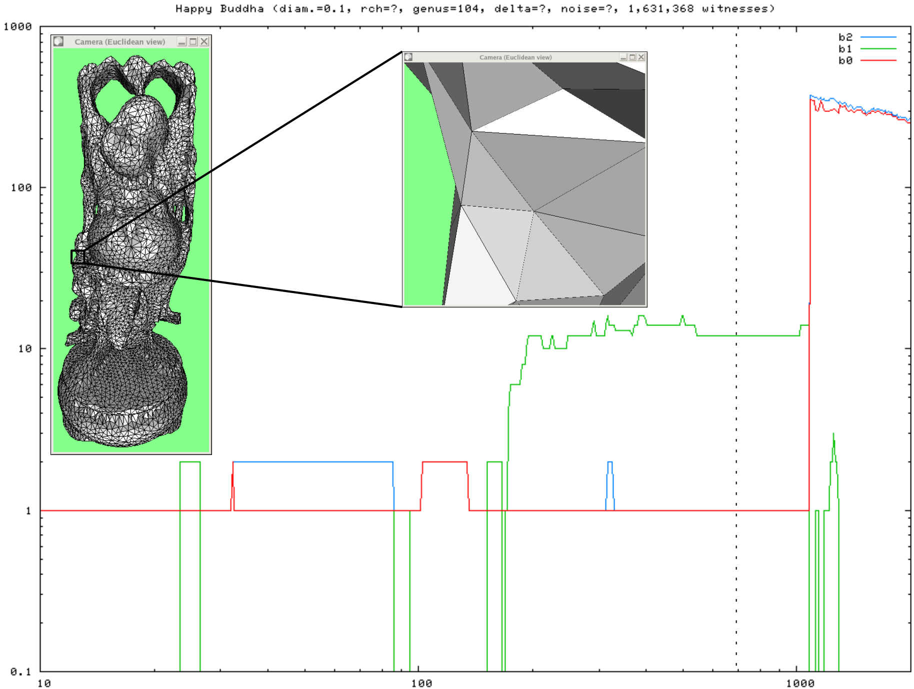

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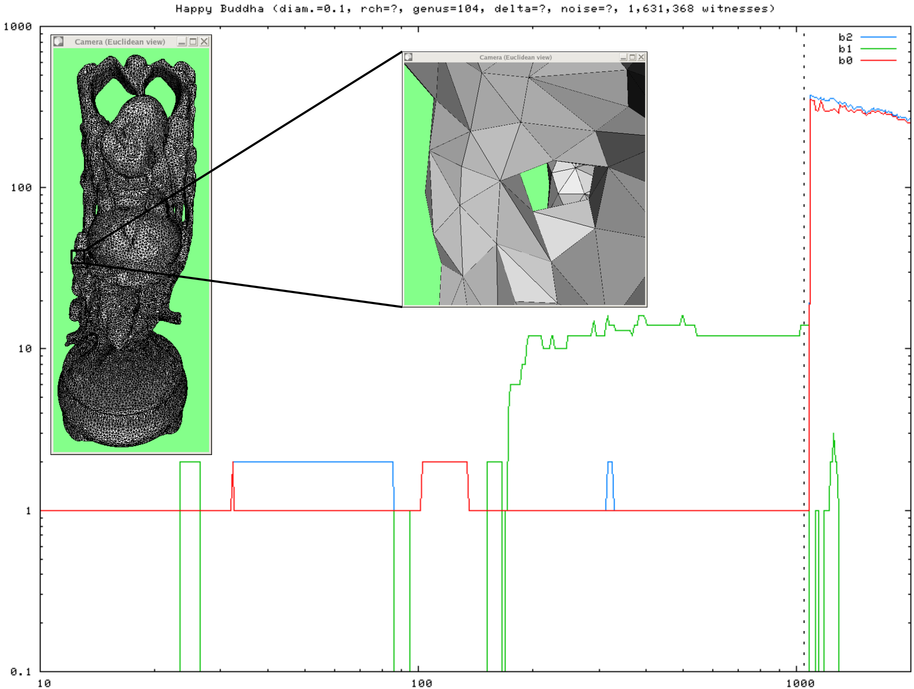

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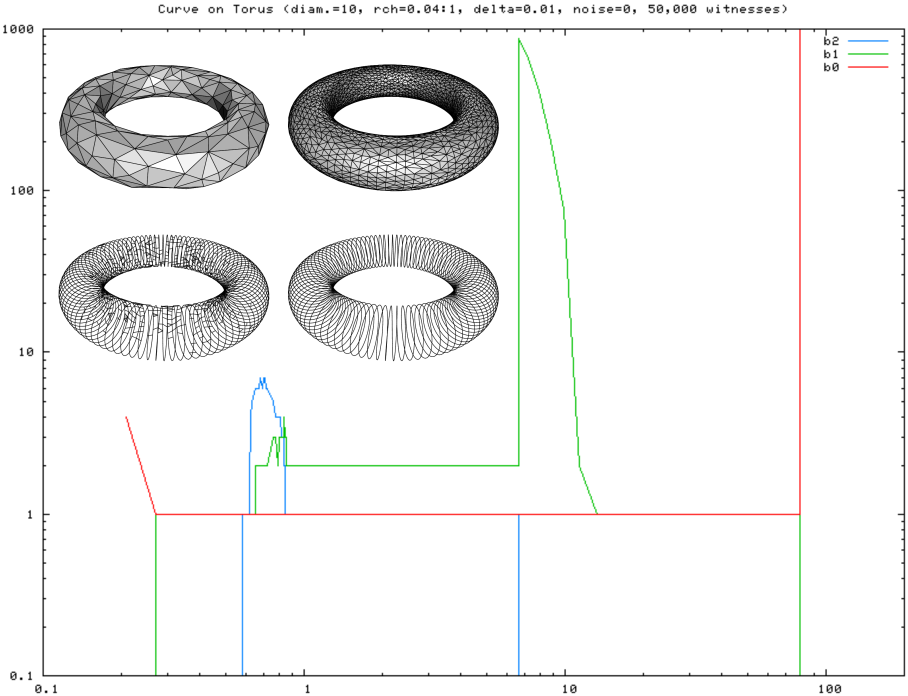

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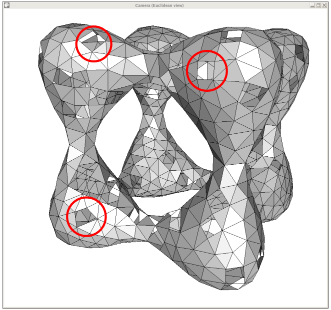

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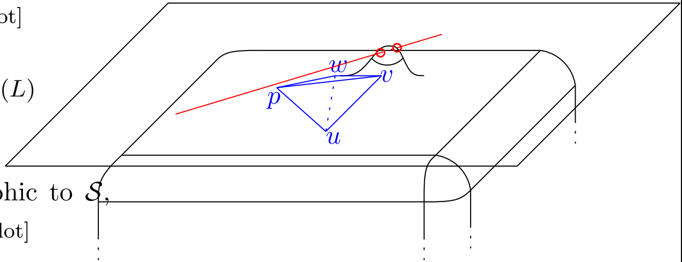

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Higher-dimensional reconstruction is still widely open