INF562, Lecture 3:
Geometric and combinatorial properties of planar graphs

mardi 22 janvier 2013

Luca Castelli Aleardi

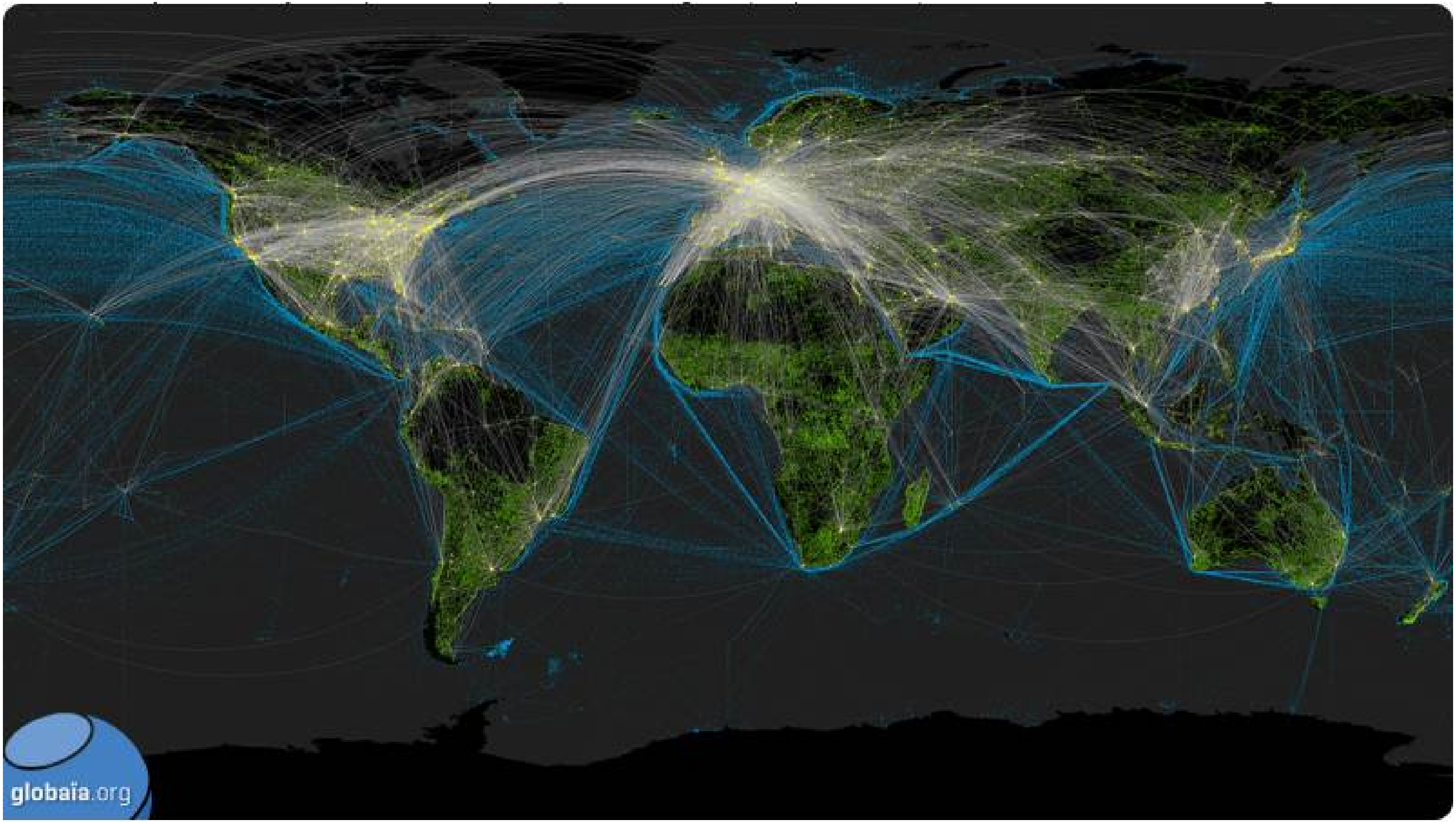
ÉCOLE POLYTECHNIQUE

Intro

Graph drawing: motivations and applications

Graph drawing and data visualization

Global transportation system



Graph drawing and data visualization

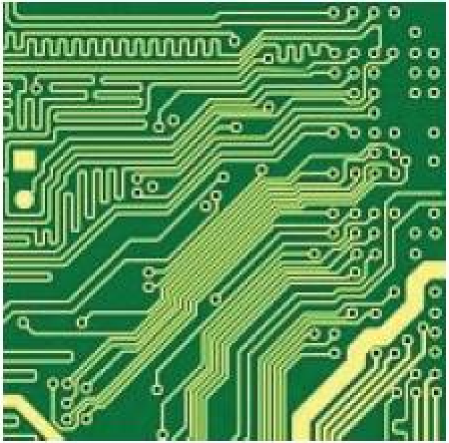
Roads, railways, ...

Graph drawing and data visualization

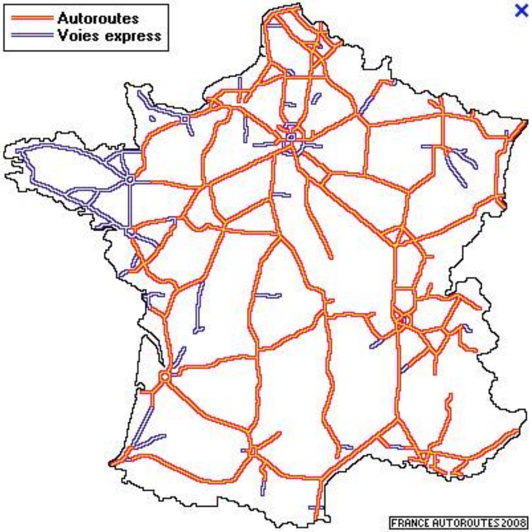
Social network graph

Planar graphs

Design of integrated circuits (VLSI)

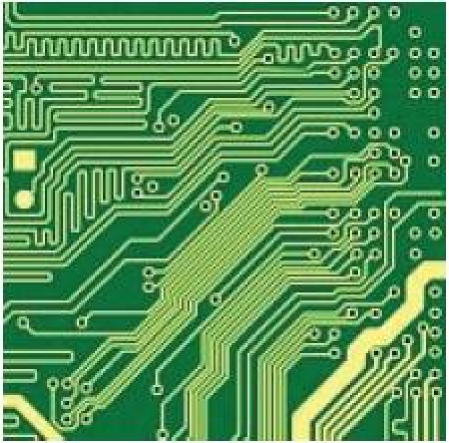


french roads network



Planar graphs

Design of integrated circuits (VLSI)



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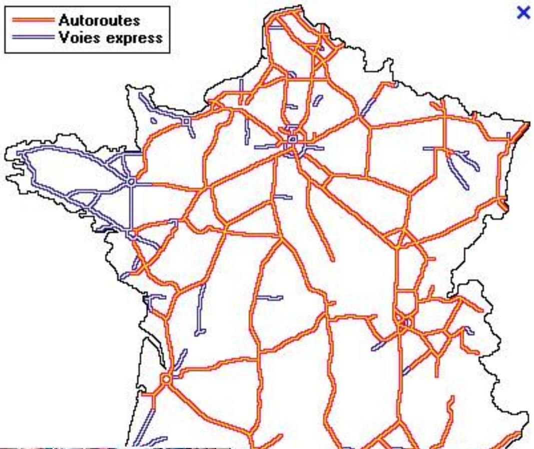
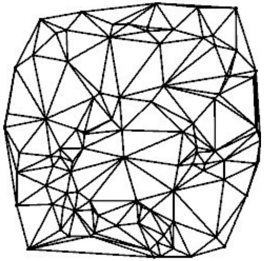


9 accidents en 2012 (last one, on 28th sptember)

www.2m40.com

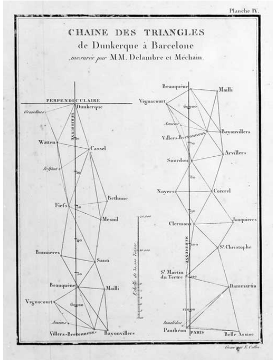
Meshes and graphs in computational geometry

Delaunay triangulations, Voronoi diagrams, planar meshes, ...

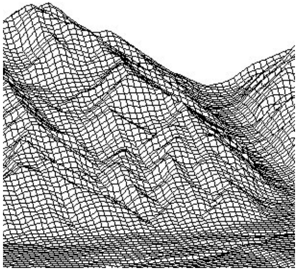


Delaunay triangulation

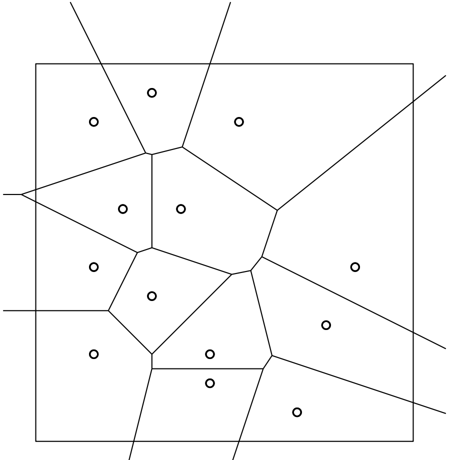
triangles meshes already used in early 19th century (Delambre et Mchain)



GIS Technology

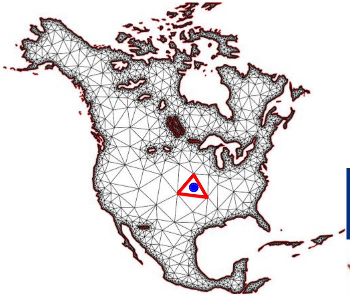


Terrain modelling

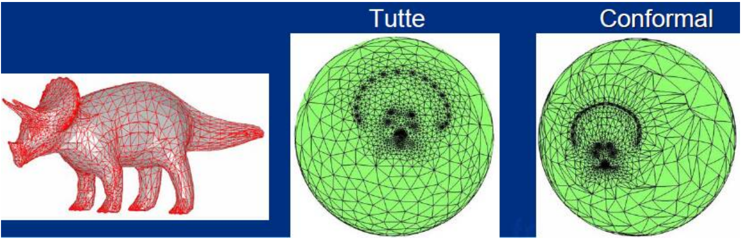


Voronoi diagram

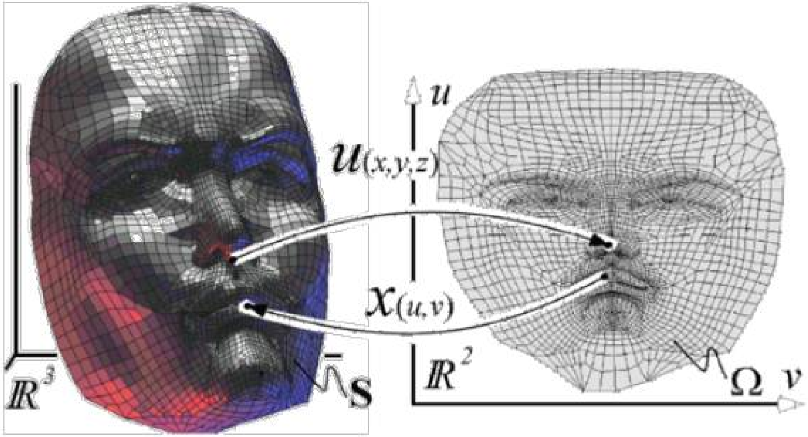
Planar mesh by L. Rineau, M. Yvinec

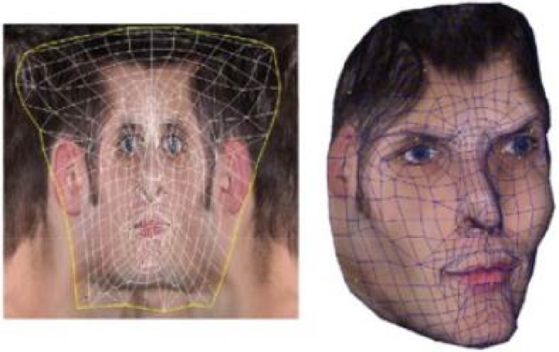


Spherical Parameterization (Sheffer Gotsman)



Mesh parameterization in geometry processing





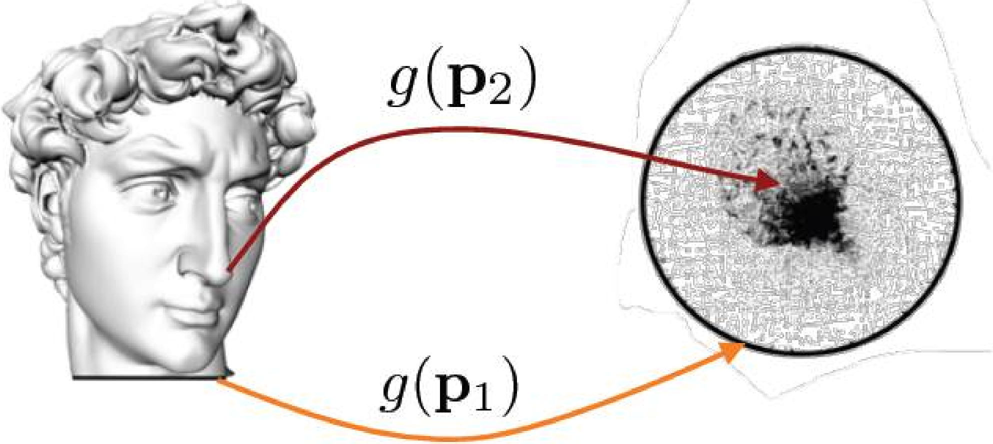
Bennis et al., 1991
Maillot et al., 1993

Mesh parameterization in geometry processing

General problem:

- Given a mesh (T, P) in 3D find a bijective mapping

$$g : P \to \mathbf{R}^2$$
$$g(\mathbf{p}_i) = \mathbf{u}_i = (u_i, v_i)$$

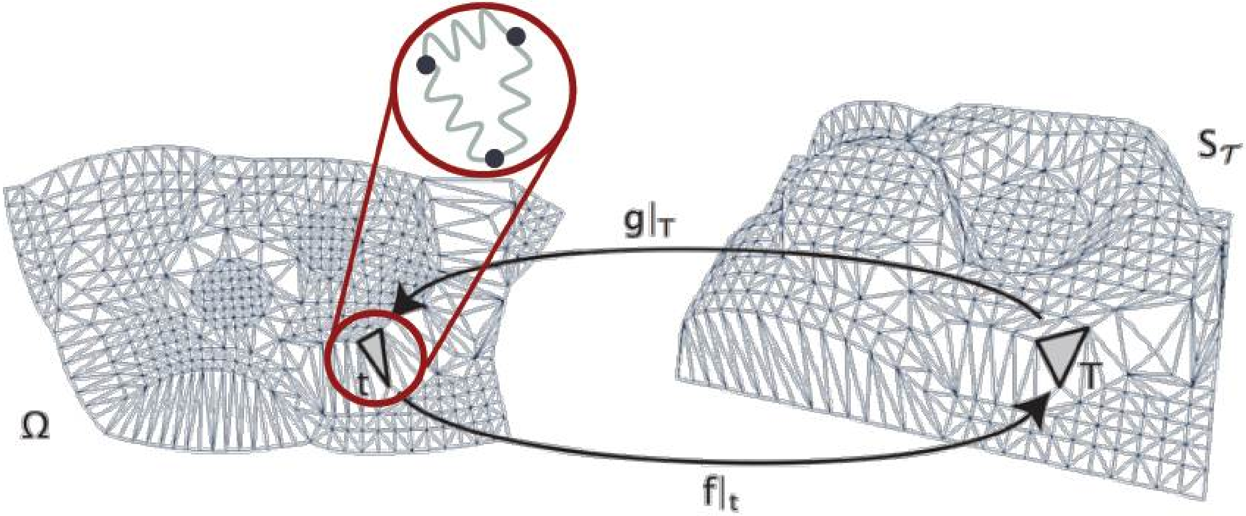


Mesh parameterization in geometry processing

Given a mesh (T, P) in 3D find a bijective mapping $g(\mathbf{p}_i) = \mathbf{u}_i$

given constraints: $g(\mathbf{b}_j) = \mathbf{u}_j$ for some $\{\mathbf{b}_j\}$

Model: imagine a **spring** at each edge of the mesh.
If the boundary is fixed, let the interior points find an **equilibrium**.



Graph drawing: motivation

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Challenge: what kind of graph does A_G represent?

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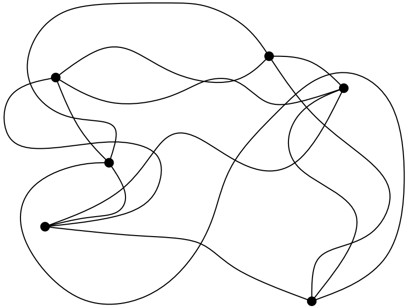

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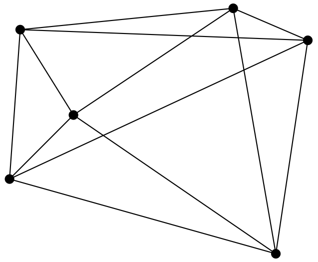

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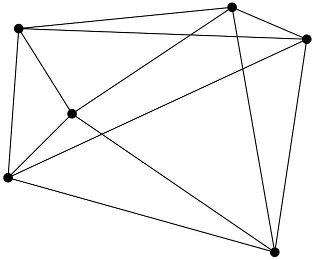

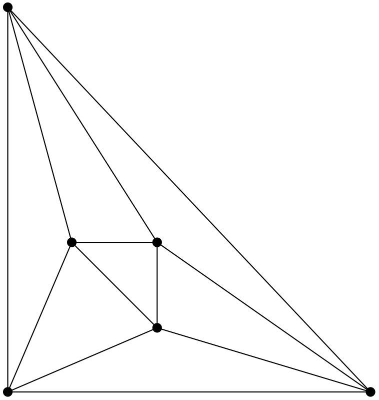

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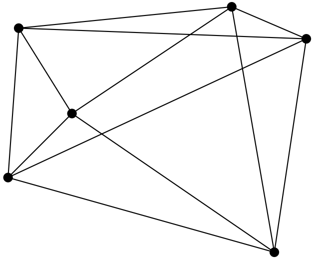

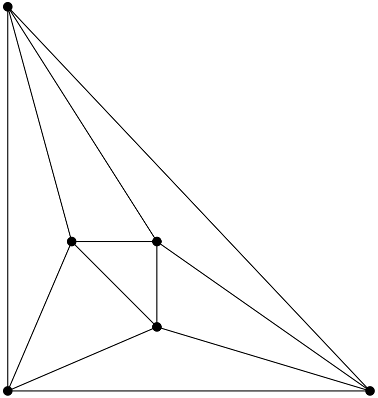

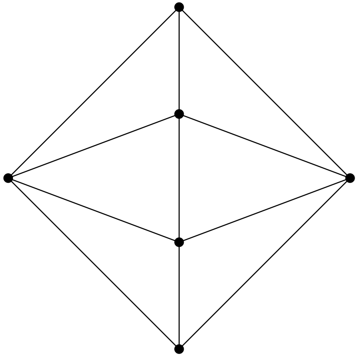

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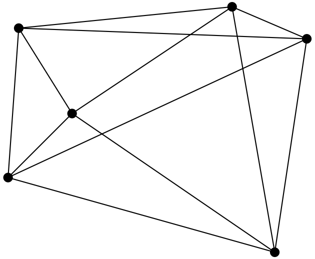

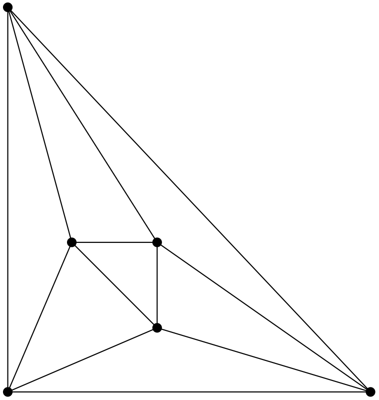

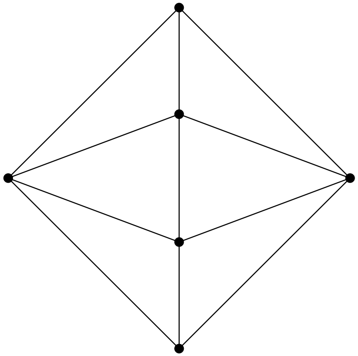

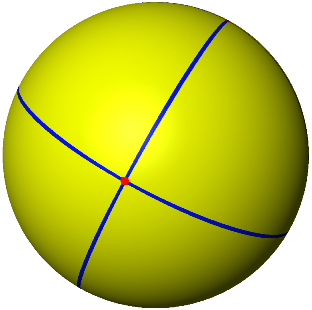

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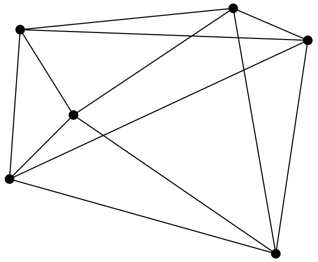

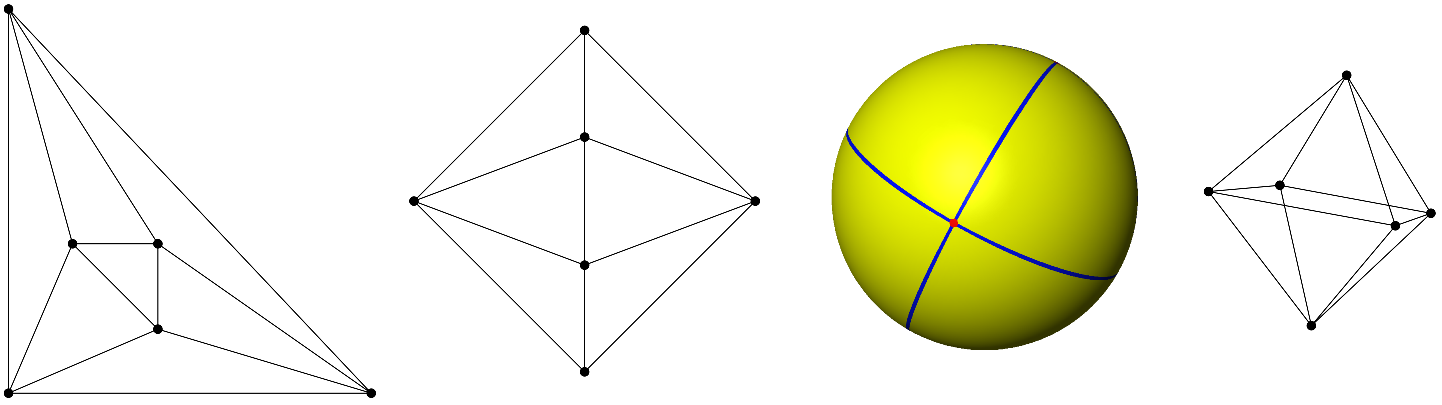

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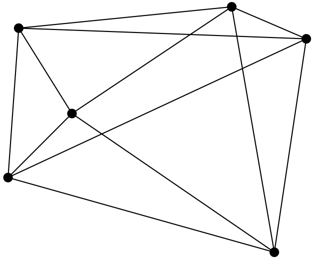

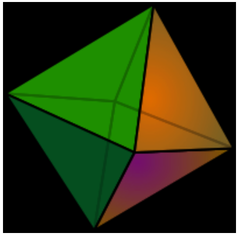

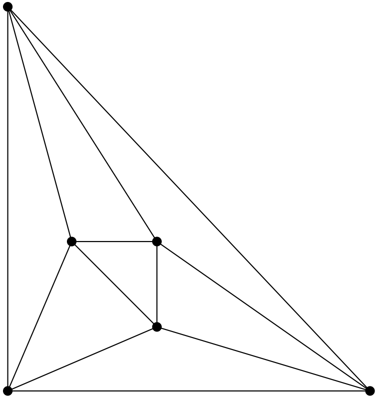

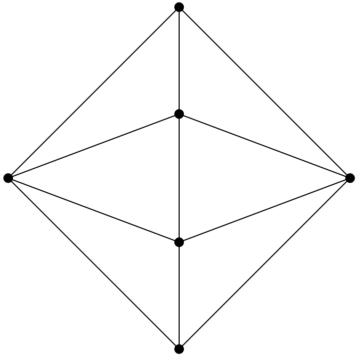

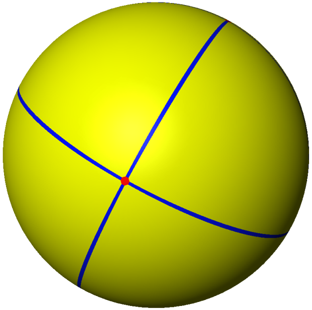

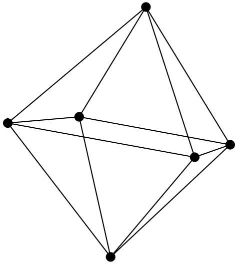

Part I

Major results in graph theory

Major results (on planar graphs) in graph theory

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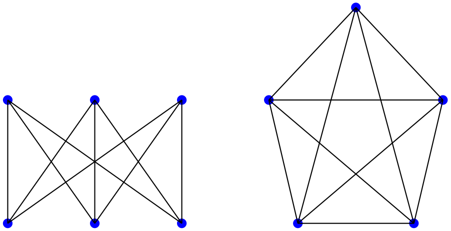
Kuratowski theorem (1930) (cfr Wagner's theorem, 1937)

- G contains neither K_5 nor $K_{3,3}$ as minors



Major results (on planar graphs) in graph theory

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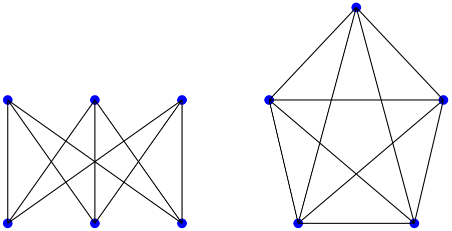
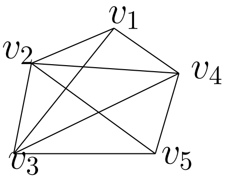
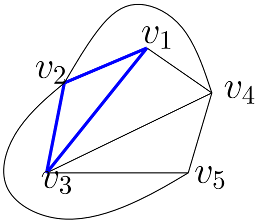


Fáry theorem (1947)

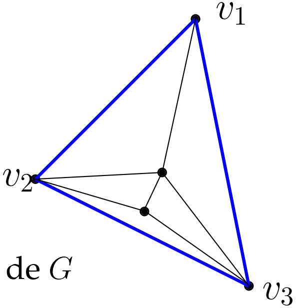
- Every (simple) planar graph admits a straight line planar embedding (no edge crossings)



dessin non planaire de G

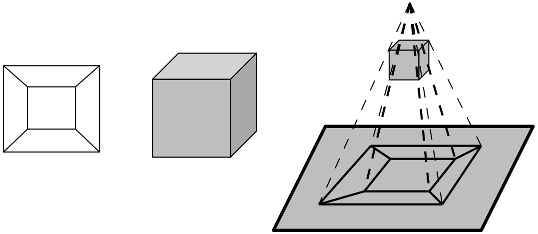


un dessin planaire de G



Major results (on planar graphs) in graph theory

Thm (Steinitz, 1916)



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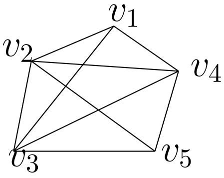

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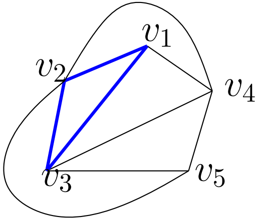

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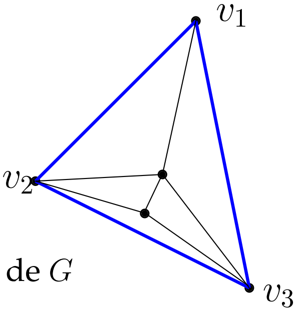
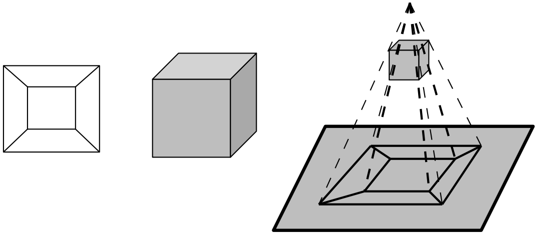


Major results (on planar graphs) in graph theory

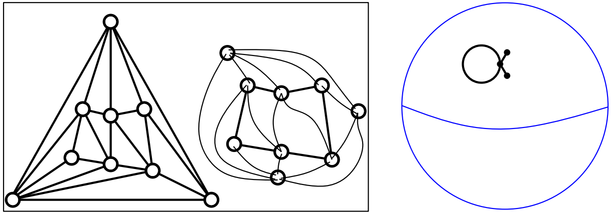
Thm (Steinitz, 1916)

3-connected planar graphs are the 1-skeletons of convex polyhedra



Thm (Whitney, 1933)

3-connected planar graphs admit a unique planar embedding (up to homeomorphism and inversion of the sphere).



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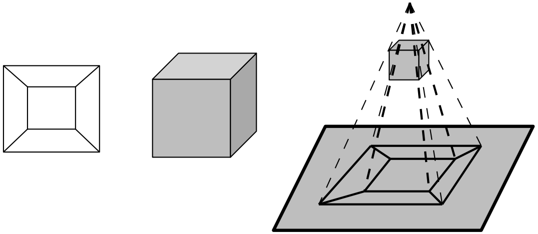
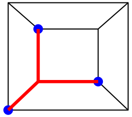


Def *G is 3-connected if*

is connected and
the removal of one or two vertices does not disconnect G



at least 3 vertices are required to disconnect the graph

Thm (Whitney, 1933)

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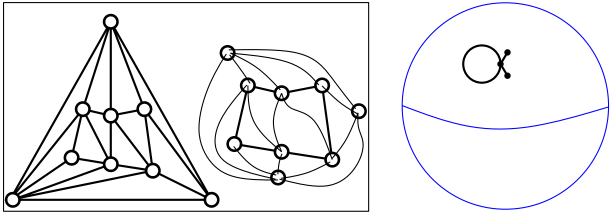
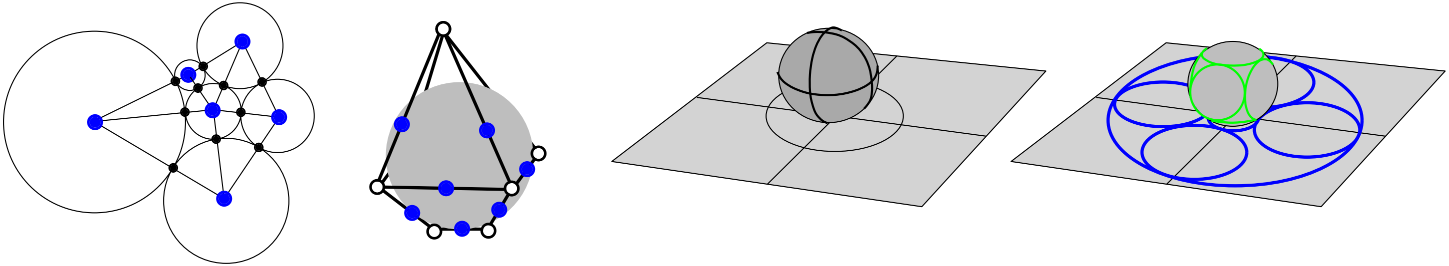


Major results (on planar graphs) in graph theory

Thm (Koebe-Andreev-Thurston)

Every planar graph with n vertices is isomorphic to the intersection graph of n disks in the plane.



Thm (Colin de Verdière, 1990)

Colin de Verdiere invariant (multiplicity of λ_2 eigenvalue of a generalized laplacian)

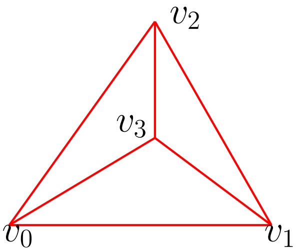
- $\mu(G) \leq 3$

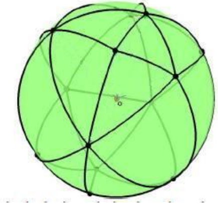
Theorem (Lovasz Schrijver '99)

Given a 3-connected planar graph G, the eigenvectors ξ_2, ξ_3, ξ_4 of a CdV matrix defines a convex polyhedron containing the origin..

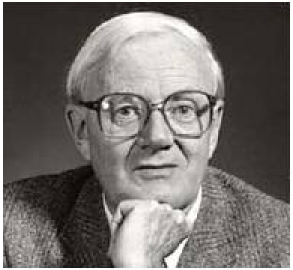
$$\lambda_1 = -4, \lambda_2 = \lambda_3 = \lambda_4 = 0$$

$$\begin{array}{c} M \\ \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \end{array} \quad \begin{array}{cccc} & \xi_x & \xi_y & \xi_z \\ \begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{array}$$





Major results (on planar graphs) in graph theory



Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a barycentric representation ρ in R^2.

$$\boxed{\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j)}$$

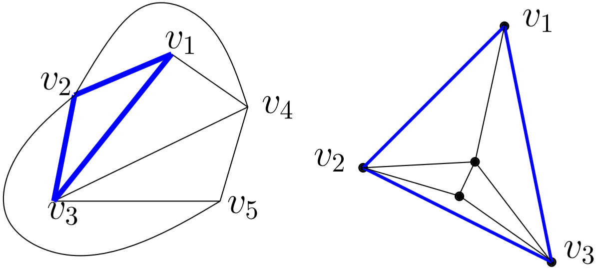
$(\sum_j w_{ij} = 1$ and $w_{ij} > 0$

$\rho : (V_G) \longrightarrow R^2$ is barycentric iff for each inner node v_i, $\rho(v_i)$ is the barycenter of the images of its neighbors

Get a straight line drawing solving a system a linear equations

$$N(v_4) = \{v_1, v_2, v_3, v_5\}$$

$$N(v_5) = \{v_2, v_3, v_4\}$$



$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

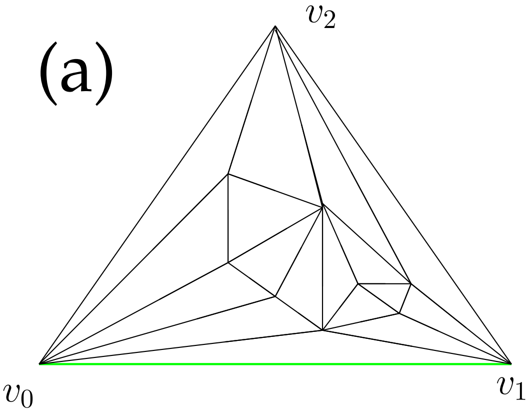
laplacian matrix

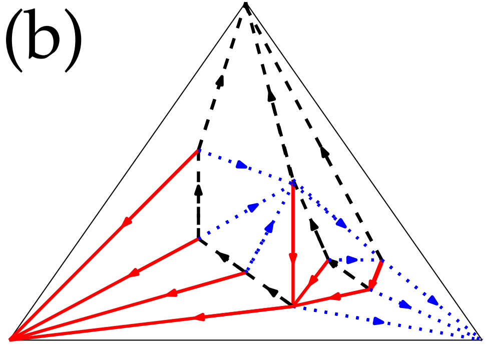
$$\begin{cases} M \cdot \underline{x} = \underline{a_x} \\ M \cdot \underline{y} = \underline{a_y} \end{cases}$$

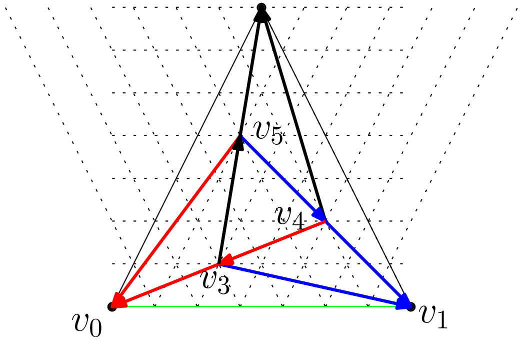
Major results (on planar graphs) in graph theory

Theorem (Schnyder '89)
A graph G is planar if and only if the dimension of its incidence poset is at most 3

Theorem (Schnyder, Soda '90)
For a triangulation $\mathcal{T}$ having n vertices, we can draw it on a grid of size $(2n-5) \times (2n-5)$, by setting $v_0 = (2n-5, 0)$, $v_1 = (0,0)$ and $v_2 = (0, 2n-5)$.



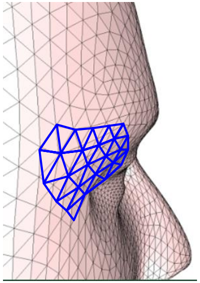




v_3	$(1,2,4)$
v_4	$(2,4,1)$
v_5	$(4,1,2)$

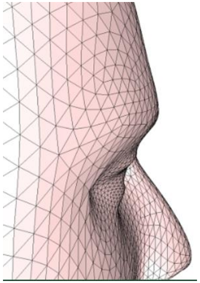
Part II

What is a surface mesh?



(a short digression on embedded graphs, simplicial complexes
and topological and combinatorial maps)

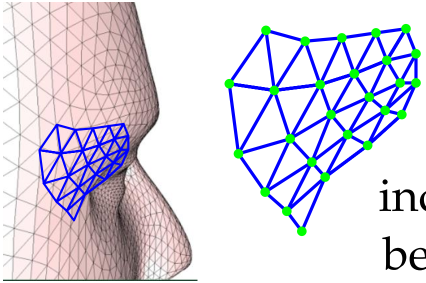
What is a (surface) mesh?



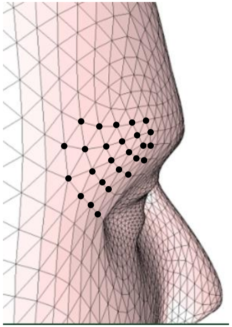
surface mesh: set of vertices, edges and faces (polygons) defining a polyhedral surface in embedded in 3D (discrete approximation of a shape)

Combinatorial structure + geometric embedding

”Connectivity”: the underlying *map*

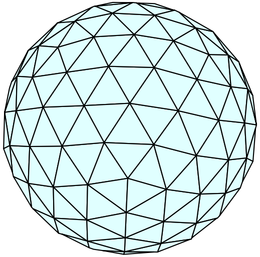


incidence relations
between triangles,
vertices and edges



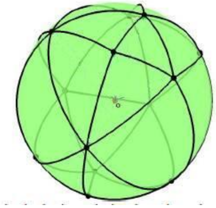
vertex
coordinates

Planar and surface meshes: definition

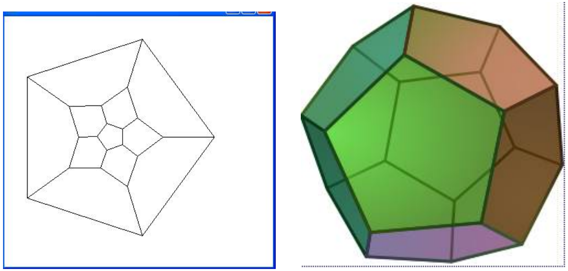


planar triangulation
embedded in R^3

triangle mesh

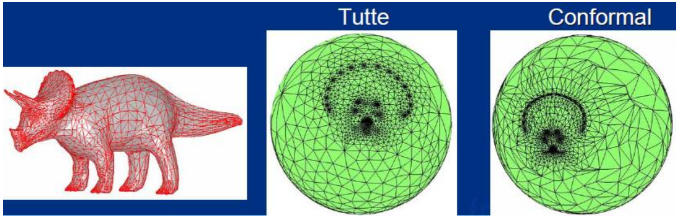


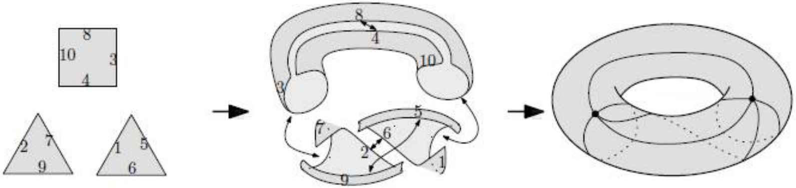
planar triangulation
spherical drawing



planar map
straight line drawing of a dodecahedron

spherical parameterizations of a
triangle mesh (Gotsman, Gu Sheffer, 2003)





toroidal map (Eric Colin de Verdière)

Surface meshes as *simplicial complexes*

abstract simplicial complex K (set of simplices)

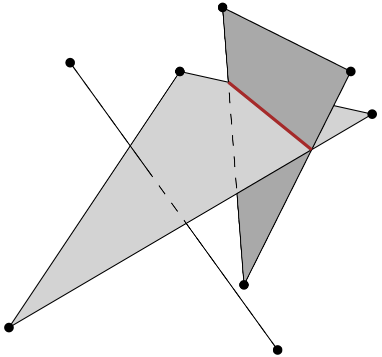
$$V = \{v_0, v_1, \ldots, v_{n-1}\}$$
$$E = \{\{i,j\}, \{k,l\}, \ldots\}$$
$$F = \{\{i,j,k\}, \{j,i,l\}, \ldots\}$$

inclusion property:

$\rho \in K$ and $\sigma \subset \rho \longrightarrow \sigma \in K$

intersection property:

given two simplices σ_1, σ_2 of K, the intersection $\sigma_1 \cup \sigma_2$ is a face of both



not valid simplicial complex

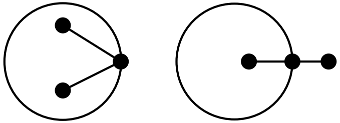
Surface meshes as *(topological) maps*

(geomettric realizations of maps)

A *graph* $G = (V, E)$ is a pair of:

- a set of *vertices* $V = (v_1, \ldots, v_n)$
- a collection of $E = (e_1, \ldots, e_m)$ elements of the cartesian product $V \times V = \{(u, v) \mid u \in V, v \in V\}$ (*edges*).

two different embeddings of the same graph



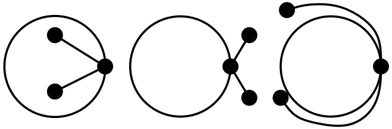
un *dessin planaire* est un plongement cellulaire de G dans R^2, qui satisfait les conditions suivantes:

(i) les sommets du graphe sont reprśentés par des points ;

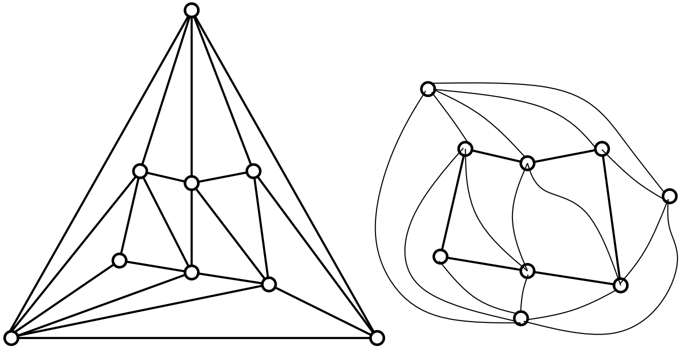
(ii) les aretes sont reprśentées par des arcs de courbes ne se coupant qu'aux sommets ;

(iii) les faces sont simplement connexes.

cellular embeddings of a graph defining the same (planar) map



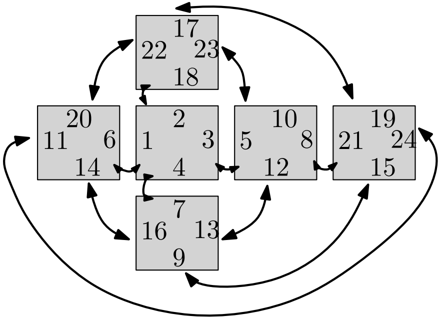
(topologica) map: cellular embedding up to homeomorphism (equivalence class)

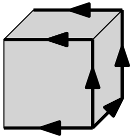


two cellular embeddings defining the same planar map

Surface meshes as *combinatorial maps*

(geomettric realizations of maps)





3 permutations on the set H of the $2n$ half-edges

(i) α involution without fixed point;

(ii) $\alpha\sigma\phi = Id$;

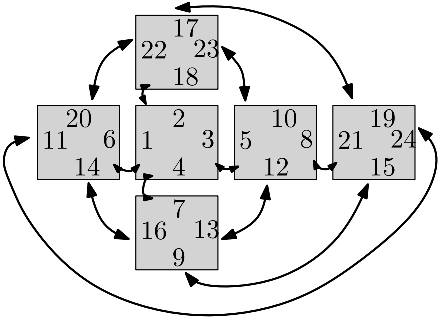
(iii) the goup generated by σ, α et ϕ transitively on H.

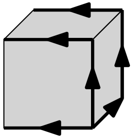
$$\phi = (1,2,3,4)(17,23,18,22)(5,10,8,12)(21,19,24,15)\ldots$$

$$\alpha = (2,18)(4,7)(12,13)(9,15)(14,16)(10,23)\ldots$$

Surface meshes as *combinatorial maps*

(geomettric realizations of maps)





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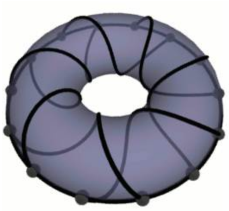
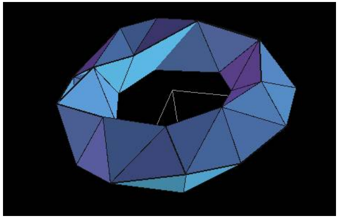
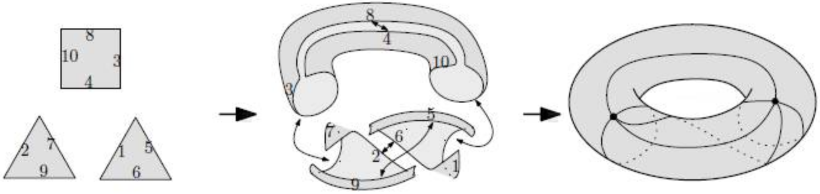
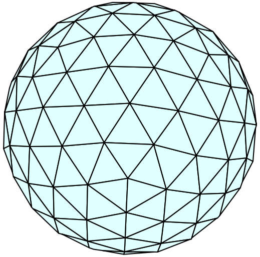




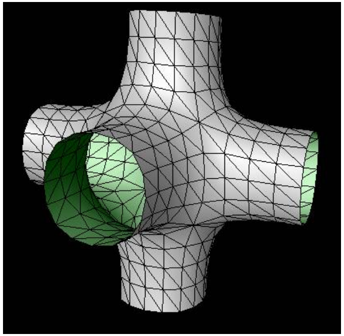


Mesh representations: classification

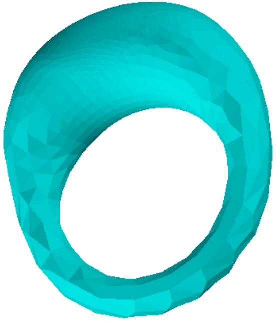
Manifold meshes



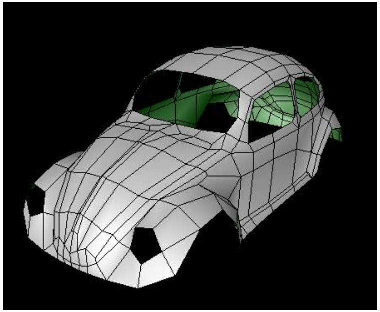
triangle meshes
no boundary



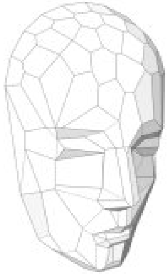
with boundaries



genus 1 mesh

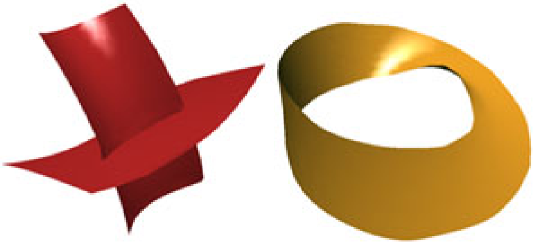


quad meshes



polygonal meshes

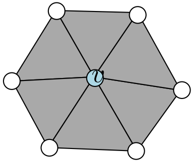
non manifold or non orientable meshes



Manifold mesh: definition

Every edge is shared by at most 2 faces

For every vertex v, the incident faces form an open or closed *fan*

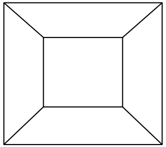


Part III

Euler formula and its consequences

Euler-Poincaré characteristic: topological invariant

$$\boxed{\chi := n - e + f}$$



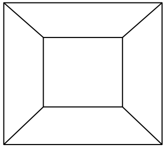
planar map

$$n - e + f = 2$$

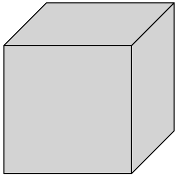
Euler's relation

Euler-Poincaré characteristic: topological invariant

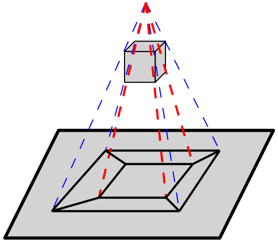
$$\boxed{\chi := n - e + f}$$



planar map



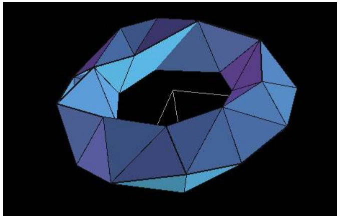
(convex) polyhedron



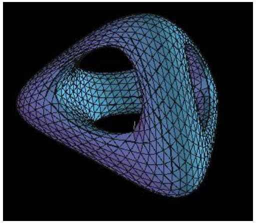
$$n - e + f = 2$$

Euler's relation

$\chi = 0$



$\chi = -4$



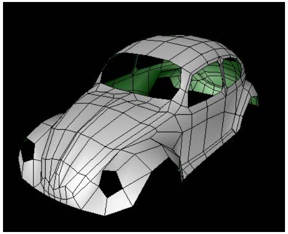
$n = 1660$

$e = 4992$

$f = 3328$

$g = 3$

$$n - e + f = 2 - 2g$$



$n = 364$

$e = 675$

$f = 302$

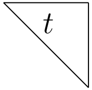
$b = 11$

$g = 0$

$$n - e + f = 2 - b$$

Euler's relation for polyhedral surfaces

$\chi := n - e + f$

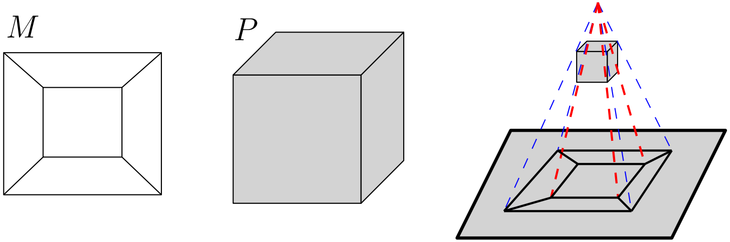


$\chi(t) = 3 - 3 + 2 = 2$

First proof: by induction

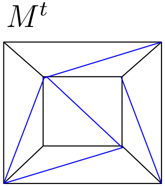
$$\boxed{n - e + f = 2}$$

Euler's relation



$\chi(M) = \chi(P) - 1$

(count exterior face)



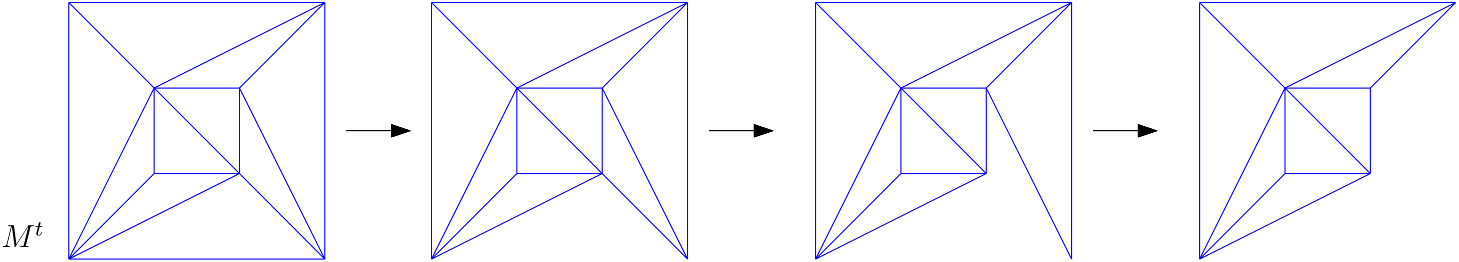
$\chi(M) = \chi(M^t)$

invariant: the boundary (exterior) is a simple cycle
perform the removal according to a shelling order

$e' = e - 1 \quad f' = f - 1$

$e'' = e' - 1 \quad f'' = f' - 1$

$e''' = e'' - 2$
$f''' = f'' - 1$
$n''' = n'' - 1$



remove a boundary edge

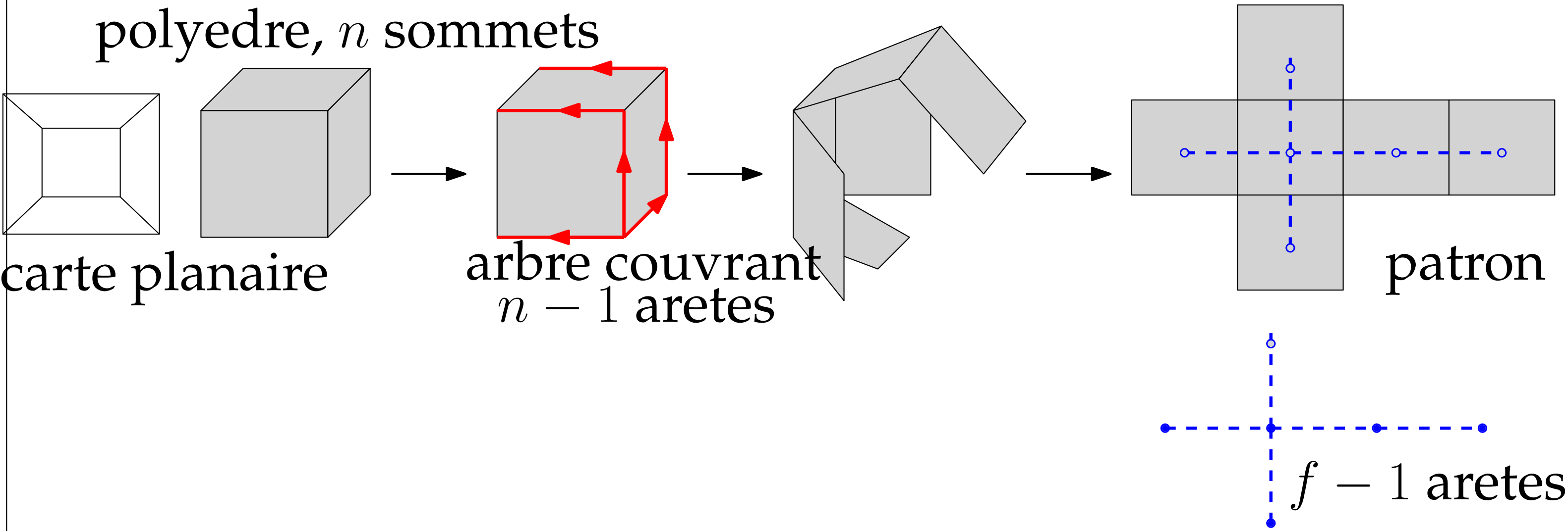
remove a boundary edge

remove a triangle

Euler’s relation for polyhedral surfaces

Overview of the proof

$$\boxed{n - e + f = 2}$$



polyedre, n sommets

carte planaire

arbre couvrant
$n-1$ aretes

patron

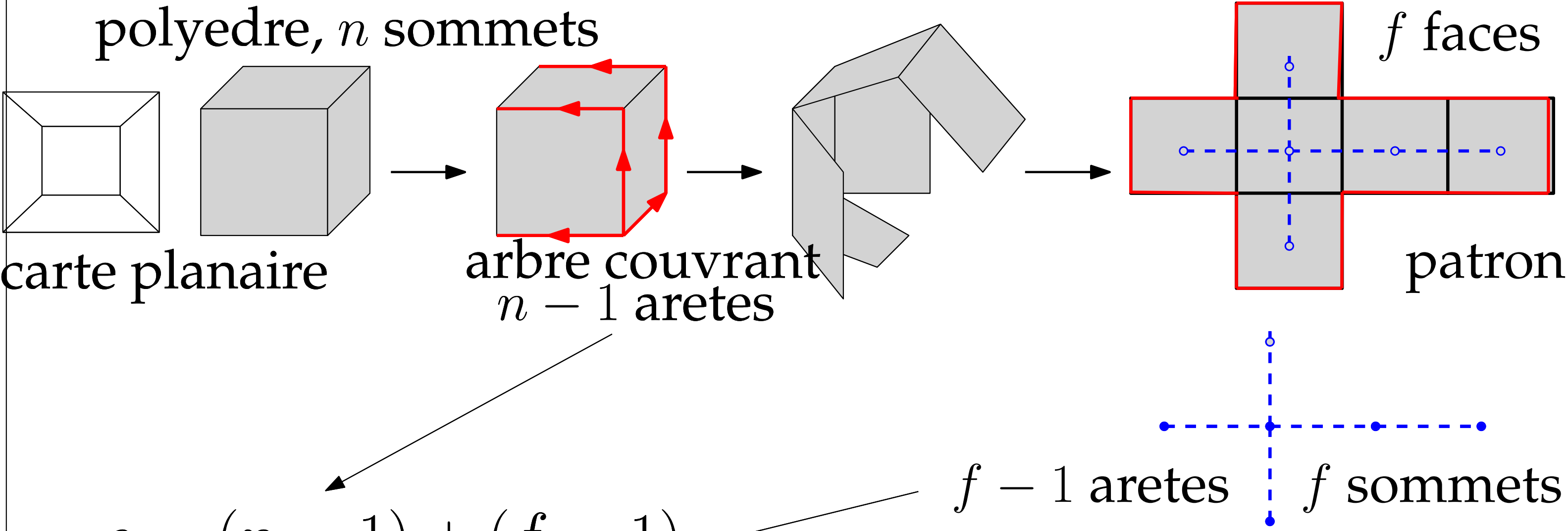
$f-1$ aretes

arbre couvrant du dual

Euler's relation for polyhedral surfaces

Overview of the proof

$$\boxed{n - e + f = 2}$$



polyedre, n sommets

carte planaire

arbre couvrant

$n-1$ aretes

f faces

patron

$f-1$ aretes

f sommets

arbre couvrant du dual

$$e = (n-1) + (f-1)$$

Euler's relation for polyhedral surfaces

Corollary: linear dependence between edges, vertices and faces

$f \leq 2n - 4$

$$\boxed{e \leq 3n - 6}$$

preuve (par double comptage)

$$f = f_1 + f_2 + f_3 + \dots$$

$$n = n_1 + n_2 + n_3 + \dots$$

toutes les faces ont degré au moins 3 ($\mathcal{G}$ est simple), on a

$$f = f_3 + f_4 + \dots$$

chaque arete apparait deux fois

$$2e = 3 \cdot f_3 + 4 \cdot f_4 + \dots$$

d'ou la relation

$$2e - 3f \geq 0$$

Euler's relation for polyhedral surfaces

Corollary: linear dependence between edges, vertices and faces

$f \leq 2n - 4$

$$\boxed{e \leq 3n - 6}$$

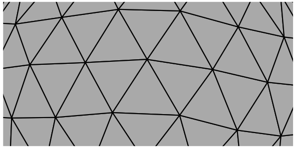
ayant prouvé $2e - 3f \geq 0$

en appliquant Euler on trouve

$$3n - 6 = 3(e - f + 2) = 3e - 3f \geq 0$$

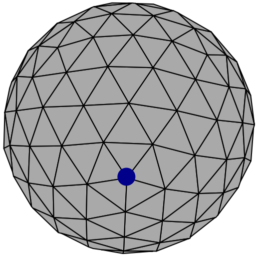
Euler's relation for polyhedral surfaces

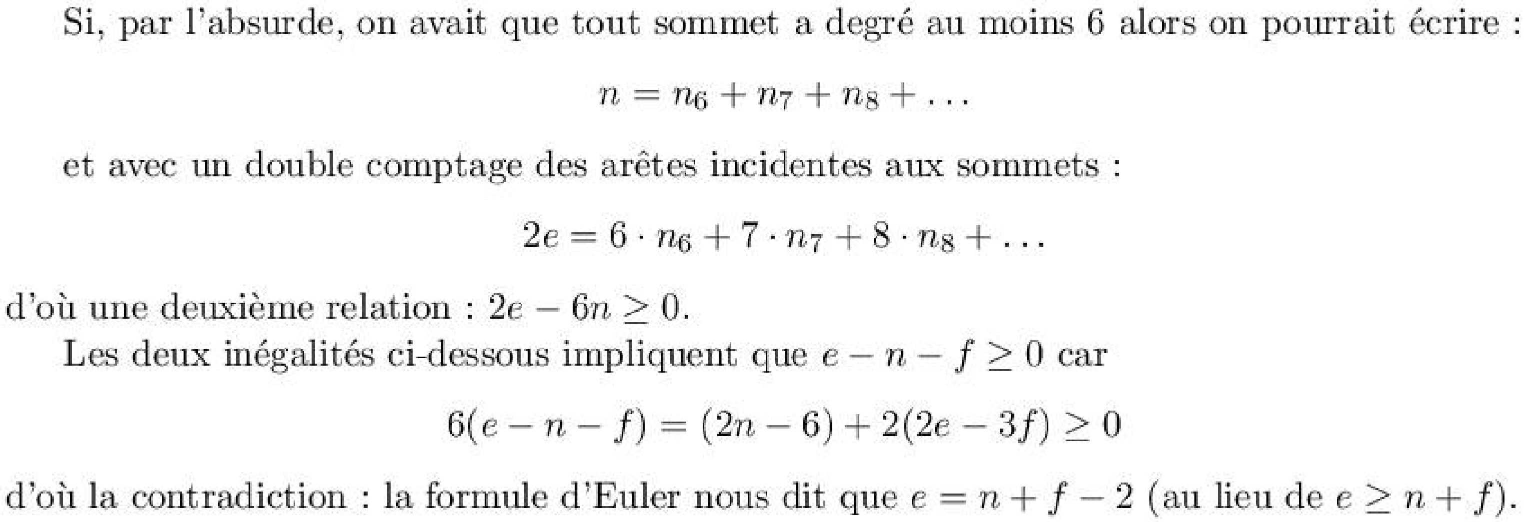
can we construct a regular mesh, where every vertex has degree 6?



Euler's relation for polyhedral surfaces

we just showed $2e - 3f \geq 0$





Si, par l'absurde, on avait que tout sommet a degré au moins 6 alors on pourrait écrire :

$$n = n_6 + n_7 + n_8 + \dots$$

et avec un double comptage des arêtes incidentes aux sommets :

$$2e = 6 \cdot n_6 + 7 \cdot n_7 + 8 \cdot n_8 + \dots$$

d'où une deuxième relation : $2e - 6n \geq 0$.

Les deux inégalités ci-dessous impliquent que $e - n - f \geq 0$ car

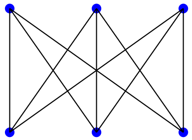
$$6(e - n - f) = (2n - 6) + 2(2e - 3f) \geq 0$$

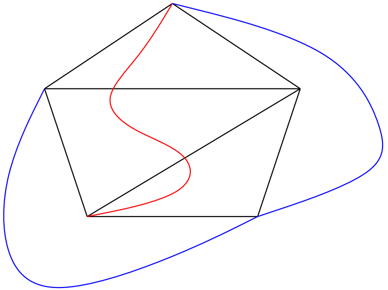
d'où la contradiction : la formule d'Euler nous dit que $e = n + f - 2$ (au lieu de $e \geq n + f$).

Major results (on planar graphs) in graph theory

Kuratowski theorem (1930) (cfr Wagner's theorem, 1937)

- G is planar iff it does not contain K_5 nor $K_{3,3}$ as minors





$K_{3,3}$ bipartite:

no cycle of length 3

$$\boxed{e \leq 2n - 4 = 8 < 9}$$

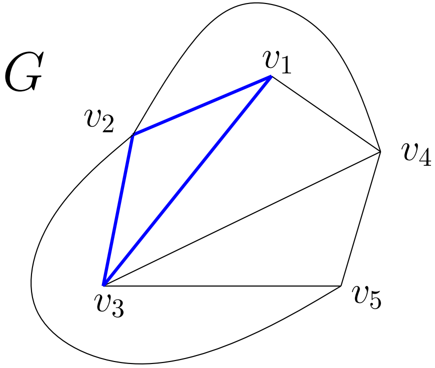
$$e \leq 3n - 6 = 9$$

but we have $e(K_5) = \binom{5}{2} = 10$

Part IV

(Some notions of) Spectral graph theory

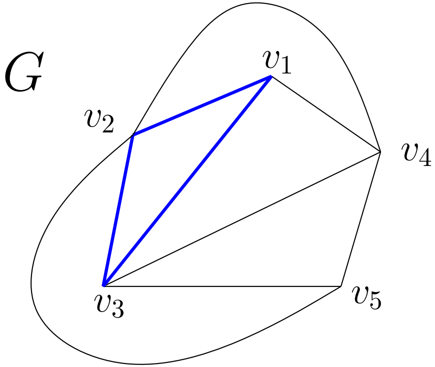
(Some notions of) Spectral graph theory



(Some notions of) Spectral graph theory

adjacency matrix

$$A_G[i,j] = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

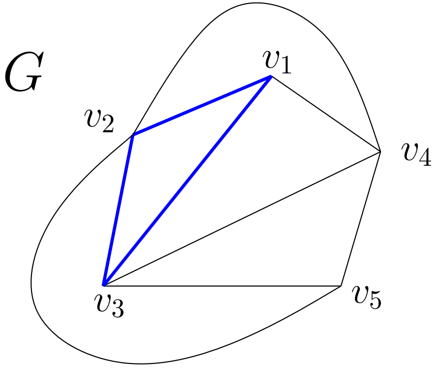


$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

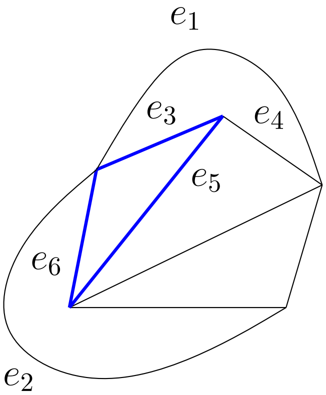
(Some notions of) Spectral graph theory

incidence matrix

$$D_G[i,k] = \begin{cases} 1 & v_i \text{ is incident to edge } e_k \\ 0 & \text{otherwise} \end{cases}$$

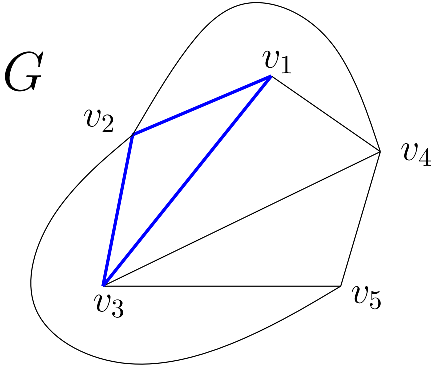


$$D_G = \begin{array}{cc} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} & \\ \left[\begin{array}{ccccccccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \cdots & \cdots & & & & & & & \\ \cdots & & & & & & & & \\ & & & & & & & & \end{array}\right] & \begin{array}{c} v_1 \\ v_2 \\ \\ \\ \\ \end{array} \end{array}$$



(Some notions of) Spectral graph theory

Laplacian matrix (simple graphs)

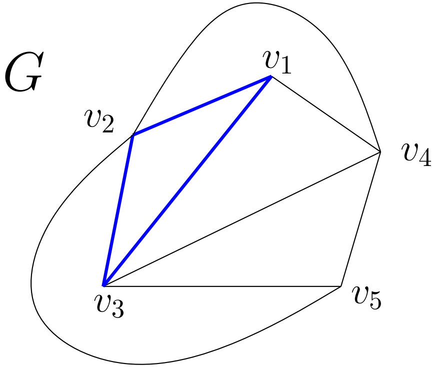


$$Q_G[i,k] = \begin{cases} deg(v_i) & \text{if } i = j \\ -A_G[i,j] & \text{otherwise} \end{cases}$$

$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

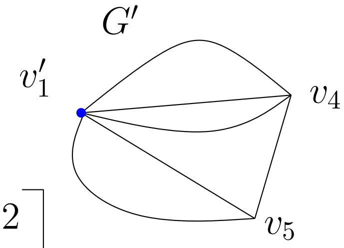
(Some notions of) Spectral graph theory

Laplacian matrix (counting multiple edges)



$$Q_G[i,k] = \begin{cases} deg(v_i) & \text{if } i = j \\ -|edges| \text{ from } v_i \text{ to } v_j & \text{otherwise} \end{cases}$$

$$Q_G[i_1, i_2, ...] = Q_G \setminus \begin{cases} \text{line } i_1, \text{ line } i_2, \cdots \\ \text{column } i_1, \text{ column } i_2, \cdots \end{cases}$$



$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

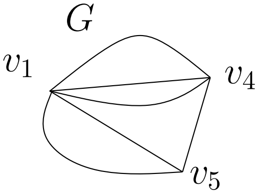
$$Q_{G'} = \begin{bmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

(Some notions of) Spectral graph theory

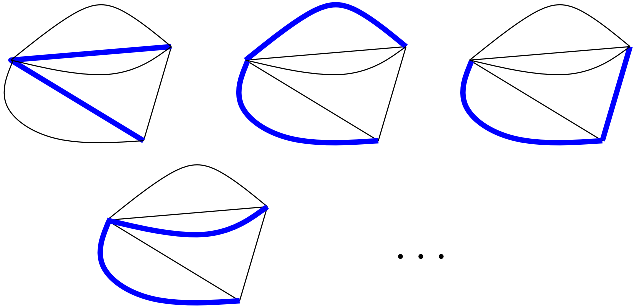
Lemma (Laplacian and the number of spanning trees)

Let Q be the laplacian of a graph G, with n vertices. Then the number of spannig trees of G is:

$$\tau(G) = det(Q[i]) \qquad (i \leq n)$$



$$Q_G \begin{bmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$



$$Q_G[1] = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} = 11$$

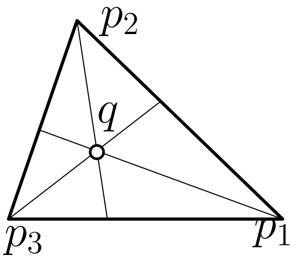
Part V

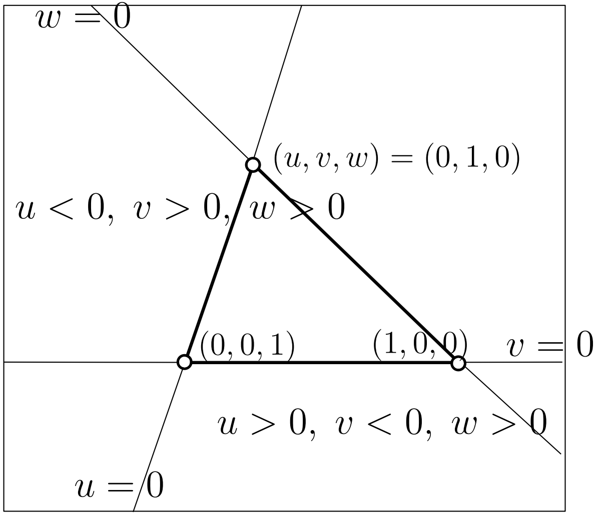
Tutte’s planar embedding

Preliminaries: barycentric coordinates

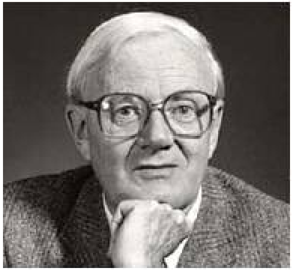
$$q = \boxed{\sum_i^n \alpha_i p_i \text{ (avec } \sum_i \alpha_i = 1)}$$

coefficients $(\alpha_1, \ldots, \alpha_n)$ are called *barycentric coordinates* of q (relative to $p_1, \ldots, p_n$)



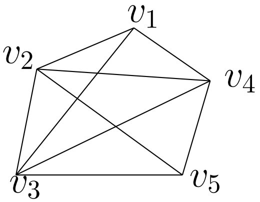


Tutte's theorem

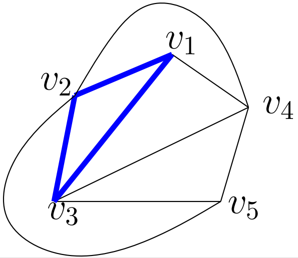


Thm (Tutte barycentric method, 1963)

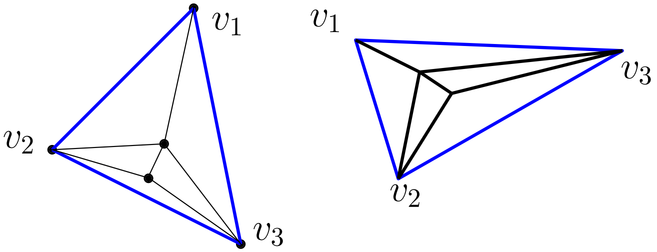
Every 3-connected planar graph G admits a convex representation ρ in R^2.



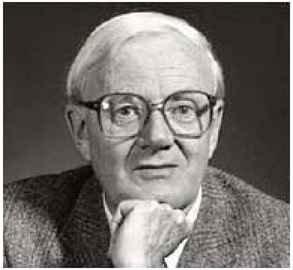
planar drawing of G



two straight-line planar drawings of G



Tutte's theorem

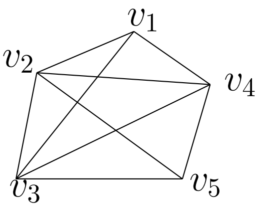


Thm (Tutte barycentric method, 1963)

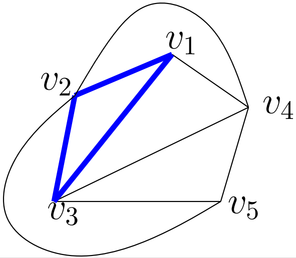
Every 3-connected planar graph G admits a convex representation ρ in R^2.

$$\rho : (V_G) \longrightarrow R^2$$

ρ est *convexe* the images of the faces of G are convex polygons



planar drawing of G



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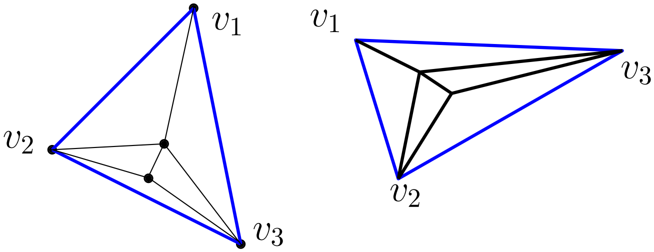
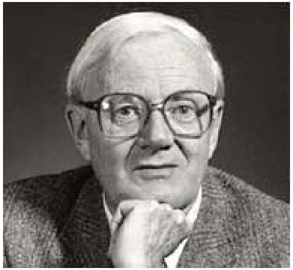


Tutte's theorem



Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a convex representation ρ in R^2.

$$\rho : (V_G) \longrightarrow R^2$$

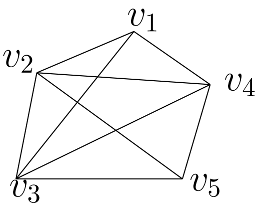
ρ is barycentric the images of interior vertices are barycenters of their neighbors

$$\boxed{\rho(v_i) = \sum_{j \in N(i)} w_{ij}\rho(v_j)}$$

where w_{ij} satisfy $\sum_j w_{ij} = 1$, and $w_{ij} > 0$

according to Tutte: $w_{ij} = \frac{1}{deg(v_i)}$

$$N(v_4) = \{v_1, v_2, v_3, v_5\}$$



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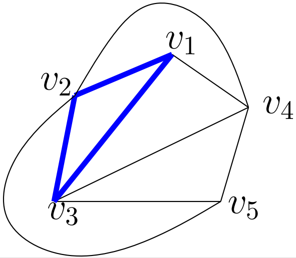

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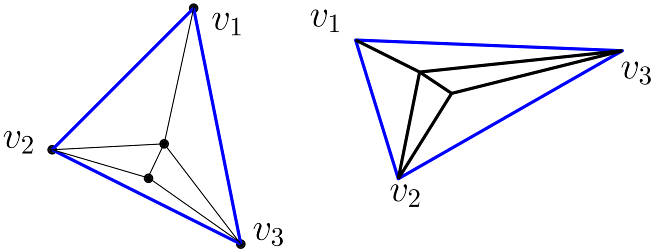
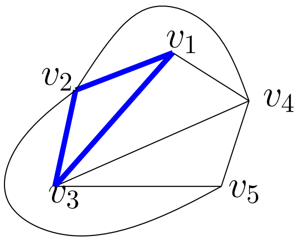


Tutte’s theorem: main steps



- find a *peripheral cycle* F (the outer face of G)

a cycle such that $G \setminus F$ is connected

(deletion of vertices and edges)

Tutte's theorem: main steps

- find a *peripheral cycle* F (the outer face of G)

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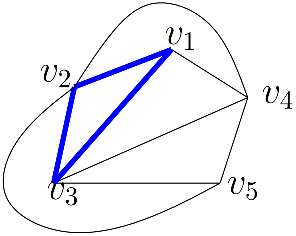
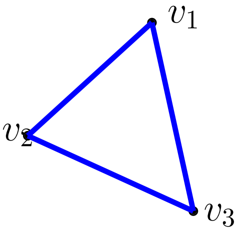


- choose a convex polygon P of size $k = |F|$

such that $\rho(F) = P$



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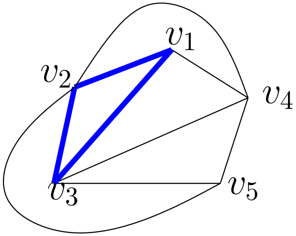

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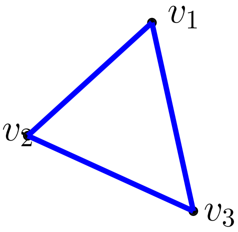

- solve equations for images of inner vertices $\rho(v_i)$:

$$\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j) \longleftrightarrow \rho(v_i) - \sum_{j \in N(i)} w_{ij} \rho(v_j) = 0$$

according to Tutte: $w_{ij} = \frac{1}{deg(v_i)}$

Tutte's theorem: main steps

- find a *peripheral cycle* F (the outer face of G)

a cycle such that $G \setminus F$ is connected

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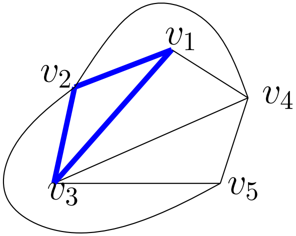

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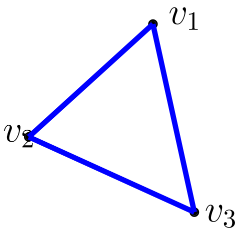
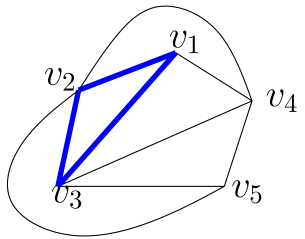


- solve a linear system:

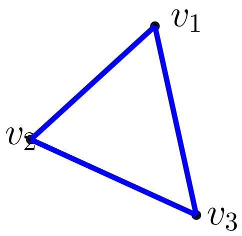
$$\begin{cases}(I-W)\cdot\underline{x}=\underline{b_x}\\ (I-W)\cdot\underline{y}=\underline{b_y}\end{cases}\Longleftrightarrow\begin{cases}\rho_x(v_i)-\sum\limits_{j\in N(i)} w_{ij}\rho_x(v_j)=0\\ \\ \rho_y(v_i)-\sum\limits_{j\in N(i)} w_{ij}\rho_y(v_j)=0\end{cases}$$

Tutte's theorem: main steps



- find a *peripheral cycle* F (the outer face of G)

a cycle such that $G \setminus F$ is connected
(deletion of vertices and edges)



- choose a convex polygon P of size $k = |F|$

such that $\rho(F) = P$

- solve a linear system:

$$N(v_4) = \{v_1, v_2, v_3, v_5\} \quad N(v_5) = \{v_2, v_3, v_4\}$$

$$\rho(v_i) := (x_i, y_i)$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_{4x} \\ b_{5x} \end{bmatrix}$$

$$\begin{matrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{matrix} \quad \begin{matrix} y_4 \\ y_5 \end{matrix} = \begin{matrix} b_{4y} \\ b_{5y} \end{matrix}$$

$$\begin{cases} \rho(v_4) - \frac{1}{4}\rho(v_5) = \frac{1}{4}\rho(v_1) + \frac{1}{4}\rho(v_2) + \frac{1}{4}\rho(v_3) \\ -\frac{1}{3}\rho(v_4) + \rho(v_5) = \frac{1}{3}\rho(v_2) + \frac{1}{3}\rho(v_3) \end{cases}$$

Validity of Tutte's theorem: main results

- show that the linear system admit a (unique) solution:

$$\begin{cases} (I - W) \cdot \underline{x} = \underline{b_x} \\ (I - W) \cdot \underline{y} = \underline{b_y} \end{cases}$$

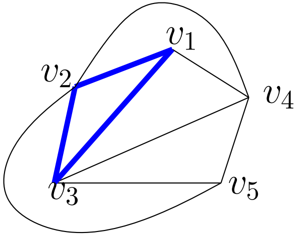
matrix $(I - W)$ is inversible

- a barycentric drawing is planar: no edge crossing

- a 3-connected planar graph G has a peripheral cycle

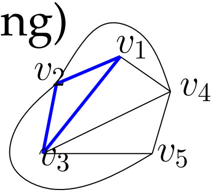
Exercice **Claim (existence of peripheral cycles)**

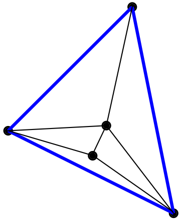
In a 3-connected planar graph peripheral cycles are exactly the faces (of the embedding)



Advantages of Tutte's drawing

- the drawing is guaranteed to be planar (no edge crossing)





- no need of the map structure

  graph structure + a peripheral cycle

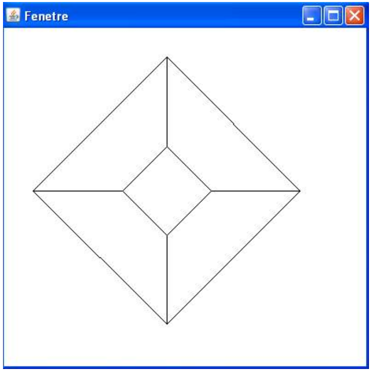
- very easy to implement: no need of sophisticated data structure or preprocessing

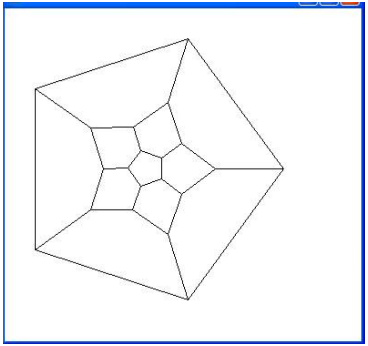
$$\begin{cases} (I - W) \cdot \underline{x} = \underline{b_x} \\ (I - W) \cdot \underline{y} = \underline{b_y} \end{cases}$$

linear systems to solves

- nice drawings

  (detection of symmetries)





Drawbacks of Tutte's drawing

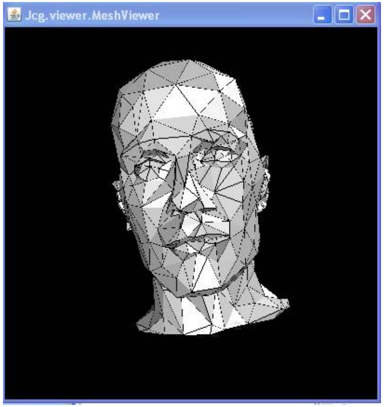
- requires to solve linear systems of equations (of size n)

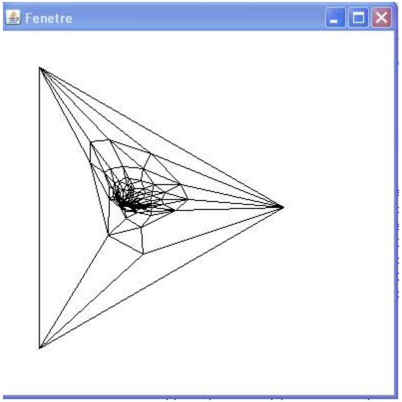
$$\begin{cases} (I - W) \cdot \underline{x} = \underline{b_x} \\ (I - W) \cdot \underline{y} = \underline{b_y} \end{cases}$$

complexity $O(n^3)$
or $O(n^{3/2})$ with methods more involved

- exponential size of the resulting vertex coordinates (with respect to n)

- drawings are not always "nice"





Tutte's spring embedder: iterative version

- choose an outer face F, and a convex polygon P
- put exterior vertices $v \in F$ on the polygon
- repeat (until convergence)

for each inner vertex $v \in V_i$ compute

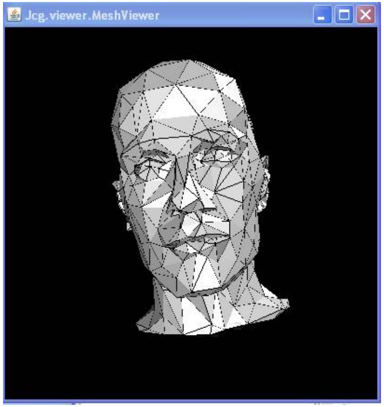
$$x_v = \frac{1}{deg(v)} \sum_{(u,v)\in E} x_u$$

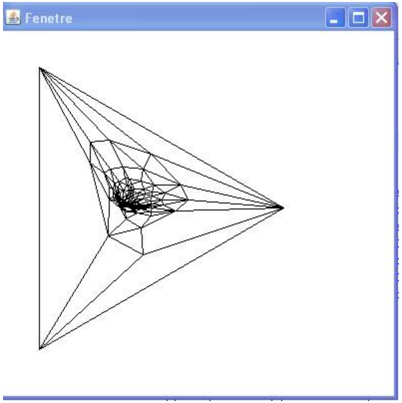
$$y_v = \frac{1}{deg(v)} \sum_{(u,v)\in E} y_u$$

$$\mathbf{F}(v) = \sum_{(u,v)\in E} (\mathbf{p}_u - \mathbf{p}_v)$$

V_i inner vertices

(u, v) edge connecting v and u





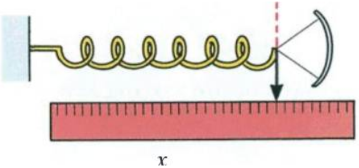
Spring drawing

- placer tous les points aléatoirement dans le plan
- repeter (jusqu'à convergence)

pour tout sommet $v \in V$ calculer

$$v = v + c_4 \cdot \mathbf{F}(v)$$

$$\text{où } \mathbf{F}(v) := F_a(v) + F_r(v)$$



force attractive (entre sommets voisins)

$$\mathbf{F}_a(v) = c_1 \cdot \sum_{(u,v)\in E} \log(dist(u,v)/c_2)$$

(u, v) arete reliant le sommet v à u

$$\mathbf{F}_r(v) = c_3 \cdot \sum_{u\in V} \frac{1}{\sqrt{dist(u,v)}}$$

force repulsive (entre tous les sommets)

$$c_1 = 2 \; c_2 = 1 \; c_3 = 1 \; c_4 = 0.01$$

Part VI

Tutte’s theorem: the proof

First: existence and uniqueness of barycentric representations

First: existence and uniqueness of barycentric representations

Theorem

Let G be a 3-connected planar graph with n vertices, and F a peripheral cycle (such that $G \setminus F$ is connected). Let P be a convex polygon, such that $\rho(F) = P$. Then the barycentric representation ρ exists (and is unique)

***Goal**: show the the systems above admit a solution (unique)*

$$\begin{cases} (I - W) \cdot \underline{x} = \underline{b_x} \\ (I - W) \cdot \underline{y} = \underline{b_y} \end{cases} \qquad \longleftrightarrow \qquad \rho(v_i) - \sum_{j \in N(i)} w_{ij} \rho(v_j) = 0$$

First: existence and uniqueness of barycentric representations

Proof

$$\begin{array}{ccc}
\rho(v_i) - \sum_{j\in N(i)} w_{ij}\rho(v_j) = 0 & & deg(v_i)\rho(v_i) - \sum_{j\in N(i)} \rho(v_j) = 0 \\
\updownarrow & \textit{the linear systems above are equivalent} & \updownarrow \\
\begin{cases} (I - W)\cdot \underline{x} = \underline{b_x} \\ (I - W)\cdot \underline{y} = \underline{b_y} \end{cases} & & \begin{cases} M\cdot \underline{x} = \underline{a_x} \\ M\cdot \underline{y} = \underline{a_y} \end{cases}
\end{array}$$

First: existence and uniqueness of barycentric representations

Proof

$$\rho(v_i) - \sum_{j\in N(i)} w_{ij}\rho(v_j) = 0 \qquad\qquad deg(v_i)\rho(v_i) - \sum_{j\in N(i)} \rho(v_j) = 0$$

the linear systems above are equivalent

$$\begin{cases} (I-W)\cdot \underline{x} = \underline{b_x} \\ (I-W)\cdot \underline{y} = \underline{b_y} \end{cases} \qquad\qquad \begin{cases} M\cdot \underline{x} = \underline{a_x} \\ M\cdot \underline{y} = \underline{a_y} \end{cases}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_{4x} \\ b_{5x} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} b_{4y} \\ b_{5y} \end{bmatrix}$$

$$M = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

First: existence and uniqueness of barycentric representations

Proof

$$\rho(v_i) - \sum_{j\in N(i)} w_{ij}\rho(v_j) = 0 \qquad\qquad deg(v_i)\rho(v_i) - \sum_{j\in N(i)} \rho(v_j) = 0$$

$$\Big\updownarrow \qquad\qquad\qquad\qquad \Big\updownarrow$$

the linear systems above are equivalent

$$\begin{cases} (I-W)\cdot\underline{x} = \underline{b_x} \\ (I-W)\cdot\underline{y} = \underline{b_y} \end{cases} \qquad\qquad \begin{cases} M\cdot\underline{x} = \underline{a_x} \\ M\cdot\underline{y} = \underline{a_y} \end{cases}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_{4x} \\ b_{5x} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} b_{4y} \\ b_{5y} \end{bmatrix}$$

$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

laplacian matrix

$$\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

$$Q_G[1,2,3] = M$$

First: existence and uniqueness of barycentric representations

Proof

$$deg(v_i)\rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0$$

$$\begin{cases} M \cdot \underline{x} = \underline{a_x} \\ M \cdot \underline{y} = \underline{a_y} \end{cases}$$

$$Q_G = \begin{bmatrix} \cancel{3} & \cancel{-1} & \cancel{-1} & \cancel{-1} & \cancel{0} \\ \cancel{-1} & \cancel{4} & \cancel{-1} & \cancel{-1} & \cancel{-1} \\ \cancel{-1} & \cancel{-1} & \cancel{4} & \cancel{-1} & \cancel{-1} \\ \cancel{-1} & \cancel{-1} & \cancel{-1} & 4 & -1 \\ \cancel{0} & \cancel{-1} & \cancel{-1} & -1 & 3 \end{bmatrix}$$

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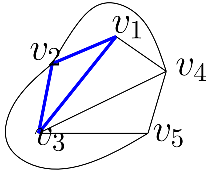
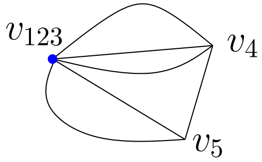
$$G \longrightarrow G/F \xrightarrow[G/F \text{ is connected}]{} det(M) = \tau(Q_{G/F}) > 0$$



$\longrightarrow$ $G/F = G/\{v_1, v_2, v_3\}$



$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix} \xrightarrow[\text{edge contraction}]{} \overset{Q_{G/F}}{\begin{bmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}}$$

$$\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

$$Q_G[1,2,3] = M$$

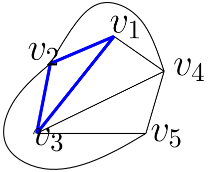
First: existence and uniqueness of barycentric representations

Proof conclusion

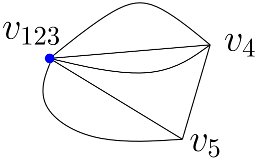
$$deg(v_i)\rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0 \qquad \begin{cases} M \cdot \underline{x} = \underline{a_x} \\ M \cdot \underline{y} = \underline{a_y} \end{cases}$$

$$G \longrightarrow G/F \xrightarrow[G/F \text{ is connected}]{} det(M) = \tau(Q_{G/F}) > 0$$

M admits inverse □



$$\longrightarrow \quad G/F = G/\{v_1, v_2, v_3\}$$



$$Q_G = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix} \xrightarrow[\text{edge contraction}]{} \overset{Q_{G/F}}{\begin{bmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}} \qquad \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

$$Q_G[1,2,3] = M$$

Second: the barycentric representation defines a planar drawing

Theorem

Let G be a 3-connected planar graph with n vertices, and F a peripheral cycle (such that $G \setminus F$ is connected). Let P be a convex polygon, such that $\rho(F) = P$.

Then the barycentric representation defines a planar drawing (no edge crossing)

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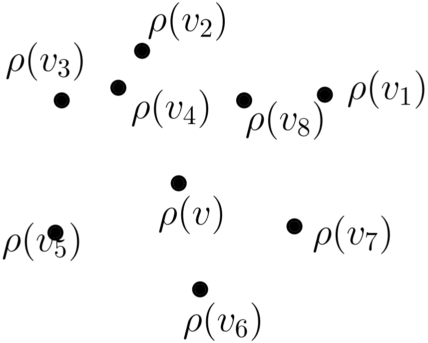
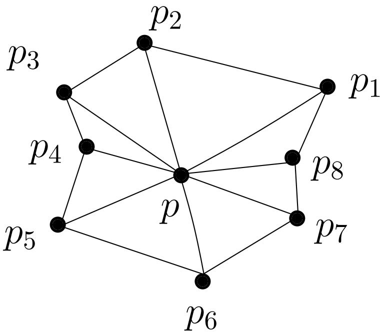


barycentric representation of G



planar drawing G

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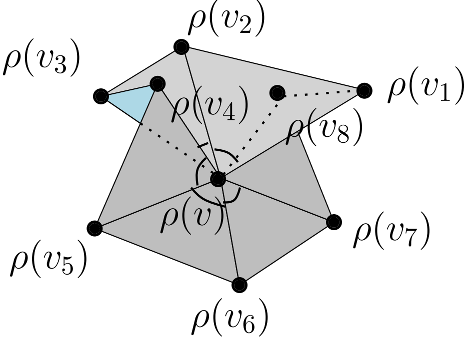
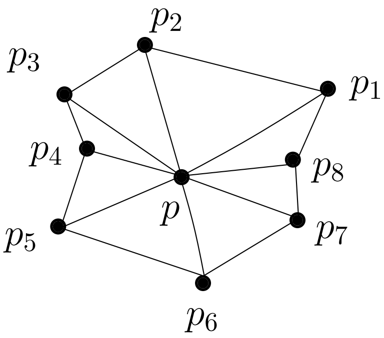


$$\sigma(v) \geq \alpha(v)$$

dessin non planaire de G

$$\boxed{\sigma(v) = \alpha(v)}$$



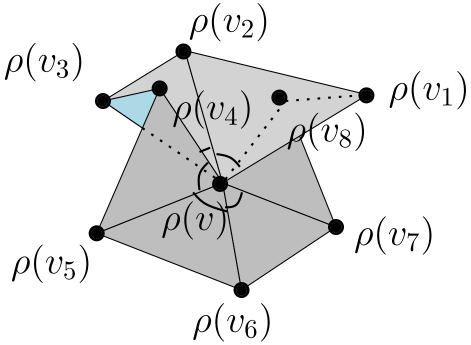
planar drawing of G

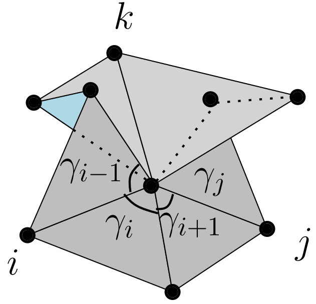
$$\alpha(v) = 2\pi$$

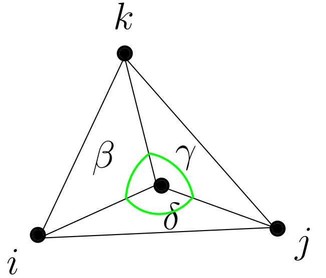
Second: the barycentric representation defines a planar drawing

Claim 1

$$\boxed{\sigma(v) \geq 2\pi = \alpha(v)}$$





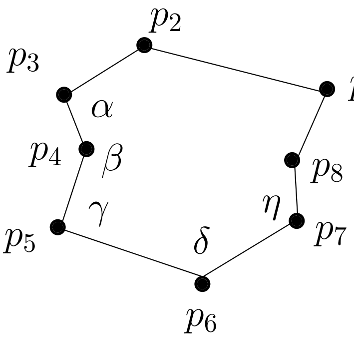


$$\sigma(v) := \sum_k \gamma_k \geq \qquad \beta + \gamma + \delta = 2\pi$$

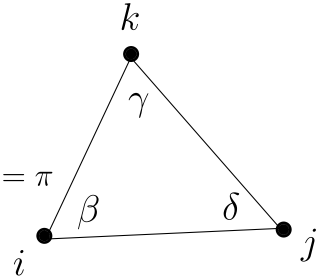
Second: the barycentric representation defines a planar drawing

Claim 2

$$\boxed{\sum_v \alpha(v) \le \sum_v \sigma(v) = \pi f}$$



$\alpha + \beta + \gamma + \delta + \ldots = (|F| - 2)\pi$



$\beta + \gamma + \delta = \pi$

$$\sum_v \alpha(v) = \sum_{v \in V \setminus F} \alpha(v) + \sum_{v \in F} \alpha(v) = 2\pi|V \setminus F| + (|F| - 2)\pi \le \boxed{\sum_v \sigma(v) = \pi f}$$

sum over inner and outer vertices

sum of the angles of triangles
(3 angles per face)

Second: the barycentric representation defines a planar drawing

Conclusion

$$\boxed{\alpha(v) = \sigma(v)}$$

Claim 2 $\sum_v \alpha(v) \leq \sum_v \sigma(v) = 2\pi$

$n - (e + |F|) + f = (|V \setminus F| + |F|) - (e + |F|) + (t + 1)$

Euler formula

Counts the number of edges $3t = 2e + |F|$

$$\left.\begin{array}{l} \sum_v \alpha(v) \leq \sum_v \sigma(v) = 2\pi \\ n - (e + |F|) + f = (|V \setminus F| + |F|) - (e + |F|) + (t + 1) \\ 3t = 2e + |F| \end{array}\right\} \longrightarrow$$

$$\sum_v \alpha(v) := 2\pi|V \setminus F| + (|F| - 2)\pi = \pi f \ = \sum_v \sigma(v)$$

Claim 1

$$\left.\begin{array}{l} \sum_v \alpha(v) = \sum_v \sigma(v) \\ 2\pi = \alpha(v) \leq \sigma(v) \end{array}\right\} \longrightarrow \quad \alpha(v) = \sigma(v) = 2\pi$$

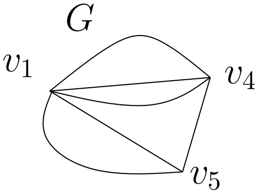
□

The missing proof

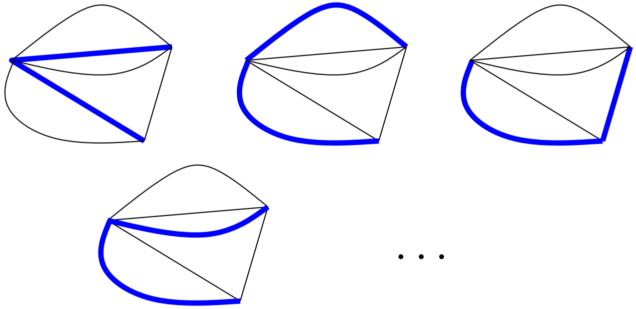
Lemma (Laplacian and the number of spanning trees)

Let Q be the laplacian of a graph G, with n vertices. Then the number of spannig trees of G is:

$$\tau(G) = det(Q[i]) \qquad (i \leq n)$$



$$Q_G \begin{bmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$



$$Q_G[1] = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} = 11$$

The missing proof

Lemma (Laplacian and the number of spanning trees)

The missing proof

Lemma (Laplacian and the number of spanning trees)

Claim 1

Pour toute arete e de G on a:

$$\tau(G) = \tau(G/e) + \tau(G \setminus e)$$

idée de la preuve

- tout arbre couvrant de G ne contenant pas e est aussi un arbre couvrant de $G \setminus e$: il y en a $\tau(G \setminus e)$
- il y a une correspondance bijective entre les arbres couvrants de G est les arbres couvrants de G/e

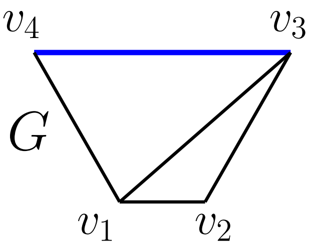
The missing proof

Lemma (Laplacian and the number of spanning trees)

Claim 2

Considérons une arete $e = (u, v)$ et la matrice E ci-dessous:

$$E \quad \begin{array}{cc} \begin{matrix} & v & \end{matrix} & \\ \begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix} & v \end{array}$$



$$Q_G \quad \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

- *on a:* $Q_G[u] = Q_{G\backslash e} + E$

$e = (v_3, v_4)$

$$Q_G[v_3] \quad \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ \cancel{-1} & \cancel{-1} & \cancel{3} & \cancel{-1} \\ -1 & 0 & -1 & 2 \end{bmatrix} \qquad \overset{Q_{G\backslash e}[v_3]}{\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ \cancel{-1} & \cancel{-1} & \cancel{\color{blue}2} & \cancel{\color{blue}0} \\ -1 & 0 & \color{blue}0 & \color{blue}1 \end{bmatrix}} \qquad \overset{E}{\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}}$$

The missing proof

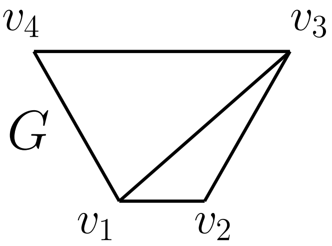
Lemma (Laplacian and the number of spanning trees)

Claim 2

observons que:

$$Q_{G\setminus e}[u,v] = Q[u,v]$$

- $$\boxed{det Q[u] = det Q_{G\setminus e}[u] + det Q_{G\setminus e}[u,v]}$$



$$Q_G \quad \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$e = (v_3, v_4)$$

- $$Q_G[v_3] \quad \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

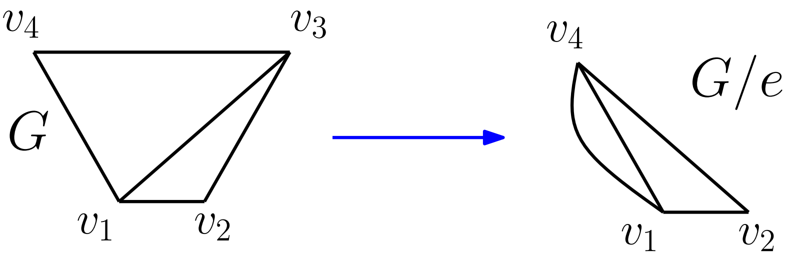
$$Q_{G\setminus e}[v_3] \quad \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$Q_{G\setminus e}[v_3, v_4] \quad \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

The missing proof

Lemma (Laplacian and the number of spanning trees)

Claim 4



$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad Q_G$$

$$\boxed{Q_{G/e}[v] = Q[u,v]}$$

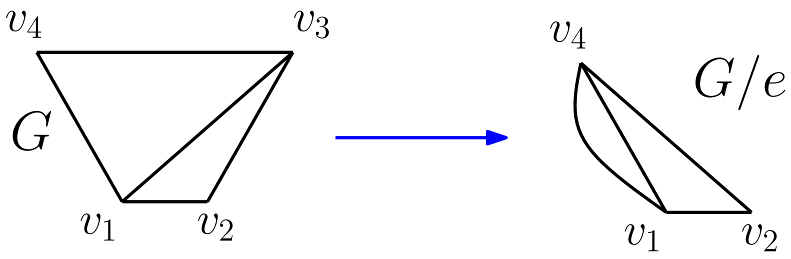
$$Q_G[v_3, v_4] \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 3 \end{bmatrix} Q_{G/e}[v_4]$$

The missing proof

Lemma (Laplacian and the number of spanning trees)

Fin: on utilise l'induction



$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad Q_G$$

$$detQ[u] = detQ_{G\setminus e}[u] + detQ_{G\setminus e}[u,v]$$

$$\big\downarrow$$

$$\boxed{detQ[u] = \underset{\tau(G\setminus e)}{detQ_{G\setminus e}[u]} + \underset{\tau(G/e)}{detQ_{G/e}[v]}} \longrightarrow \tau(G) = detQ[u]$$

□