INF562, Lecture 3: Geometric and combinatorial properties of planar graphs
mardi 22 janvier 2013

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Intro
Graph drawing: motivations and applications
Graph drawing and data visualization

Global transportation system
Graph drawing and data visualization
Rocks, railways, ...
Graph drawing and data visualization

Social network graph
Planar graphs

Design of integrated circuits (VLSI)
Planar graphs

Design of integrated circuits (VLSI)

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9 accidents en 2012 (last one, on 28th september)
Meshes and graphs in computational geometry
Delaunay triangulations, Voronoi diagrams, planar meshes, ...

Delaunay triangulation

Voronoi diagram

triangles meshes already used in early 19th century (Delambre et Mchain)

Planar mesh by L. Rineau, M. Yvinec

GIS Technology

Terrain modelling

Spherical Parameterization (Sheffer Gotsman)
Mesh parameterization in geometry processing
Mesh parameterization in geometry processing

General problem:
- Given a mesh \((T, P)\) in 3D find a bijective mapping

\[
g : P \rightarrow \mathbb{R}^2 \\
g(p_i) = u_i = (u_i, v_i)
\]
Mesh parameterization in geometry processing

Given a mesh \((T, P)\) in 3D find a bijective mapping \(g(p_i) = u_i\)
given constraints: \(g(b_j) = u_j\) for some \(\{b_j\}\)

Model: imagine a **spring** at each edge of the mesh.
If the boundary is fixed, let the interior points find an **equilibrium**.
Graph drawing: motivation

\[ A_G = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix} \]

Challenge: what kind of graph does \( A_G \) represent?
Graph drawing: motivation

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1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix} \]

Challenge: what kind of graph does \( A_G \) represent?
Part I
Major results in graph theory
Major results (on planar graphs) in graph theory
Major results (on planar graphs) in graph theory

Kuratowski theorem (1930) (cfr Wagner’s theorem, 1937)

- $G$ contains neither $K_5$ nor $K_{3,3}$ as minors
Major results (on planar graphs) in graph theory

**Kuratowski theorem (1930)** (cfr Wagner’s theorem, 1937)

- $G$ contains neither $K_5$ nor $K_{3,3}$ as minors

**Fáry theorem (1947)**

- Every (simple) planar graph admits a straight line planar embedding (no edge crossings)
Major results (on planar graphs) in graph theory

Thm (Steinitz, 1916)

Fáry theorem (1947)
- Every (simple) planar graph admits a straight line planar embedding (no edge crossings)
Major results (on planar graphs) in graph theory

**Thm (Steinitz, 1916)**
3-connected planar graphs are the 1-skeletons of convex polyhedra

**Thm (Whitney, 1933)**
3-connected planar graphs admit a unique planar embedding (up to homeomorphism and inversion of the sphere).
**Major results (on planar graphs) in graph theory**

**Thm (Steinitz, 1916)**
3-connected planar graphs are the 1-skeletons of convex polyhedra

**Def**  
$G$ is 3-connected if  

is connected and 
the removal of one or two vertices does not disconnect $G$

at least 3 vertices are required to disconnect the graph

**Thm (Whitney, 1933)**
3-connected planar graphs admit a unique planar embedding (up to homeomorphism and inversion of the sphere).
Major results (on planar graphs) in graph theory

Thm (Koebe-Andreev-Thurston)
Every planar graph with \( n \) vertices is isomorphic to the intersection graph of \( n \) disks in the plane.

\[
\begin{align*}
\mu(G) &\leq 3 \\
M &= 
\begin{bmatrix}
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
\end{bmatrix} \\
\xi_x &= 
\begin{bmatrix}
-1 \\
0 \\
0 \\
1 \\
\end{bmatrix} \\
\xi_y &= 
\begin{bmatrix}
-1 \\
1 \\
0 \\
0 \\
\end{bmatrix} \\
\xi_z &= 
\begin{bmatrix}
-1 \\
0 \\
1 \\
0 \\
\end{bmatrix}
\end{align*}
\]

\( \lambda_1 = -4, \lambda_2 = \lambda_3 = \lambda_4 = 0 \)

Thm (Colin de Verdière, 1990)
Colin de Verdière invariant (multiplicity of \( \lambda_2 \) eigenvalue of a generalized laplacian)

\[
\begin{align*}
\lambda_1 &= -4, \lambda_2 = \lambda_3 = \lambda_4 = 0
\end{align*}
\]

Theorem (Lovasz Schrijver '99)
Given a 3-connected planar graph \( G \), the eigenvectors \( \xi_2, \xi_3, \xi_4 \) of a CdV matrix defines a convex polyhedron containing the origin.
Major results (on planar graphs) in graph theory

Thm (Tutte barycentric method, 1963)
Every 3-connected planar graph $G$ admits a barycentric representation $\rho$ in $\mathbb{R}^2$.

$\rho : (V_G) \rightarrow \mathbb{R}^2$ is barycentric iff for each inner node $v_i$, $\rho(v_i)$ is the barycenter of the images of its neighbors.

$N(v_4) = \{v_1, v_2, v_3, v_5\}$
$N(v_5) = \{v_2, v_3, v_4\}$

$$\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j)$$

(\sum_j w_{ij} = 1 \text{ and } w_{ij} > 0)

Get a straight line drawing solving a system a linear equations

$$L = \begin{bmatrix}
3 & -1 & -1 & -1 & 0 \\
-1 & 4 & -1 & -1 & 1 \\
-1 & -1 & 4 & -1 & 1 \\
-1 & -1 & -1 & 3 & 3 \\
0 & -1 & -1 & -1 & 3
\end{bmatrix} \begin{cases}
M \cdot x = a_x \\
M \cdot y = a_y
\end{cases}$$

laplacian matrix
Major results (on planar graphs) in graph theory

**Theorem** (Schnyder ’89)
A graph $G$ is planar if and only if the dimension of its incidence poset is at most 3

**Theorem (Schnyder, Soda ’90)**
For a triangulation $T$ having $n$ vertices, we can draw it on a grid of size $(2n - 5) \times (2n - 5)$, by setting $v_0 = (2n - 5, 0)$, $v_1 = (0, 0)$ and $v_2 = (0, 2n - 5)$.
Part II
What is a surface mesh?
(a short digression on embedded graphs, simplicial complexes and topological and combinatorial maps)
What is a (surface) mesh?

*Surface mesh*: set of vertices, edges and faces (polygons) defining a polyhedral surface in embedded in 3D (discrete approximation of a shape)

**Combinatorial structure** + geometric embedding

"Connectivity": the underlying *map*

incidence relations between triangles, vertices and edges

vertex coordinates
Planar and surface meshes: definition

- Planar triangulation embedded in $\mathbb{R}^3$
- Spherical drawing
- Planar map
- Straight line drawing of a dodecahedron

Spherical parameterizations of a triangle mesh (Gotsman, Gu Sheffer, 2003)

Tutte
Conformal

Toroidal map (Eric Colin de Verdière)
Surface meshes as simplicial complexes

abstract simplicial complex $K$ (set of simplices)

$V = \{v_0, v_1, \ldots, v_{n-1}\}$
$E = \{\{i, j\}, \{k, l\}, \ldots\}$
$F = \{\{i, j, k\}, \{j, i, l\}, \ldots\}$

inclusion property:

$\rho \in K$ and $\sigma \subset \rho \rightarrow \sigma \in K$

intersection property:

given two simplices $\sigma_1, \sigma_2$ of $K$, the intersection $\sigma_1 \cup \sigma_2$ is a face of both
A graph $G = (V, E)$ is a pair of:

- a set of vertices $V = (v_1, \ldots, v_n)$
- a collection of $E = (e_1, \ldots, e_m)$ elements of the cartesian product $V \times V = \{(u, v) \mid u \in V, v \in V\}$ (edges).

A surface mesh is a geometric realization of a map.

Un dessin planaire est un plongement cellulaire de $G$ dans $\mathbb{R}^2$, qui satisfait les conditions suivantes:

(i) les sommets du graphe sont représentés par des points ;
(ii) les arêtes sont représentées par des arcs de courbes ne se coupant qu’aux sommets ;
(iii) les faces sont simplement connexes.

A (topological) map is a cellular embedding up to homeomorphism (equivalence class).
Surface meshes as *combinatorial maps* (geometric realizations of maps)

3 permutations on the set $H$ of the $2n$ half-edges

(i) $\alpha$ involution without fixed point;

(ii) $\alpha \sigma \phi = \text{Id}$;

(iii) the group generated by $\sigma$, $\alpha$ et $\phi$ transitively on $H$.

$\phi = (1, 2, 3, 4)(17, 23, 18, 22)(5, 10, 8, 12)(21, 19, 24, 15) \ldots$

$\alpha = (2, 18)(4, 7)(12, 13)(9, 15)(14, 16)(10, 23) \ldots$
Surface meshes as *combinatorial maps*  
(geometric realizations of maps)

3 permutations on the set $H$ of the $2n$ half-edges

(i) $\alpha$ involution without fixed point;

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$\phi = (1, 2, 3, 4)(17, 23, 18, 22)(5, 10, 8, 12)(21, 19, 24, 15) \ldots$

$\alpha = (2, 18)(4, 7)(12, 13)(9, 15)(14, 16)(10, 23) \ldots$
Mesh representations: classification

**Manifold meshes**
- triangle meshes
  - no boundary
- quad meshes
- polygonal meshes
- with boundaries
- genus 1 mesh

**Non manifold or non orientable meshes**

**Manifold mesh: definition**
- Every edge is shared by at most 2 faces
- For every vertex $v$, the incident faces form an open or closed fan
Part III
Euler formula and its consequences
Euler-Poincaré characteristic: topological invariant

\[ \chi := n - e + f \]

planar map

\[ n - e + f = 2 \]

Euler’s relation
Euler-Poincaré characteristic: topological invariant

\[ \chi := n - e + f \]

planar map \hspace{1cm} (convex) polyhedron

\[ n - e + f = 2 \]
Euler’s relation

\[ \chi = 0 \]
\[ \chi = -4 \]

\[ n = 1660 \]
\[ e = 4992 \]
\[ f = 3328 \]
\[ g = 3 \]

\[ n = 364 \]
\[ e = 675 \]
\[ f = 302 \]
\[ b = 11 \]
\[ g = 0 \]
Euler’s relation for polyhedral surfaces

\[ \chi(M) = \chi(P) - 1 \]
(count exterior face)

First proof: by induction

\[ \chi(t) = 3 - 3 + 2 = 2 \]

\[ n - e + f = 2 \]

Euler’s relation

\[ \chi := n - e + f \]

\[ M \]

\[ P \]

\[ M' \]

\[ M'' \]

\[ M''' \]

\[ e' = e - 1 \quad f' = f - 1 \quad e'' = e' - 1 \quad f'' = f' - 1 \]

removing a boundary edge

removing a boundary edge

removing a triangle

invariant: the boundary (exterior) is a simple cycle
perform the removal according to a shelling order

\[ e''' = e'' - 2 \]

\[ f''' = f'' - 1 \]

\[ n''' = n'' - 1 \]
Euler’s relation for polyhedral surfaces

Overview of the proof

\[ n - e + f = 2 \]
Euler’s relation for polyhedral surfaces

Overview of the proof

\[ n - e + f = 2 \]

polyedre, \( n \) sommets

carte planaire

arbre couvrant, \( n - 1 \) aretes

\[ e = (n - 1) + (f - 1) \]

arbre couvrant du dual

\[ f - 1 \text{ aretes} \]

\[ f \text{ sommets} \]

\[ f \text{ faces} \]

patron
Euler’s relation for polyhedral surfaces
Corollary: linear dependence between edges, vertices and faces
\[ e \leq 3n - 6 \]

\[ f \leq 2n - 4 \]

\[ f = f_1 + f_2 + f_3 + \ldots \]
\[ n = n_1 + n_2 + n_3 + \ldots \]

\begin{enumerate}
\item toutes les faces ont degré au moins 3 (\( G \) est simple), on a
\[ f = f_3 + f_4 + \ldots \]
\item chaque arête apparaît deux fois
\[ 2e = 3 \cdot f_3 + 4 \cdot f_4 + \ldots \]
\item d’où la relation
\[ 2e - 3f \geq 0 \]
\end{enumerate}
Euler’s relation for polyhedral surfaces

Corollary: linear dependence between edges, vertices and faces

\[ e \leq 3n - 6 \]

having proved \( 2e - 3f \geq 0 \)

by applying Euler we find

\[ 3n - 6 = 3(e - f + 2) = 3e - 3f \geq 0 \]
Euler’s relation for polyhedral surfaces

can we construct a regular mesh, where every vertex has degree 6?
Euler’s relation for polyhedral surfaces
we just showed $2e - 3f \geq 0$

Si, par l’absurde, on avait que tout sommet a degré au moins 6 alors on pourrait écrire :

$$n = n_6 + n_7 + n_8 + \ldots$$

et avec un double comptage des arêtes incidentes aux sommets :

$$2e = 6 \cdot n_6 + 7 \cdot n_7 + 8 \cdot n_8 + \ldots$$

d’où une deuxième relation : $2e - 6n \geq 0$.

Les deux inégalités ci-dessous impliquent que $e - n - f \geq 0$ car

$$6(e - n - f) = (2n - 6) + 2(2e - 3f) \geq 0$$

d’où la contradiction : la formule d’Euler nous dit que $e = n + f - 2$ (au lieu de $e \geq n + f$).
Major results (on planar graphs) in graph theory

Kuratowski theorem (1930) (cfr Wagner’s theorem, 1937)

- $G$ is planar iff it does not contain $K_5$ nor $K_{3,3}$ as minors

$K_{3,3}$ bipartite:
- no cycle of length 3
- $e \leq 2n - 4 = 8 < 9$

but we have $e(K_5) = \binom{5}{2} = 10$
Part IV
(Some notions of) Spectral graph theory
(Some notions of) Spectral graph theory

\[ G \]
(Some notions of) Spectral graph theory

adjacency matrix

\[ A_G[i, j] = \begin{cases} 
1 & \text{if } v_i \text{ is adjacent to } v_j \\
0 & \text{otherwise} 
\end{cases} \]

\[
A_G = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 
\end{bmatrix}
\]
(Some notions of) Spectral graph theory

incidence matrix

\[ D_G[i, k] = \begin{cases} 
1 & \text{if } v_i \text{ is incident to edge } e_k \\
0 & \text{otherwise}
\end{cases} \]

\[
D_G = \begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

\[ G \]

\[ v_1 \]

\[ v_2 \]

\[ v_3 \]

\[ v_4 \]

\[ v_5 \]

\[ e_1 \]

\[ e_2 \]

\[ e_3 \]

\[ e_4 \]

\[ e_5 \]

\[ e_6 \]
(Some notions of) Spectral graph theory

Laplacian matrix (simple graphs)

\[ Q_G[i, k] = \begin{cases} 
\deg(v_i) & \text{if } i = j \\
-A_G[i, j] & \text{otherwise}
\end{cases} \]

\[ Q_G = \begin{bmatrix}
3 & -1 & -1 & -1 & 0 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
0 & -1 & -1 & -1 & 3
\end{bmatrix} \]
(Some notions of) Spectral graph theory

Laplacian matrix (counting multiple edges)

\[
Q_G[i, k] = \begin{cases} 
\deg(v_i) & \text{if } i = j \\
-|\text{edges}| \text{ from } v_i \text{ to } v_j & \text{otherwise}
\end{cases}
\]

\[
Q_G[i_1, i_2, \ldots] = Q_G \setminus \begin{cases} 
\text{line } i_1, \text{ line } i_2, \ldots \\
\text{column } i_1, \text{ column } i_2, \ldots
\end{cases}
\]

\[
Q_G = \begin{bmatrix}
3 & -1 & -1 & -1 & 0 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
0 & -1 & -1 & -1 & 3
\end{bmatrix}
\]

\[
Q_{G'} = \begin{bmatrix}
5 & -3 & -2 \\
-3 & 4 & -1 \\
-2 & -1 & 3
\end{bmatrix}
\]
(Some notions of) Spectral graph theory

Lemma (Laplacian and the number of spanning trees)

Let $Q$ be the laplacian of a graph $G$, with $n$ vertices. Then the number of spanning trees of $G$ is:

$$
\tau(G) = \det(Q[i]) \quad (i \leq n)
$$

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$$
\tau(G) = \det(Q[i]) \quad (i \leq n)
$$

Lemma (Laplacian and the number of spanning trees)
Part V

Tutte’s planar embedding
Preliminaries: barycentric coordinates

\[ q = \sum_{i}^{n} \alpha_i p_i \ (\text{avec } \sum_{i}^{n} \alpha_i = 1) \]

coefficients \((\alpha_1, \ldots, \alpha_n)\) are called **barycentric coordinates** of \(q\) (relative to \(p_1, \ldots, p_n\))
Tutte’s theorem

Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph $G$ admits a convex representation $\rho$ in $\mathbb{R}^2$. 

planar drawing of $G$

two straight-line planar drawings of $G$
Thm (Tutte barycentric method, 1963)
Every 3-connected planar graph $G$ admits a convex representation $\rho$ in $R^2$.

\[ \rho : (V_G) \longrightarrow R^2 \]

$\rho$ est convexe  the images of the faces of $G$ are convex polygons
**Tutte’s theorem**

Thm (Tutte barycentric method, 1963)

*Every 3-connected planar graph* \( G \) *admits a convex representation* \( \rho \) *in* \( \mathbb{R}^2 \).

\[
\rho : (V_G) \longrightarrow \mathbb{R}^2
\]

\( \rho \) is barycentric: the images of interior vertices are barycenters of their neighbors

\[
\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j)
\]

where \( w_{ij} \) satisfy \( \sum_j w_{ij} = 1 \), and \( w_{ij} > 0 \)

According to Tutte: \( w_{ij} = \frac{1}{\deg(v_i)} \)

\( N(v_4) = \{v_1, v_2, v_3, v_5\} \)

planar drawing of \( G \)

two straight-line planar drawings of \( G \)
Tutte’s theorem: main steps

- find a *peripheral cycle* $F$ (the outer face of $G$)

  a cycle such that $G \setminus F$ is connected
  (deletion of vertices and edges)
Tutte’s theorem: main steps

- find a peripheral cycle $F$ (the outer face of $G$)
  a cycle such that $G \setminus F$ is connected
  (deletion of vertices and edges)

- choose a convex polygon $P$ of size $k = |F|
  such that $\rho(F) = P$
Tutte’s theorem: main steps

- find a peripheral cycle $F$ (the outer face of $G$)
  - a cycle such that $G \setminus F$ is connected
    (deletion of vertices and edges)

- choose a convex polygon $P$ of size $k = |F|$
  - such that $\rho(F) = P$

- solve equations for images of inner vertices $\rho(v_i)$:

$$
\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j) \quad \rightarrow \quad \rho(v_i) - \sum_{j \in N(i)} w_{ij} \rho(v_j) = 0
$$

according to Tutte: $w_{ij} = \frac{1}{\deg(v_i)}$
Tutte’s theorem: main steps

- find a *peripheral cycle* $F$ (the outer face of $G$)
  a cycle such that $G \setminus F$ is connected
  (deletion of vertices and edges)

- choose a convex polygon $P$ of size $k = |F|
  such that $\rho(F) = P$

- solve a linear system:
\[
\begin{align*}
(I - W) \cdot x &= b_x \\
(I - W) \cdot y &= b_y
\end{align*}
\]
\[
\begin{align*}
\rho_x(v_i) - \sum_{j \in N(i)} w_{ij} \rho_x(v_j) &= 0 \\
\rho_y(v_i) - \sum_{j \in N(i)} w_{ij} \rho_y(v_j) &= 0
\end{align*}
\]
Tutte’s theorem: main steps

- find a peripheral cycle $F$ (the outer face of $G$)
  a cycle such that $G \setminus F$ is connected
  (deletion of vertices and edges)

- choose a convex polygon $P$ of size $k = |F|
  such that $\rho(F) = P$

- solve a linear system:

$$
\begin{bmatrix}
1 & -\frac{1}{4} \\
-\frac{1}{3} & 1 \\
1 & -\frac{1}{4} \\
-\frac{1}{3} & 1
\end{bmatrix}
\begin{bmatrix}
x_4 \\
x_5 \\
y_4 \\
y_5
\end{bmatrix} =
\begin{bmatrix}
b_{4x} \\
b_{5x} \\
b_{4y} \\
b_{5y}
\end{bmatrix}
$$

$$
\rho(v_i) := (x_i, y_i)
$$

N($v_4$) = \{v_1, v_2, v_3, v_5\}  N($v_5$) = \{v_2, v_3, v_4\}

$$
\begin{align*}
\rho(v_4) - \frac{1}{4}\rho(v_5) &= \frac{1}{4}\rho(v_1) + \frac{1}{4}\rho(v_2) + \frac{1}{4}\rho(v_3) \\
-\frac{1}{3}\rho(v_4) + \rho(v_5) &= \frac{1}{3}\rho(v_2) + \frac{1}{3}\rho(v_3)
\end{align*}
$$
Validity of Tutte’s theorem: main results

• show that the linear system admit a (unique) solution:
\[\begin{align*}
(I - W) \cdot x &= b_x \\
(I - W) \cdot y &= b_y
\end{align*}\]

matrix \((I - W)\) is inversible

• a barycentric drawing is planar: no edge crossing

• a 3-connected planar graph \(G\) has a peripheral cycle

Exercice  Claim (existence of peripheral cycles)

In a 3-connected planar graph peripheral cycles are exactly the faces (of the embedding)
Advantages of Tutte’s drawing

- the drawing is guaranteed to be planar (no edge crossing)
- no need of the map structure
- graph structure + a peripheral cycle
- very easy to implement: no need of sophisticated data structure or preprocessing

\[
\begin{align*}
(I - W) \cdot x &= b_x \\
(I - W) \cdot y &= b_y
\end{align*}
\]

- nice drawings
  (detection of symmetries)
Drawbacks of Tutte’s drawing

- requires to solve linear systems of equations (of size $n$)
  \[
  \begin{align*}
  (I - W) \cdot x &= b_x \\
  (I - W) \cdot y &= b_y
  \end{align*}
  \]
  complexity $O(n^3)$ or $O(n^{3/2})$ with methods more involved

- exponential size of the resulting vertex coordinates (with respect to $n$)

- drawings are not always "nice"
Tutte’s spring embedder: iterative version

- choose an outer face $F$, and a convex polygon $P$
- put exterior vertices $v \in F$ on the polygon
- repeat (until convergence)
  - for each inner vertex $v \in V_i$ compute
    \[
    x_v = \frac{1}{\deg(v)} \sum_{(u,v) \in E} x_u \\
    y_v = \frac{1}{\deg(v)} \sum_{(u,v) \in E} y_u
    \]

\[
F(v) = \sum_{(u,v) \in E} (p_u - p_v)
\]

$V_i$ inner vertices
$(u, v)$ edge connecting $v$ and $u$
Spring drawing

• placer tous les points aléatoirement dans le plan
• repeter (jusqu’à convergence)

pour tout sommet \( v \in V \) calculer

\[
v = v + c_4 \cdot F(v)
\]

où \( F(v) := F_a(v) + F_r(v) \)

\[
F_a(v) = c_1 \cdot \sum_{(u,v) \in E} \log(d_{st}(u, v)/c_2)
\]

\((u, v)\) arete reliant le sommet \( v \) à \( u \)

\[
F_r(v) = c_3 \cdot \sum_{u \in V} \frac{1}{\sqrt{d_{st}(u, v)}}
\]

force repulsive (entre tous les sommets)

\( c_1 = 2 \ c_2 = 1 \ c_3 = 1 \ c_4 = 0.01 \)
Part VI

Tutte’s theorem: the proof
First: existence and uniqueness of barycentric representations
First: existence and uniqueness of barycentric representations

**Theorem**
Let $G$ be a 3-connected planar graph with $n$ vertices, and $F$ a peripheral cycle (such that $G \setminus F$ is connected). Let $P$ be a convex polygon, such that $\rho(F) = P$. Then the barycentric representation $\rho$ exists (and is unique)

**Goal:** show the the systems above admit a solution (unique)

\[
\begin{align*}
(I - W) \cdot x &= b_x \\
(I - W) \cdot y &= b_y
\end{align*}
\Rightarrow \rho(v_i) - \sum_{j \in N(i)} w_{ij} \rho(v_j) = 0
\]
First: existence and uniqueness of barycentric representations

**Proof**

\[
\rho(v_i) - \sum_{j \in N(i)} w_{ij} \rho(v_j) = 0 \quad \text{deg}(v_i) \rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0
\]

\[
(I - W) \cdot x = b_x
\]

\[
(I - W) \cdot y = b_y
\]

\[
\begin{align*}
M \cdot y &= a_y \\
M \cdot x &= a_x
\end{align*}
\]

the linear systems above are equivalent
First: existence and uniqueness of barycentric representations

**Proof**

\[
\rho(v_i) - \sum_{j \in N(i)} w_{ij} \rho(v_j) = 0, \quad \text{deg}(v_i) \rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0
\]

\[
(I - W) \cdot x = b_x \\
(I - W) \cdot y = b_y
\]

The linear systems above are equivalent

\[
\begin{bmatrix}
1 & -\frac{1}{4} \\
-\frac{1}{3} & 1 \\
1 & -\frac{1}{4} \\
-\frac{1}{3} & 1
\end{bmatrix}
\begin{bmatrix}
x_4 \\
x_5 \\
y_4 \\
y_5
\end{bmatrix}
= 
\begin{bmatrix}
b_{4x} \\
b_{5x} \\
b_{4y} \\
b_{5y}
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
4 & -1 \\
-1 & 3
\end{bmatrix}
\]

\[
M \cdot x = a_x \\
M \cdot y = a_y
\]
First: existence and uniqueness of barycentric representations

**Proof**

\[
\rho(v_i) - \sum_{j \in N(i)} w_{ij} \rho(v_j) = 0 \quad \text{deg}(v_i) \rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0
\]

the linear systems above are equivalent

\[
\begin{align*}
(I - W) \cdot x &= b_x \\
(I - W) \cdot y &= b_y
\end{align*}
\]

\[
\begin{align*}
M \cdot x &= a_x \\
M \cdot y &= a_y
\end{align*}
\]

\[
Q_G = \begin{bmatrix}
3 & -1 & -1 & -1 & 0 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
0 & -1 & -1 & -1 & 3
\end{bmatrix}
\]

laplacian matrix

\[
Q_G[1, 2, 3] = M
\]
First: existence and uniqueness of barycentric representations

Proof

\[ \text{deg}(v_i) \rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0 \]

\[
\begin{align*}
Q_G &= \begin{bmatrix}
3 & -1 & -1 & -1 & 0 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
0 & -1 & -1 & -1 & 3
\end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix}
4 & -1 \\
-1 & 3
\end{bmatrix}
\]

\[Q_G[1,2,3] = M\]
First: existence and uniqueness of barycentric representations

Proof

\[ \text{deg}(v_i) \rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0 \]

\[
\begin{align*}
\begin{bmatrix}
3 & 1 & 1 & 1 & 0 \\
1 & 4 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & 4 & -1 \\
0 & 1 & 1 & -1 & 3
\end{bmatrix} & \quad Q_G = \begin{bmatrix}
5 & -3 & -2 \\
3 & 4 & -1 \\
-2 & -1 & 3
\end{bmatrix} \\
Q_G[1, 2, 3] & = M
\end{align*}
\]

\[ G \xrightarrow[g/f \text{is connected}]{} G/F \]

\[ \det(M) = \tau(Q_{G/F}) > 0 \]
First: existence and uniqueness of barycentric representations

**Proof**

$$deg(v_i)\rho(v_i) - \sum_{j \in N(i)} \rho(v_j) = 0$$

\[
\begin{align*}
G & \quad \rightarrow \quad G/F \\
& \quad \text{G/F is connected} \\
\det(M) = \tau(Q_{G/F}) > 0 \\
M & \quad \text{admits inverse} \\
\end{align*}
\]

\[
\begin{bmatrix}
3 & -1 & -1 & -1 & 0 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
0 & -1 & -1 & -1 & 3 \\
\end{bmatrix}
\]

edge contraction

\[
\begin{bmatrix}
5 & -3 & -2 \\
-3 & 4 & -1 \\
-2 & -1 & 3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
4 & -1 \\
-1 & 3 \\
\end{bmatrix}
\]

\[
Q_G[1, 2, 3] = M
\]
Second: the barycentric representation defines a planar drawing

**Theorem**

Let $G$ be a 3-connected planar graph with $n$ vertices, and $F$ a peripheral cycle (such that $G \setminus F$ is connected). Let $P$ be a convex polygon, such that $\rho(F) = P$.

Then the barycentric representation defines a planar drawing (no edge crossing)
Second: the barycentric representation defines a planar drawing

**Theorem**
Let $G$ be a 3-connected planar graph with $n$ vertices, and $F$ a peripheral cycle (such that $G \setminus F$ is connected). Let $P$ be a convex polygon, such that $\rho(F) = P$.
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![Diagram showing the barycentric representation of $G$ and the planar drawing $G$]
Second: the barycentric representation defines a planar drawing

**Theorem**

Let $G$ be a 3-connected planar graph with $n$ vertices, and $F$ a peripheral cycle (such that $G \setminus F$ is connected). Let $P$ be a convex polygon, such that $\rho(F) = P$.

Then the barycentric representation defines a planar drawing (no edge crossing)

$$\sigma(v) \geq \alpha(v)$$

dessin non planaire de $G$

$$\sigma(v) = \alpha(v)$$

planar drawing of $G$

$$\alpha(v) = 2\pi$$
Second: the barycentric representation defines a planar drawing

Claim 1

\[ \sigma(v) \geq 2\pi = \alpha(v) \]

\[ \sigma(v) := \sum_k \gamma_k \geq \beta + \gamma + \delta = 2\pi \]
Second: the barycentric representation defines a planar drawing

Claim 2

\[ \sum_v \alpha(v) \leq \sum_v \sigma(v) = \pi f \]

\[ \alpha + \beta + \gamma + \delta + \ldots = (|F| - 2)\pi \]

\[ \sum_v \alpha(v) = \sum_{v \in V \setminus F} \alpha(v) + \sum_{v \in F} \alpha(v) = 2\pi |V \setminus F| + (|F| - 2)\pi \leq \sum_v \sigma(v) = \pi f \]

sum over inner and outer vertices

sum of the angles of triangles
(3 angles per face)
Second: the barycentric representation defines a planar drawing

**Conclusion**

\[ \alpha(v) = \sigma(v) \]

**Claim 2**

\[ \sum_v \alpha(v) \leq \sum_v \sigma(v) = 2\pi \]

\[ n - (e + |F|) + f = (|V \setminus F| + |F|) - (e + |F|) + (t + 1) \]

**Euler formula**

Counts the number of edges

\[ 3t = 2e + |F| \]

\[ \sum_v \alpha(v) := 2\pi|V \setminus F| + (|F| - 2)\pi = \pi f = \sum_v \sigma(v) \]

**Claim 1**

\[ 2\pi = \alpha(v) \leq \sigma(v) \]

\[ \alpha(v) = \sigma(v) = 2\pi \]
The missing proof

Lemma (Laplacian and the number of spanning trees)

Let $Q$ be the laplacian of a graph $G$, with $n$ vertices. Then the number of spanning trees of $G$ is:

$$\tau(G) = \text{det}(Q[i]) \quad (i \leq n)$$

Let $Q$ be the laplacian of a graph $G$, with $n$ vertices. Then the number of spanning trees of $G$ is:

$$\tau(G) = \text{det}(Q[i]) \quad (i \leq n)$$

$$Q_G = \begin{bmatrix}
5 & -3 & -2 \\
-3 & 4 & -1 \\
-2 & -1 & 3
\end{bmatrix}$$

$$Q_G[1] = \begin{bmatrix}
4 & -1 \\
-1 & 3
\end{bmatrix} = 11$$
The missing proof

Lemma (Laplacian and the number of spanning trees)
The missing proof
Lemma (Laplacian and the number of spanning trees)

Claim 1
Pour toute arete e de G on a:
\[ \tau(G) = \tau(G/e) + \tau(G \setminus e) \]

idée de la preuve

- tout arbre couvrant de G ne contenant pas e est aussi un arbre couvrant de G \ e: il y en a \( \tau(G \setminus e) \)

- il y a une correspondance bijective entre les arbres couvrants de G est les arbres couvrants de G/e
The missing proof

Lemma (Laplacian and the number of spanning trees)

Claim 2

Considérons une arete $e = (u, v)$ et la matrice $E$ ci-dessous:

$$E = \begin{bmatrix} 0 & v \\ 0 & 1 & 0 \end{bmatrix}$$

- on a: $Q_G[u] = Q_{G\setminus e} + E$

$$Q_G[v_3] = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$Q_{G\setminus e}[v_3] = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
The missing proof

Lemma (Laplacian and the number of spanning trees)

Claim 2

\[ Q_{G \setminus e}[u, v] = Q[u, v] \]

\[ \text{observons que:} \]
\[ Q_{G \setminus e}[u, v] = Q[u, v] \]

\[ \text{det} Q[u] = \text{det} Q_{G \setminus e}[u] + \text{det} Q_{G \setminus e}[u, v] \]

\[ Q_G \]

\[ e = (v_3, v_4) \]

\[ Q_{G \setminus e}[v_3] \]

\[ Q_{G \setminus e}[v_3, v_4] \]
The missing proof

Lemma (Laplacian and the number of spanning trees)

Claim 4

\[ Q_{G/e}[v] = Q[u, v] \]

\[
\begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 3 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}
\quad Q_G
\]

\[
\begin{bmatrix}
3 & -1 & -2 \\
-1 & 2 & -1 \\
-2 & -1 & 3
\end{bmatrix}
\quad Q_{G/e}[v_4]
\]

\[
\begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 3 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}
\quad Q_G[v_3, v_4]
\]

\[
\begin{bmatrix}
3 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 3 \\
-1 & 0 & -1
\end{bmatrix}
\quad Q_G[e]
\]
The missing proof

Lemma (Laplacian and the number of spanning trees)

Fin: on utilise l’induction

\[ \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{vmatrix} \]

\[ Q_G \]

\[ \text{det} Q[u] = \text{det} Q_{G \setminus e}[u] + \text{det} Q_{G \setminus e}[u, v] \]

\[ \text{det} Q[u] = \text{det} Q_{G \setminus e}[u] + \text{det} Q_{G/e}[v] \]

\[ \tau(G \setminus e) \quad \tau(G/e) \]

\[ \tau(G) = \text{det} Q[u] \]

\[ \blacksquare \]