Mode Seeking

Input: \(I = \{\rho_1, \ldots, \rho_m\} \subset \mathbb{R}^d\)

Hyp: the \(\rho_i\) are sampled i.i.d. according to some unknown probability distribution \(\pi\) with (unknown) density \(f\) w.r.t. the Lebesgue measure.

- \(f\) is regular enough, typically a Morse function: twice differentiable, finitely many critical points, non-degenerate (Hessian matrix is non-singular), all distinct critical values.

Note: the gradient vector field \(x \rightarrow \nabla f(x)\) is Lipschitz continuous \(\Rightarrow\) it can be integrated into a gradient flow \(\Phi: \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) whose trajectories are solutions of the ODE

\[
\begin{align*}
\dot{x}(t) &= \nabla f \circ \Phi(t) \\
\Phi(0) &= x
\end{align*}
\]

(Comm. Lipschitz Thm.)

Thus: if \(f\) is Morse, then almost every point of \(\mathbb{R}^d\) ends up at a maximum of \(f\) when following the gradient flow (integration of gradient vector field) of \(f\).

\[\text{principle: cluster (almost) } \mathbb{R}^d \text{ by the ascending regions of the peaks of } f.\]

\(\text{in practice: simulation by hill-climbing.}\)