Degree-0 persistence (a.k.a. Size Theory)

<u>Input</u>: $f : X \to \mathbb{R}$

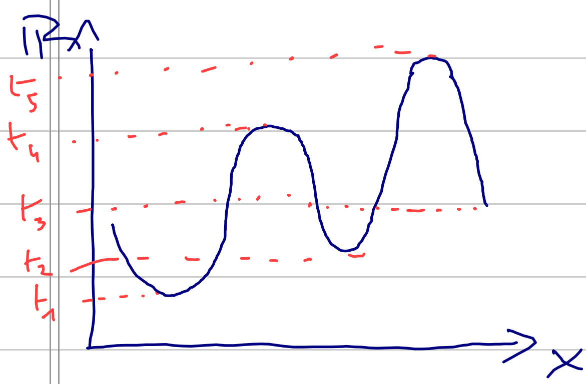
<u>Idea</u>: look at (path-) connected components of <u>excursion sets</u> of f:

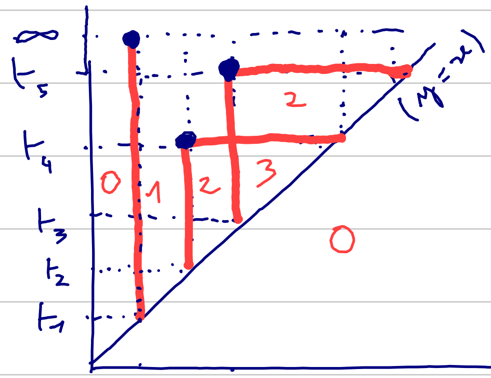
- sublevel sets: $f^{-1}((-\infty, t])$
- superlevel sets: $f^{-1}([t, +\infty))$

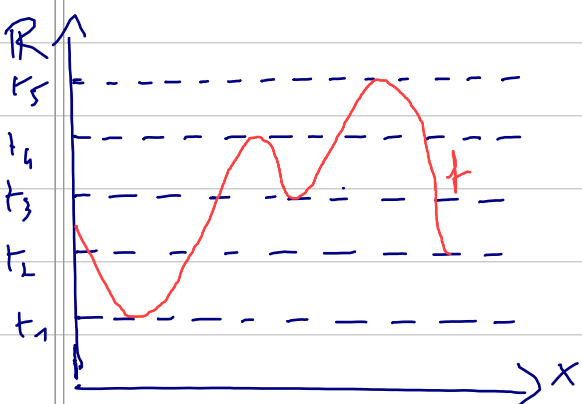
Hyp: f is <u>tame</u>, i.e. every excursion set has finitely many c.c.s.

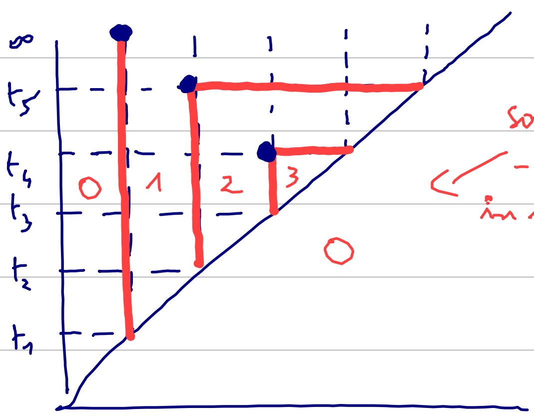
Def: (size function) $\ell : \mathbb{R}^2 \to \mathbb{N}$

$(s,t) \mapsto \#\{$c.c.s of $f^{-1}((-\infty, s])$ that are still independent in $f^{-1}((-\infty, t])\}$.







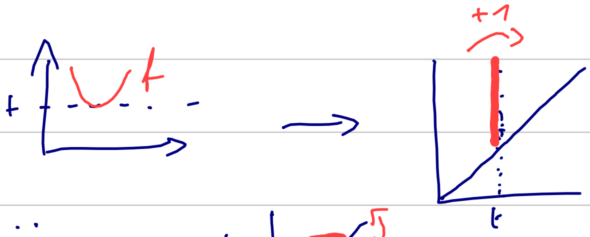


same example as in the slides

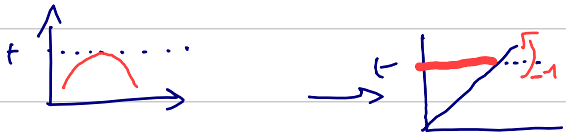
Prop: β is piecewise constant, with jumps across horizontal and vertical lines.

→ proof idea:

- local minimum:



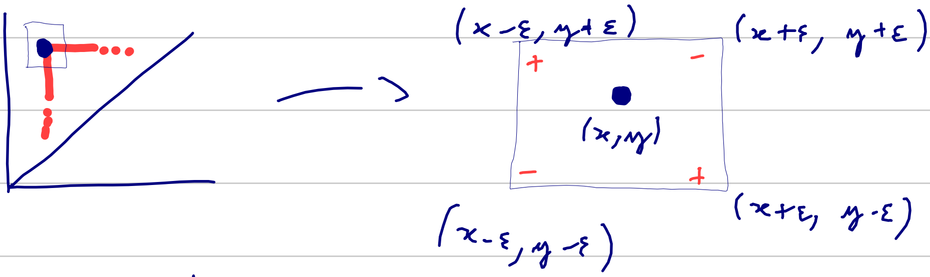
- saddle:



↳ arrangement of horizontal and vertical lines ∎

Def: $\mathrm{Dg}\, f := \{$outward corner points of the graph of $\beta\}$.
(⚠ multiset)

$$\mathrm{mult}_{\mathrm{Dg} f}((x,y)) := \lim_{\varepsilon \to 0} \beta(x+\varepsilon, y-\varepsilon) - \beta(x+\varepsilon, y+\varepsilon) + \beta(x-\varepsilon, y+\varepsilon) - \beta(x-\varepsilon, y-\varepsilon).$$



(cf. figures on previous page for examples)

Thm (stability): For any tame functions $f, g : X \to \mathbb{R}$,

$$d_B^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq \|f-g\|_\infty$$

$\boxed{\text{computing degree-0 persistence:}}$

(enough to record connected components)

Input: graph $G=(V,E)$, map $f: V \sqcup E \to \mathbb{R}$

Hyp: graph filtration, i.e. $f(u,v) \geq \max\{f(u), f(v)\}$ $\forall (u,v) \in E$.

Preproc:
- sort $V \sqcup E$ by increasing f-values
  - ↳ get a sequence $(\sigma_1, \dots, \sigma_m)$ of vertices/edges.
- initialize a union-find data structure