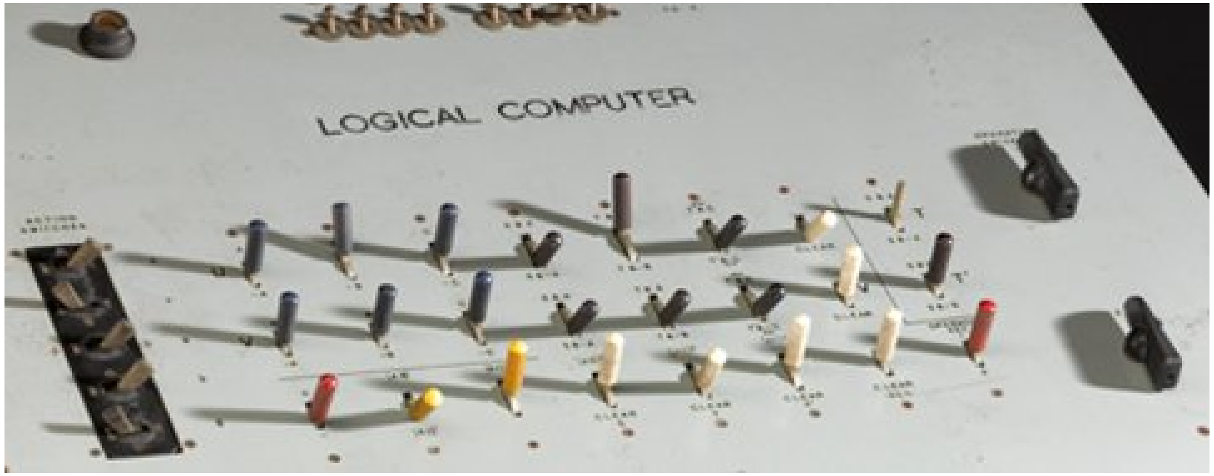
INF551

Computational Logic:

Artificial Intelligence in Mathematical Reasoning



Stéphane Graham-Lengrand

Stephane.Lengrand@Polytechnique.edu

Lecture IV

Semi-decision procedures for proof-search:

goal-directed techniques and unification

Automated reasoning techniques you know so far

For propositional logic: DPLL

For propositional logic

+ decision procedure for universally quantified conjunctions of literals: DPLL($\mathcal{T}$)

When general quantifiers are involved:

- **For some specific theories** (if lucky):

  decision procedure given by quantifier elimination techniques (cf Lecture 3)

- **Otherwise**:

  in general, problem not decidable

Semi-decidability

BUT:

the problem of provability (in predicate logic) is still semi-decidable

Semi-decision algorithm for provability of $\mathcal{T} \vdash A$:

Enumerate all trees (where nodes are labelled by sequents)

Check every time if it is a proof of $\mathcal{T} \vdash A$

Correctness:

If $\mathcal{T} \vdash A$ is provable, then at some point one of its proofs will be enumerated

Otherwise the algorithm does not terminate

Comments:

Sufficient to prove semi-decidability and Gödel's theorem

Useless in practice

Today and next week: better semi-decision algorithms, to be used in practice

Contents

I. Proof-search: from Natural Deduction to Sequent Calculus

The proof system you know so far: Natural Deduction

Introduction rules

Elimination rules

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \Rightarrow Q} \qquad \frac{\Gamma \vdash P \Rightarrow Q \quad \Gamma \vdash P}{\Gamma \vdash Q}$$

$$\frac{\Gamma \vdash P_1 \quad \Gamma \vdash P_2}{\Gamma \vdash P_1 \wedge P_2} \qquad \frac{\Gamma \vdash P_1 \wedge P_2}{\Gamma \vdash P_i}\, i \in \{1,2\}$$

$$\frac{\Gamma \vdash P_i}{\Gamma \vdash P_1 \vee P_2}\, i \in \{1,2\} \qquad \frac{\Gamma \vdash P_1 \vee P_2 \quad \Gamma, P_1 \vdash Q \quad \Gamma, P_2 \vdash Q}{\Gamma \vdash Q}$$

$$\frac{}{\Gamma \vdash \top} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash P}$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x P}\, x \notin \mathsf{FV}(\Gamma) \qquad \frac{\Gamma \vdash \forall x P}{\Gamma \vdash \{{}^{t}\!/\!{}_{x}\} P}$$

$$\frac{\Gamma \vdash \{{}^{t}\!/\!{}_{x}\} P}{\Gamma \vdash \exists x P} \qquad \frac{\Gamma \vdash \exists x P \quad \Gamma, P \vdash Q}{\Gamma \vdash Q}$$

$$\frac{}{\Gamma, P \vdash P} \qquad \frac{\Gamma \vdash \neg\neg P}{\Gamma \vdash P} \qquad \frac{\Gamma, P \vdash \bot}{\Gamma \vdash \neg P} \qquad \frac{\Gamma \vdash \neg P \quad \Gamma \vdash P}{\Gamma \vdash \bot}$$

Principles of goal-directed proof-search

Instead of enumerating all trees & then check if they are correct proof-trees of your goal...

...try to construct trees

- where each node (and its children) form a correct instance of inference rule
- whose root is labelled with your goal

Construct such a tree incrementally from its root, i.e. the goal Goal-directed

Incrementally = At each leaf, extend the proof-tree with new leaf:

the old leaf should be the conclusion of an inference rule

the new leaves should be its premisses

$$\dfrac{\dfrac{\dfrac{}{P, Q \vdash P}\,\textit{axiom} \qquad \dfrac{}{P, Q \vdash Q}\,\textit{axiom}}{P, Q \vdash P \wedge Q}\,\wedge\text{-intro}}{P \vdash Q \Rightarrow (P \wedge Q)}\,\Rightarrow\text{-intro}$$

Let's try an introduction rule. Another one. Let's try an axiom.

Elimination rules

Situation is not as good

Can always apply

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-elim}$$

B must be guessed (not present in the conclusion),

so all possible B should be enumerated?

An example:

$$\frac{\overline{P \wedge Q \vdash P \wedge Q}}{P \wedge Q \vdash P} \wedge\text{-elim}$$

How were $\wedge$ and Q guessed?

An asymmetry

In natural deduction, while trying to prove $\mathcal{T} \vdash A$,

Shape of A guides the choice of introduction rules

Shape of hypotheses in $\mathcal{T}$ does not guide the choice of elimination rules (at least directly)

The spirit of Sequent Calculus (Gentzen 1935)

Introduction rules (work well): they are kept = right rules

Elimination rules:

replaced by introduction rules for hypotheses = left rules

Example:

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge\text{-left}$$

Back to our example:

$$\frac{P, Q \vdash P}{P \wedge Q \vdash P} \wedge\text{-left}$$

Other left rules

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \vee\text{-left}$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B} \neg\text{-left}$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C} \Rightarrow\text{-left}$$

$$\frac{}{\Gamma, \bot \vdash A} \bot\text{-left}$$

Elimination of Double Negation

In Natural Deduction: EDN saves copies of formulae on the left.

Could be kept in Sequent Calculus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}\,\text{axiom} \qquad \dfrac{}{P, Q \vdash Q}\,\text{axiom}}{P, P \Rightarrow Q \vdash Q}\,\Rightarrow\text{-left}}{P, \neg Q, P \Rightarrow Q \vdash \bot}\,\neg\text{-left}}{P, \neg Q \vdash \neg(P \Rightarrow Q)}\,\neg\text{-right}}{\neg\neg(P \Rightarrow Q), P, \neg Q \vdash \bot}\,\neg\text{-left}}{\neg\neg(P \Rightarrow Q), P \vdash \neg\neg Q}\,\neg\text{-right}}{\neg\neg(P \Rightarrow Q), P \vdash Q}\,\text{EDN}$$

. . . or you can keep saved copies on the right

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}\,\text{axiom} \qquad \dfrac{}{P, Q \vdash Q}\,\text{axiom}}{P, P \Rightarrow Q \vdash Q}\,\Rightarrow\text{-left}}{P, P \Rightarrow Q \vdash (\bot,)Q}}{P \vdash \neg(P \Rightarrow Q), Q}\,\neg\text{-right}}{\neg\neg(P \Rightarrow Q), P \vdash (\bot,)Q}\,\neg\text{-left}}{\neg\neg(P \Rightarrow Q), P \vdash Q}}{\neg\neg(P \Rightarrow Q), P \vdash Q}$$

Sequents with several propositions on the right:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}\,\text{axiom} \qquad \dfrac{}{P, Q \vdash Q}\,\text{axiom}}{P, P \Rightarrow Q \vdash Q}\,\Rightarrow\text{-left}}{P \vdash \neg(P \Rightarrow Q), Q}\,\neg\text{-right}}{\neg\neg(P \Rightarrow Q), P \vdash Q}\,\neg\text{-left}$$

II. The rules of Sequent Calculus

Logical rules

$$\frac{}{\Gamma \vdash \top, \Delta}\ \top\text{-right} \qquad \frac{}{\Gamma, \bot \vdash \Delta}\ \bot\text{-left}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge\text{-right} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge\text{-left}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee\text{-right} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee\text{-left}$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \Rightarrow\text{-right} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow\text{-left}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg\text{-right} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg\text{-left}$$

The axiom rule: $\dfrac{}{\Gamma, A \vdash A, \Delta}\ \text{axiom}$

From bottom to top: every rule decreases the number of connectives

Height of proof-trees is bound!

Decision algorithm

. . . for propositional logic

What about quantifiers?

$$\frac{\Gamma \vdash P, \Delta}{\Gamma \vdash (\forall x\ P), \Delta}\ x \notin \mathsf{FV}(\Gamma, \Delta) \qquad \frac{\Gamma, (\forall x\ P), \{t/x\}P \vdash \Delta}{\Gamma, (\forall x\ P) \vdash \Delta}$$

$$\frac{\Gamma \vdash \{t/x\}P, (\exists x\ P), \Delta}{\Gamma \vdash (\exists x\ P), \Delta} \qquad \frac{\Gamma, P \vdash \Delta}{\Gamma, (\exists x\ P) \vdash \Delta}\ x \notin fv(\Gamma, \Delta)$$

Natural Deduction: hypotheses are permanent

can be used several times

$\forall$-left and $\exists$-right need to keep a *duplicata* of the formula

The drinker's theorem

Consider the following statement:

"There is always someone such that, if he drinks, everybody drinks"

Proof - Informal

Take the first guy you see, call it Bob.

Either Bob does not drink,

in which case he satisfies the predicate “if he drinks, everybody drinks”

. . . or Bob drinks, in which case we have to check that everybody else drinks

If this is the case, then again Bob is the person we are looking for

If we find someone who does not drink, call it Derek,

we change our mind and say that the guy we are looking for is Derek

Proof - Formal

Formal proof in sequent calculus:

Let A be the formula $\exists x\ (\text{drinks}(x) \Rightarrow \forall y\ \text{drinks}(y))$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\text{drinks}(\text{Bob}), \text{drinks}(y) \vdash \text{drinks}(y), \forall y'\ \text{drinks}(y'), A$}
\UnaryInfC{$\text{drinks}(\text{Bob}) \vdash \text{drinks}(y), \text{drinks}(y) \Rightarrow \forall y'\ \text{drinks}(y'), A$}
\RightLabel{$t = y$}
\UnaryInfC{$\text{drinks}(\text{Bob}) \vdash \text{drinks}(y), A$}
\UnaryInfC{$\text{drinks}(\text{Bob}) \vdash \forall y\ \text{drinks}(y), A$}
\UnaryInfC{$\vdash \text{drinks}(\text{Bob}) \Rightarrow \forall y\ \text{drinks}(y), A$}
\RightLabel{$t = \text{Bob}$}
\UnaryInfC{$\vdash A$}
\end{prooftree}
$$

How to translate a proof from Natural Deduction?

By induction on the size of the proof

Translating a proof of the form

$$\dfrac{\dfrac{\pi}{\Gamma \vdash A \wedge B}}{\Gamma \vdash A}\ \wedge\text{-elim}$$

By induction hypothesis, we get a proof-tree π' (in Sequent Calculus) of $\Gamma \vdash A \wedge B$

$$\dfrac{\dfrac{\pi'}{\Gamma \vdash A \wedge B} \quad \dfrac{\overline{\Gamma, A, B \vdash A}\ \text{axiom}}{\Gamma, A \wedge B \vdash A}\ \wedge\text{-left}}{\Gamma \vdash A}\ \color{red}{\text{cut}}$$

The cut rule

First, we add rule

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}$$

Secondly, show that this rule is superfluous

cut-elimination

Why eliminate cuts?

Destroys all advantages of Sequent Calculus

Two theorems

Equivalence with Natural Deduction:

$\Gamma \vdash A$ provable in ND iff provable in SC with cuts

Translations both ways

Cut-elimination:

$\Gamma \vdash \Delta$ has a proof in SC with cuts iff it has a cut-free proof

Cut-elimination by step-by-step reduction of proof

A typical case

$$
\dfrac{
\dfrac{\dfrac{\pi_1}{\Gamma, A, B \vdash \Delta}}{\Gamma, {\color{red}A \wedge B} \vdash \Delta}\,\wedge\text{-left}
\qquad
\dfrac{\dfrac{\pi_2}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_3}{\Gamma \vdash B, \Delta}}{\Gamma \vdash {\color{red}A \wedge B}, \Delta}\,\wedge\text{-right}
}{\Gamma \vdash \Delta}\,\text{cut}
$$

$$
\dfrac{
\dfrac{\dfrac{\pi_1}{\Gamma, A, B \vdash \Delta} \quad \dfrac{\pi_3}{\Gamma, A \vdash B, \Delta}}{\Gamma, A \vdash \Delta}\,\text{cut}
\qquad
\dfrac{\pi_2}{\Gamma \vdash A, \Delta}
}{\Gamma \vdash \Delta}\,\text{cut}
$$

The cut-elimination proof is by induction on the size of the formula been “cut”

(here, a cut on $A \wedge B$ is replaced by two “smaller cuts”, on A and on B, resp.)

III. Proof-search in the cut-free Sequent Calculus

Choices

Never need to enumerate all possible propositions

But... some choices still have to be made

1. Choice of sequent

$$\dfrac{\dfrac{}{P, Q \vdash P}\,{\color{red}?} \qquad \dfrac{}{P, Q \vdash Q}\,{\color{red}?}}{P, Q \vdash P \wedge Q}\ \wedge\text{-right}$$

2. Choice of Proposition to decompose or application of axiom

$$P \wedge Q \vdash Q \vee R$$

3. Choice of term

$$\dfrac{\Gamma, \forall x\ A, \{{}^{t}\!/\!_{x}\}A \vdash B}{\Gamma, \forall x\ A \vdash B}\ \forall\text{-left}$$

Several kinds of choices

Imagine you have to do “either $\mathfrak{A}$ or $\mathfrak{B}$”

General case: don’t know

choose $\mathfrak{A}$, then if $\mathfrak{A}$ fails choose $\mathfrak{B}$ (*backtrack*)

Irrelevant choice: don’t care

no matter whether you choose $\mathfrak{A}$ or $\mathfrak{B}$, the result will be the same

(sequentialising independent tasks to be done)

1. Choice of Sequent don’t care

$$\dfrac{\dfrac{}{P, Q \vdash P}\,? \qquad \dfrac{}{P, Q \vdash Q}\,?}{P, Q \vdash P \wedge Q}\,\wedge\text{-right}$$

2. Choice of Proposition to decompose or application of axiom don’t care!?

$$P \wedge Q \vdash Q \vee R$$

3. Choice of Term (disregarding duplicata) don’t know

$$\dfrac{\Gamma, \{{}^{t}\!/\!_{x}\} A \vdash B}{\Gamma, \forall x\, A \vdash B}\,\forall\text{-left}$$

Finite and infinite choice

Choice of Sequent: finite

Choice of Proposition: finite (if $\mathcal{T}$ finite)

Choice of term: infinite

Duplicata allows us to prevent backtracking

(can always re-attack formula with different term instead of backtracking)

Still, each time we have to choose 1 term among infinitely many:

$$\dfrac{\dfrac{}{p(f(f(c))) \vdash p(f(t)), \exists x\, p(f(x))}\text{ axiom}}{p(f(f(c))) \vdash \exists x\, p(f(x))}\ \exists\text{-right}$$

try c, $f(c)$, $f(f(c))$,... for term t

Which term to choose? How have we guessed it?

Can we delay the choice of the term?

Let's use a *meta-variable* X

Sequent Calculus with meta-variables: 1st attempt

$$\frac{\Gamma \vdash P, \Delta}{\Gamma \vdash (\forall x\, P), \Delta}\; x \notin fv(\Gamma, \Delta) \qquad \frac{\Gamma, (\forall x\, P), \{{}^{\mathbf{X}}\!/\!_x\} P \vdash \Delta}{\Gamma, (\forall x\, P) \vdash \Delta}\; \mathbf{X} \notin mv(\Gamma, \Delta)$$

$$\frac{\Gamma \vdash \{{}^{\mathbf{X}}\!/\!_x\} P, (\exists x\, P), \Delta}{\Gamma \vdash (\exists x\, P), \Delta}\; \mathbf{X} \notin mv(\Gamma, \Delta) \qquad \frac{\Gamma, P \vdash \Delta}{\Gamma, (\exists x\, P) \vdash \Delta}\; x \notin fv(\Gamma, \Delta)$$

Syntax must be enriched:

$$t ::= x \mid X \mid f(t_1, \ldots, t_n) \qquad \text{if } f/n \in \Sigma$$

$mv(t)$ and $mv(\Gamma)$ are the equivalent of $fv(t)$ and $fv(\Gamma)$ for meta-variables

Axiom and Substitution

Example: $$\frac{r(986) \vdash r(Y), (\exists y\, r(y))}{r(986) \vdash \exists y\, r(y)}$$

We wish to say: it's done by instantiating Y by 986

More generally, what to do for an axiom?

$$\frac{}{\Gamma, p(t_1, \ldots, t_n) \vdash p(u_1, \ldots, u_n), \Delta}$$

It would be good to instantiate all meta-variables so that,

for all i such that $1 \leq i \leq n$, we have $t_i = u_i$.

Substitution: partial function from meta-variables to terms ($\sigma(X) = t$), which can easily be extended to terms ($\sigma(u) = t$) as follows:

$$\begin{array}{lll}
\sigma(f(t_1, \ldots, t_n)) & = f(\sigma(t_1), \ldots, \sigma(t_n)) & \\
\sigma(x) & = x & \\
\sigma(X) & = X & \text{si } X \notin \text{domain}(\sigma)
\end{array}$$

Unifier and Propagation

Formally, we look for a substitution σ such that for all i with $1 \leq i \leq n$, we have $\sigma(t_i) = \sigma(u_i)$.

σ is a *unifier*, i.e.

σ is a solution to the *unification* problem $t_1 = u_1, \ldots, t_n = u_n$

Example Let $A := \forall x\, p(x,x)$ and $B := \exists y\, (p(y,0) \wedge p(y,S(0)))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{ok with } \sigma(Y) = \sigma(X) = 0}{A, p(X,X) \vdash p(Y,0), B}}{A \vdash p(Y,0), B} \qquad \dfrac{\dfrac{\text{ok with } \sigma'(Y) = \sigma'(X') = S(0)}{A, p(X',X') \vdash p(Y,S(0)), B}}{A \vdash p(Y,S(0)), B}}{A \vdash p(Y,0) \wedge p(Y,S(0)), B}}{A \vdash B}
$$

σ and σ' incompatible: impossible to reconstruct a proof. As soon as one of them is chosen, we have to propagate this choice into the other branch.

Sequent Calculus with meta-variables: 2nd attempt

The state of all open branches $(\Gamma_1 \vdash \Delta_1) \ldots (\Gamma_n \vdash \Delta_n)$ is grouped in a datastructure :

$$\Gamma_1 \vdash \Delta_1 \;\wr\; \ldots \;\wr\; \Gamma_n \vdash \Delta_n$$

$$\frac{\mathcal{S} \;\wr\; \Gamma \vdash P, \Delta}{\mathcal{S} \;\wr\; \Gamma \vdash (\forall x\, P), \Delta}\, x \notin fv(\Gamma, \Delta) \qquad \frac{\mathcal{S} \;\wr\; \Gamma, (\forall x\, P), \left\{{}^{\mathbf{X}}\!/\!_{x}\right\} P \vdash \Delta}{\mathcal{S} \;\wr\; \Gamma, (\forall x\, P) \vdash \Delta}\, \mathbf{X} \notin mv(\Gamma, \Delta)$$

$$\frac{\mathcal{S} \;\wr\; \Gamma \vdash \left\{{}^{\mathbf{X}}\!/\!_{x}\right\} P, (\exists x\, P), \Delta}{\mathcal{S} \;\wr\; \Gamma \vdash (\exists x\, P), \Delta}\, \mathbf{X} \notin mv(\Gamma, \Delta) \qquad \frac{\mathcal{S} \;\wr\; \Gamma, P \vdash \Delta}{\mathcal{S} \;\wr\; \Gamma, (\exists x\, P) \vdash \Delta}\, x \notin fv(\Gamma, \Delta)$$

$$\frac{\sigma(\mathcal{S})}{\mathcal{S} \;\wr\; \Gamma, p(t_1, \ldots, t_n) \vdash p(u_1, \ldots, u_n), \Delta}\, \sigma \text{ unifier of } (t_i = u_i)_{1 \leq i \leq n}$$

…and the logical rules are adapted

Connect.	Left-rules	Right-rules
$\top, \bot$	$\dfrac{\mathcal{S}}{\mathcal{S} \wr \Gamma, \bot \vdash \Delta}$	$\dfrac{\mathcal{S}}{\mathcal{S} \wr \Gamma \vdash \top, \Delta}$
$\neg$	$\dfrac{\mathcal{S} \wr \Gamma \vdash A, \Delta}{\mathcal{S} \wr \Gamma, \neg A \vdash \Delta}$	$\dfrac{\mathcal{S} \wr \Gamma, A \vdash \Delta}{\mathcal{S} \wr \Gamma \vdash \neg A, \Delta}$
$\vee$	$\dfrac{\mathcal{S} \wr \Gamma, A \vdash \Delta \;\wr\; \Gamma, B \vdash \Delta}{\mathcal{S} \wr \Gamma, A \vee B \vdash \Delta}$	$\dfrac{\mathcal{S} \wr \Gamma \vdash A, B, \Delta}{\mathcal{S} \wr \Gamma \vdash A \vee B, \Delta}$
$\wedge$	$\dfrac{\mathcal{S} \wr \Gamma, A, B \vdash \Delta}{\mathcal{S} \wr \Gamma, A \wedge B \vdash \Delta}$	$\dfrac{\mathcal{S} \wr \Gamma \vdash A, \Delta \;\wr\; \Gamma \vdash B, \Delta}{\mathcal{S} \wr \Gamma \vdash A \wedge B, \Delta}$
$\Rightarrow$	$\dfrac{\mathcal{S} \wr \Gamma \vdash A, \Delta \;\wr\; \Gamma, B \vdash \Delta}{\mathcal{S} \wr \Gamma, A \Rightarrow B \vdash \Delta}$	$\dfrac{\mathcal{S} \wr \Gamma, A \vdash B, \Delta}{\mathcal{S} \wr \Gamma \vdash A \Rightarrow B, \Delta}$

Unification

Let's come back to unifiers.

Questions:

Is there always a unifier for a problem $t_1 = t'_1, \ldots, t_n = t'_n$?

How to find it in non-trivial cases?

. . . **Robinson's unification algorithm**

Unification algorithm: an example

Solutions to problem

$$p(f(X)) = p(f(f(c)))$$

are the same as those of problem

$$f(X) = f(f(c))$$

are the same as those of problem

$$X = f(c)$$

and this problem has one solution: the substitution $X \mapsto f(c)$.

Unification algorithm: general case

Choose an equation in the system

- $f(t_1, ..., t_n) = f(u_1, ..., u_n)$ $\longrightarrow$ replace by $t_1 = u_1, ..., t_n = u_n$
- $f(t_1, ..., t_n) = g(u_1, ..., u_m)$ $\longrightarrow$ fail
- $X = X$ $\longrightarrow$ suppress
- $X = t$ (or $t = X$), X featured in t, t distinct from X, $\longrightarrow$ fail
- $X = t$ (or $t = X$), X not featured in t,

  $\longrightarrow$ substitute X by t in the rest of the system

  if solving it returns a substitution σ

  then return $\sigma \cup \{X \mapsto \sigma(t)\}$

The result is denoted $mgu(t_1 = t'_1, \ldots, t_n = t'_n)$ (most general unifier)

Is it finished?

Example Let $P_1 := \ \forall z\, p(z,z)$ and $P_2 := \ \exists x\, \forall y\, p(x, S(y))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{P_1, p(Z,Z) \vdash (p(X,S(y))), P_2}}{P_1 \vdash (p(X,S(y))), P_2}}{P_1 \vdash (\forall y\, p(X,S(y))), P_2}}{P_1 \vdash P_2}}{\vdash P_1 \Rightarrow P_2}
$$

with

$$
\begin{aligned}
mgu(Z = X, Z = S(y)): \quad & Z \mapsto S(y) \\
& X \mapsto S(y)
\end{aligned}
$$

The term for x could not use y, which is freed at a later stage!

The trick

Example Let $P_1 := \ \forall z\, p(z,z)$ and $P_2 := \ \exists x\, \forall y\, p(x, S(y))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P_1, p(Z,Z) \vdash_X (p(X, S(y(X)))), P_2}}{P_1 \vdash_X (p(X, S(y(X)))), P_2}}{P_1 \vdash_X (\forall y\, p(X, S(y))), P_2}}{P_1 \vdash P_2}}{\vdash P_1 \Rightarrow P_2}$$

not ok, since there are no unifiers for $Z = X, Z = S(y(X))$
$(mgu(Z = X, Z = S(y(X))) =$ Fail)

Just a technical trick, or deeper remark?

Compare $\vdash \exists x_1 \ldots \exists x_n \forall y\; P$ with $\vdash \exists x_1 \ldots \exists x_n \left\{{}^{Y(x_1,\ldots,x_n)}\!/_{y}\right\} P$

Equiprovable! (cf next week)

Sequent Calculus with meta-variables

This time it's the correct version!

$$\frac{\mathcal{S} \wr \Gamma \vdash_\Phi \left\{{}^{x(\Phi)}\!/_{x}\right\} P, \Delta}{\mathcal{S} \wr \Gamma \vdash_\Phi (\forall x\; P), \Delta}\; x \notin fv(\Gamma, \Delta)$$

$$\frac{\mathcal{S} \wr \Gamma, (\forall x\; P), \left\{{}^{\mathbf{X}}\!/_{x}\right\} P \vdash_{\Phi, \mathbf{X}} \Delta}{\mathcal{S} \wr \Gamma, (\forall x\; P) \vdash_\Phi \Delta}\; \mathbf{X} \notin mv(\Gamma, \Delta)$$

$$\frac{\mathcal{S} \wr \Gamma \vdash_{\Phi, \mathbf{X}} \left\{{}^{\mathbf{X}}\!/_{x}\right\} P, (\exists x\; P), \Delta}{\mathcal{S} \wr \Gamma \vdash_\Phi (\exists x\; P), \Delta}\; \mathbf{X} \notin mv(\Gamma, \Delta)$$

$$\frac{\mathcal{S} \wr \Gamma, \left\{{}^{x(\Phi)}\!/_{x}\right\} P \vdash_\Phi \Delta}{\mathcal{S} \wr \Gamma, (\exists x\; P) \vdash_\Phi \Delta}\; x \notin fv(\Gamma, \Delta)$$

$$\frac{\sigma(\mathcal{S})}{\mathcal{S} \wr \Gamma, p(t_1, \ldots, t_n) \vdash_\Phi p(u_1, \ldots, u_n), \Delta}\; \sigma = mgu(t_1 = u_1, \ldots t_n = u_n)$$

Final remarks

Using this Sequent Calculus for propositional logic, how does it compare to DPLL?

Not good (why?)

. . . but there are ways to adapt Sequent Calculus to be as efficient!

Next week:

- Using the above algorithmics to program: an introduction to **logic programming**
- Another proving technique that is not goal-directed: the **resolution method**

Questions?